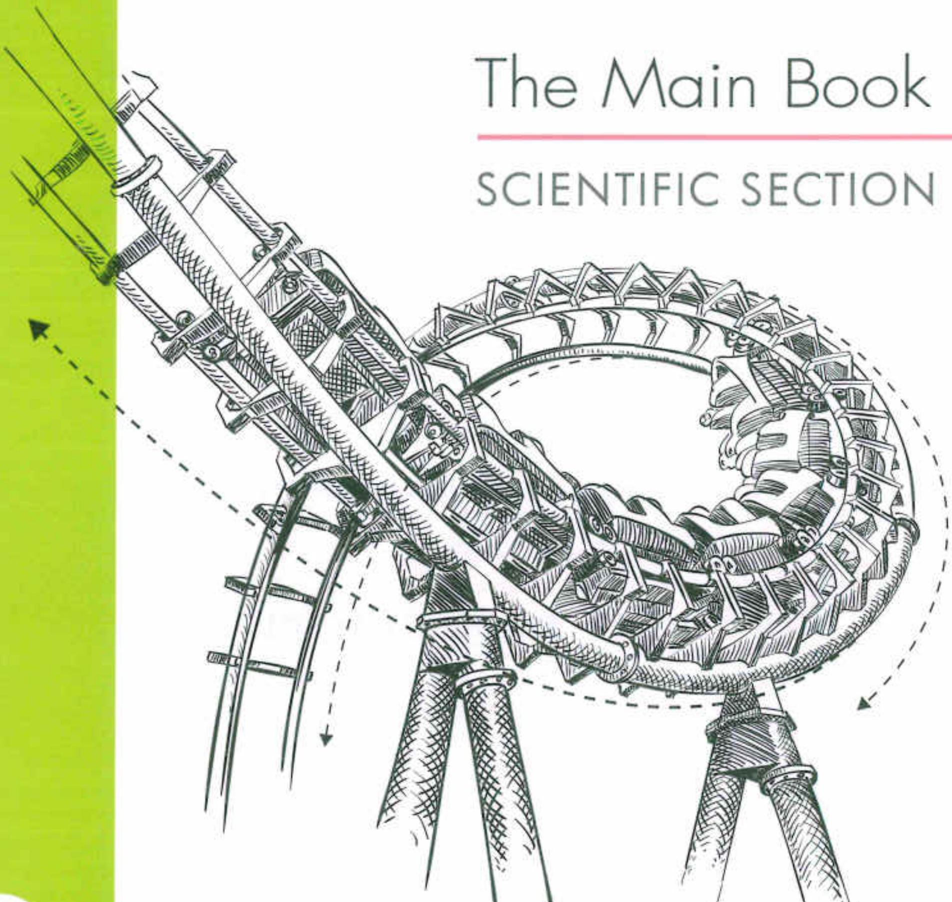
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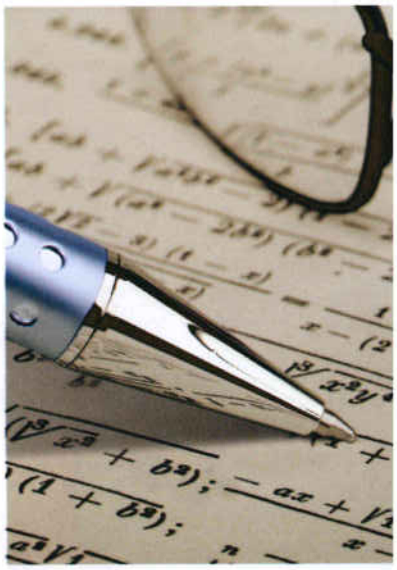
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First Algebra

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UNIT 2 Exponents , logarithms and their applications.



Second Calculus and Trigonometry

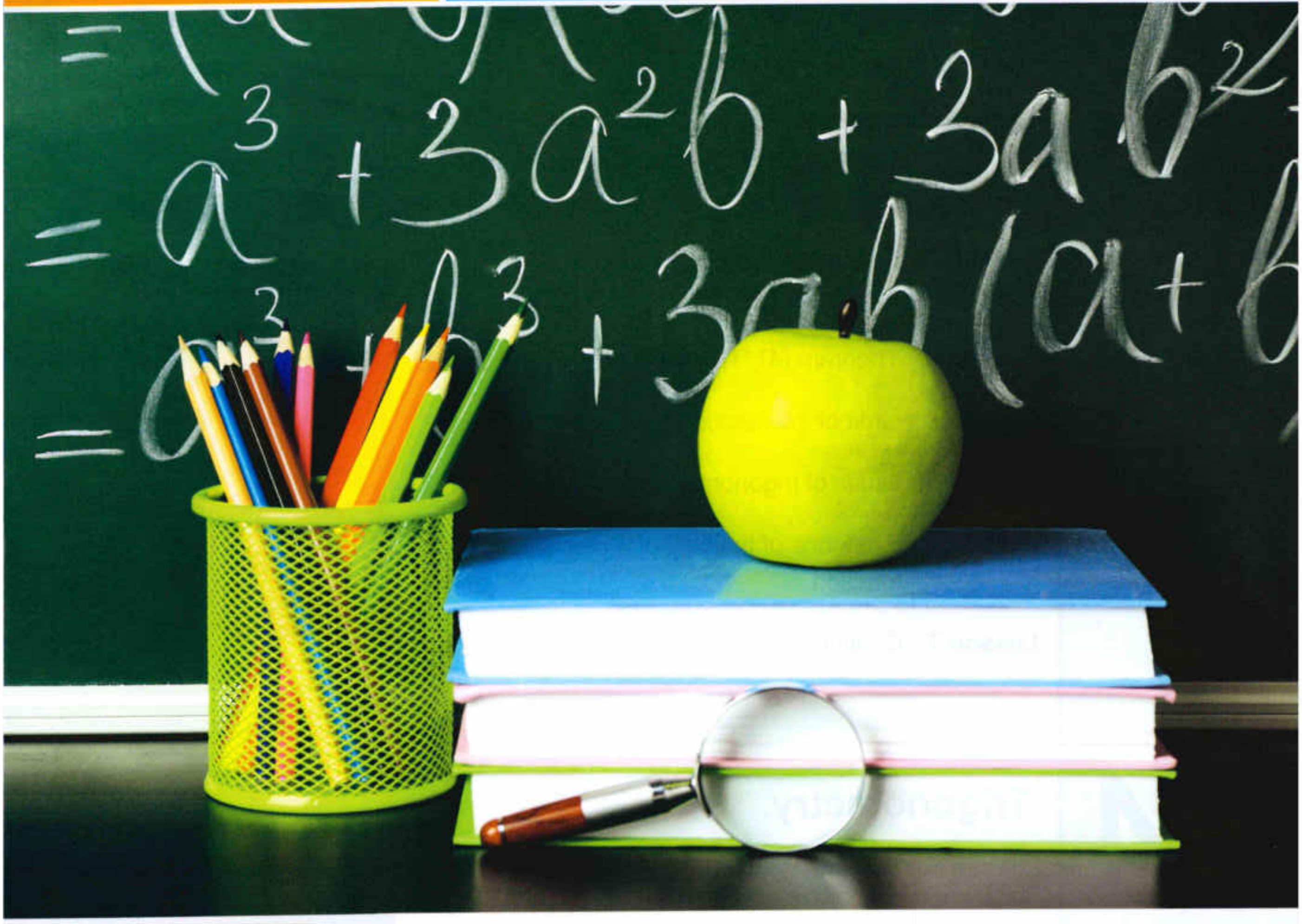
UNIT 3 Limits and continuity.



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First Algebra



UNIT 1 Functions of a real variable and drawing curves.

UNIT 2 Exponents , logarithms and their applications.

UNIT 1

Functions of a real variable and drawing curves.

Unit Lessons

- Pre-requirements for unit one.

Lesson 1 Real functions.
(Determination the domain and range – Discuss the monotony).

Lesson 2 Operations on functions – Composition of functions.

Lesson 3 Some properties of functions (even and odd functions / one - to - one functions)

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Pre-requirements for unit one

* If X and Y are two non-empty sets , then :

  It is said that the relation from the set X to the set Y is a function if each element in X is related with one and only one element in Y

  The set X is called «the domain of the function».

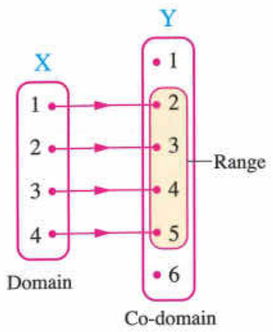
  The set Y is called «the co-domain of the function».

  The set of images of elements in the domain X is called «the range of the function» and it is subset of the co-domain Y

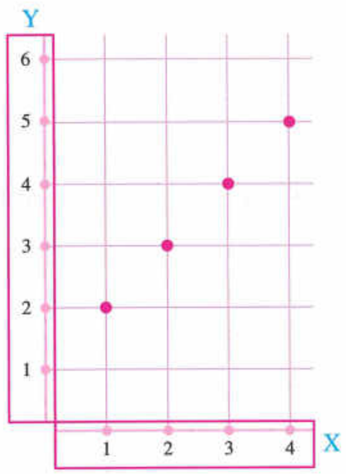
* The function f is written as $f : X \longrightarrow Y$, and the rule of the function is written as $y = f(x)$
* The set $\{(x , y) : x \in X , y \in Y , y = f(x)\}$ is called the set of ordered pairs of the function.
* The function can be represented by an arrow diagram or cartesian diagram.

For example :

* If $X = \{1 , 2 , 3 , 4\}$, $Y = \{1 , 2 , 3 , 4 , 5 , 6\}$ and the function $f : X \longrightarrow Y$ where $f(x) = x + 1$, then the set of ordered pairs of the function = $\{(1 , 2) , (2 , 3) , (3 , 4) , (4 , 5)\}$



Arrow diagram



Cartesian diagram

Notice that :

- Not each relation from X to Y is a function but all functions from X to Y are relations satisfy that :
  - Each element in X appears once as a first projection in one of the ordered pairs of the relation.
  - Each element in X has only one arrow going out to an element of Y in the arrow diagram which represents the relation.
  - Each vertical line has only one point from the points of the relation.

* The function $f : f(X) = a_0 + a_1 X + a_2 X^2 + a_3 X^3 + \cdots + a_n X^n$ where :
  $a_0 , a_1 , a_2 , a_3 , \ldots , a_n$ are constants , $a_n \in \mathbb{R} - \{0\}$
  is called polynomial function of nth degree and its domain and range are $\mathbb{R}$ if its not mention other than that.
* If $f(X) = a$ where $a \neq 0$, then the function f is called constant polynomial function and if $f(X) =$ zero , then the function f is called zero function and is represented graphically by the X-axis.
* The polynomial function is named according to the degree of the algebraic expression of the function.
* The polynomial function of first degree is called linear function , of second degree is called quadratic function and of third degree is called cubic function and so on.
* Set of zeroes of polynomial function f is the set of values of X that make $f(X) = 0$ and equals the set of X-coordinates of the points of intersection of the curve of the function with X-axis.
* Two polynomial functions f and g are equal in their common domain if :

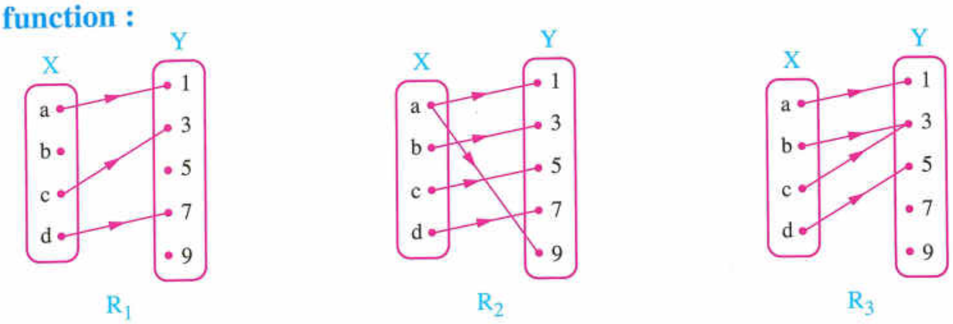
$f(X) = g(X)$

i.e. If they have the same degree and the coefficients of corresponding powers of X in them are equal.

* The function $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(X) = a X^n$ where $a \in \mathbb{R}^*$, $n \in \mathbb{Z}^+$ is called power function , so at adding or subtracting power functions with constants , we get a polynomial function.

Example 1

Show with reasons , which of the following relations (represented by the shown arrow diagrams) represents a function , if so , mention each of the domain and the range for every function :



Solution

- R_1 is not a function because there is no arrow from $b \in X$ to an element in Y
- R_2 is not a function because there are two arrows going from $a \in X$ to two elements in Y
- R_3 is a function because there is one and only one arrow drawn from each element in X to a corresponding element in Y

, the domain = $\{a , b , c , d\}$ and the range = $\{1 , 3 , 5\}$

Example 2

Determine the values of a , b and c which make $f(X) = g(X)$ where :

$f(X) = (a + b) X^2 - 3X - 4$, $g(X) = X^2 + (a + c) X + b$

Solution

$\because$ $f(X)$ and $g(X)$ have the same degree (second).

$\therefore$ $f(X) = g(X)$ when the coefficients of the corresponding powers of X are equal.

$\therefore a + b = 1$ (1)

, $a + c = -3$ (2)

, $b = -4$

Substituting in (1) : $\therefore a - 4 = 1$ $\therefore a = 5$

Substituting in (2) : $\therefore 5 + c = -3$ $\therefore c = -8$



Lesson One

Real functions

(Determination the domain and range - Discuss the monotony)

Real function

The function $f : X \longrightarrow Y$ is called a real function if each of the domain (X) and the co-domain (Y) is the set of the real numbers or a proper subset of it.

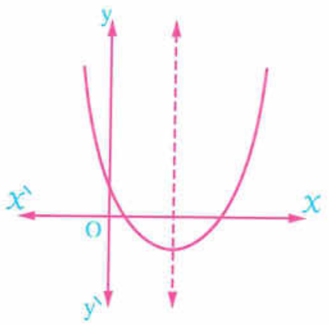
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Determining whether the relation from $X \longrightarrow Y$ is a function or not :

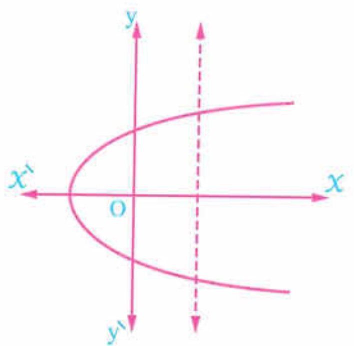
(1) Algebraically : The relation is a function if every value of the variable $x \in X$ is related with only one value of the variable $y \in Y$

(2) Graphically (The vertical line test) :

The relation is not a function if there exists at least one straight line parallel to y-axis and intersects the graph of the relation at more than one point.



Function from $X \longrightarrow Y$



Not Function from $X \longrightarrow Y$

Example 1

Show giving reasons , which of the following two relations is a function on $\mathbb{R}$:

(1) $y = x^2 + 3$ (2) $y^2 = x^2 + 9$

Solution

(1) The relation $y = x^2 + 3$ is a function because every real value of the variable x is related with a unique value of the variable y

For example : When $x = 3$, then $y = 12$ and when $x = -2$, then $y = 7$ and so on.

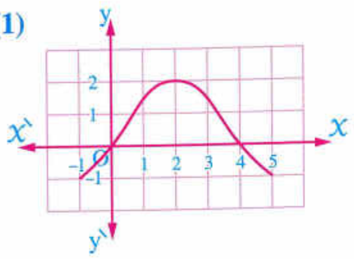
(2) The relation $y^2 = x^2 + 9$ is not a function because there is at least one real value of the variable x is related with two different values of the variable y

For example : When $x = 4$, then $y^2 = 25$ $\therefore y = \pm 5$

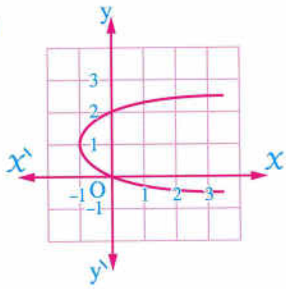
Example 2

Show which of the following graphs represents a function on $\mathbb{R}$, which doesn't represent a function giving reasons :

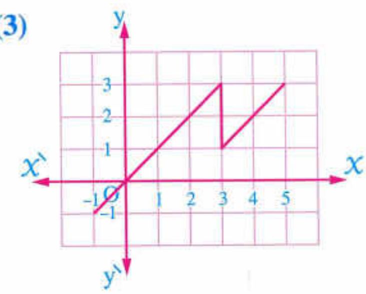
(1)



(2)



(3)



Solution

(1) Represents a function for each vertical line intersects the curve at one point at most.

(2) Does not represent a function for there are many vertical lines intersect the curve at two points.

(3) Does not represent a function for there is a vertical line passing through the point (3 , 0) and intersect the curve at a set of points.

Remarks

1. The relation $y = 4$ (represented by a horizontal straight line parallel to x-axis) is a function from X to Y because each element in X is related with only one element in Y

2. The relation $x = 4$ (represented by a vertical straight line parallel to y-axis is not a function from X to Y because the element $x = 4$ is related with infinite number of elements in Y

Identifying the domain of the real functions

The domain of the function is identified by its rule or its graph.

First Identifying the domain of the function from its rule

(1) Polynomial function

$f : f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$ where $a_0 , a_1 , a_2 , ... , a_n$ are real constants , $a_n \in \mathbb{R} - \{0\}$ is called polynomial function of nth degree , then the domain of the polynomial function is $\mathbb{R}$ unless it is defined on a subset of it.

For example : $f : f(x) = 3$ (Constant polynomial) , its domain $= \mathbb{R}$

, $f : f(x) = 2x + 1 ,\ x \leq 1$ (First degree polynomial) , its domain $=]-\infty , 1]$

, $f : f(x) = x^2 - 4x + 3$ (Second degree polynomial) , its domain $= \mathbb{R}$

(2) Rational function

If f is a rational function where $f(x) = \frac{h(x)}{g(x)}$, h and g are two polynomials , then the domain of the function $f = \mathbb{R}$ – the set of zeroes of the denominator.

Example (3)

State the domain of each of the rational functions defined by the following rules :

(1) $f(x) = \frac{1}{x}$

(2) $f(x) = \frac{3}{x - 2}$

(3) $f(x) = \frac{x - 1}{2x^2 + 5x}$

(4) $f(x) = \frac{x - 3}{x^2 - 5x + 6}$

(5) $f(x) = \frac{x + 1}{x^2 - 4x + 4}$

(6) $f(x) = \frac{x}{x^2 + 25}$

Solution

(1) The domain $= \mathbb{R} - \{0\}$

(2) The domain $= \mathbb{R} - \{2\}$

(3) Let $2x^2 + 5x = 0$

$\therefore x(2x + 5) = 0$

$\therefore x = 0$ or $x = \frac{-5}{2}$

$\therefore$ The domain $= \mathbb{R} - \{0 , \frac{-5}{2}\}$

(4) Let $x^2 - 5x + 6 = 0$

$\therefore (x - 2)(x - 3) = 0$

$\therefore x = 2$ or $x = 3$

$\therefore$ The domain $= \mathbb{R} - \{2 , 3\}$

(5) Let $x^2 - 4x + 4 = 0$ $\therefore (x-2)^2 = 0$

$\therefore x = 2$ $\therefore$ The domain $= \mathbb{R} - \{2\}$

(6) Let $x^2 + 25 = 0$

and this equation has no solution in $\mathbb{R}$ *i.e.* There are no real zeroes of the denominator

$\therefore$ The domain $= \mathbb{R}$

3 The nth root function

If $f(x) = \sqrt[n]{h(x)}$ where $n \in \mathbb{Z}^+$, $n > 1$, $h(x)$ is a polynomial

First : When (n) is an odd number , then the domain of $f = \mathbb{R}$

Second : When (n) is an even number , then :

The domain of f is the set of all values of x which satisfy $h(x) \geq 0$, n is called the index of the root.

Example 4

State the domain of each of the real functions which are defined by the following rules :

(1) $f(x) = \sqrt{x+2}$

(2) $f(x) = \sqrt{x^2+5}$

(3) $f(x) = \sqrt{-2x+3}$

(4) $f(x) = \sqrt[3]{9-x^2}$

(5) $f(x) = \sqrt{4x^2 - 12x + 9}$

(6) $f(x) = \sqrt{x^2 - 4}$

(7) $f(x) = \dfrac{1}{\sqrt{4 + 3x - x^2}}$

Solution

(1) $\because$ The index of the root is an even number.

$\therefore$ The function is defined where $x + 2 \geq 0$

$\therefore x \geq -2$ $\therefore$ The domain $= [-2 , \infty[$

(2) $\because$ The index of the root is even.

$\therefore$ The function is defined where $x^2 + 5 \geq 0$ and that is true for all real values.

$\therefore$ The domain is $\mathbb{R}$

(3) $\because$ The index of the root is an even number.

$\therefore -2x + 3 \geq 0$ $\therefore x \leq \frac{3}{2}$

$\therefore$ The domain $=]-\infty , \frac{3}{2}]$

(4) $\because$ The index of the root is an odd number. $\therefore$ The domain $= \mathbb{R}$

(5) $\because$ The index of the root is an even number.

$\therefore$ The function is defined where : $4x^2 - 12x + 9 \geq 0$

$\therefore (2x - 3)^2 \geq 0$ $\therefore$ The domain of the function $= \mathbb{R}$

(6) $\because$ The index of the root is an even number.

$\therefore$ The function is defined where : $x^2 - 4 \geq 0$

$\therefore (x - 2)(x + 2) \geq 0$

, $\because$ S.S. of the inequality $= \mathbb{R} -]-2 , 2[$

$\therefore$ The domain of the function $= \mathbb{R} -]-2 , 2[$

(7) $\because$ The function is defined where : $4 + 3x - x^2 > 0$

$\therefore x^2 - 3x - 4 < 0$

$\therefore (x - 4)(x + 1) < 0$

, $\because$ S.S. of the inequality $=]-1 , 4[$

$\therefore$ The domain of the function $=]-1 , 4[$

Remember that *(Solving inequalities of second degree in one variable)*

If ℓ and m where $\ell <$ m are two real roots of the equation : $a x^2 + b x + c = 0$, $a > 0$, then the S.S. in $\mathbb{R}$ of the inequality :

(1) $a x^2 + b x + c \geq 0$ equals $\mathbb{R} -]\ell , m[$

(2) $a x^2 + b x + c > 0$ equals $\mathbb{R} - [\ell , m]$

(3) $a x^2 + b x + c \leq 0$ equals $[\ell , m]$

(4) $a x^2 + b x + c < 0$ equals $]\ell , m[$

4 Piecewise function

It is the function that is defined by different rules for different parts of its domain.

Example 5

Determine the domain of each of the two functions defined by the following rules :

(1) $f(x) = \begin{cases} 2 - x & , \quad x < 0 \\ x - 2 & , \quad x > 0 \end{cases}$

(2) $f(x) = \begin{cases} x^2 & , \quad -2 \leq x < 0 \\ x & , \quad 0 \leq x \leq 1 \\ \frac{1}{x} & , \quad x > 1 \end{cases}$

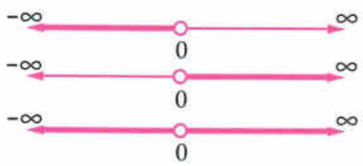
Solution

(1) The function f is defined on two intervals as the following :

Defined when $x \in]-\infty , 0[$

, defined when $x \in]0 , \infty[$

$\therefore$ Domain of $f =]-\infty , 0[\cup]0 , \infty[= \mathbb{R} - \{0\}$



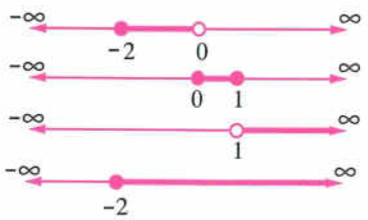
(2) The function f is defined on three intervals as the following :

Defined when $x \in [-2 , 0[$

, defined when $x \in [0 , 1]$

, defined when $x \in]1 , \infty[$

$\therefore$ Domain of $f = [-2 , 0[\cup [0 , 1] \cup]1 , \infty[= [-2 , \infty[$

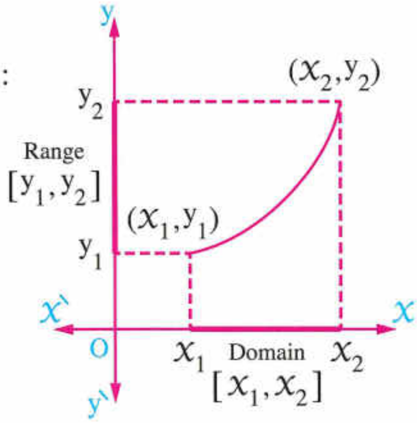


Second Identifying the domain and range of the function from its graph

From the graph of the function we can deduce the domain and the range of the function to be :

(1) Domain of the function is the set of the x-coordinates of all the points that lie on the curve of the function.

(2) Range of the function is the set of the y-coordinates of all the points that lie on the curve of the function.





Example 6

Determine the domain and range for each function represented by the following figures :

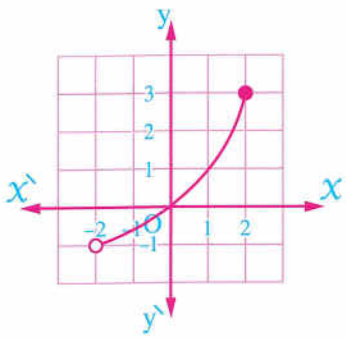


Fig. (1)

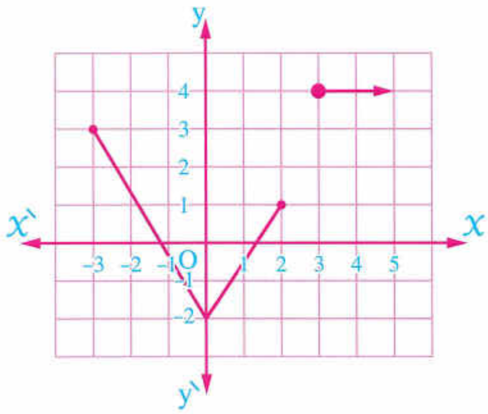


Fig. (2)

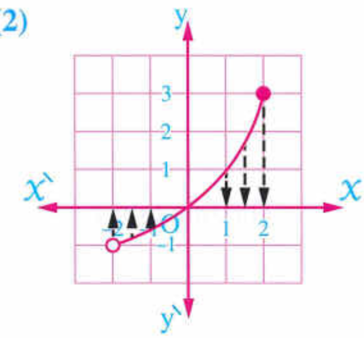
Solution

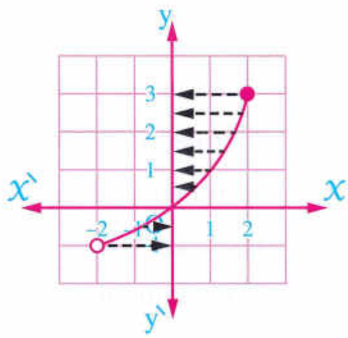
In fig. (1) : * The x-coordinates of all points on the curve of the function are on the interval $]-2, 2]$

$\therefore$ The domain $=]-2, 2]$

* The y-coordinates of all points on the curve of the function are on the interval $]-1, 3]$

$\therefore$ The range $=]-1, 3]$



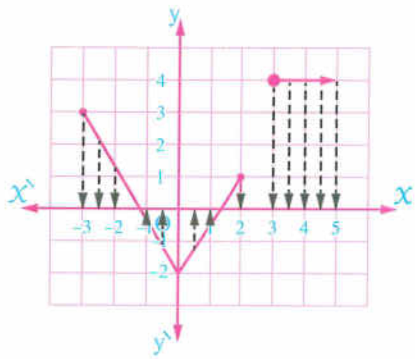


Notice that :

* The unshaded circle at point $(-2, -1)$ shows that the point $\notin$ the function and so $-2 \notin$ the domain of the function and $-1 \notin$ the range of the function.
* The shaded circle at point $(2, 3)$ shows that the point $\in$ the function and so $2 \in$ the domain of the function and $3 \in$ the range of the function.

In fig. (2) : * The x-coordinates of all points on the curve of the function are on the two intervals $[-3, 2]$ and $[3, \infty[$

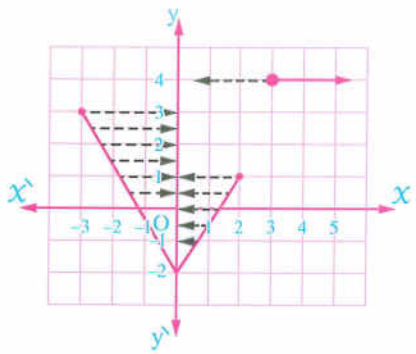
$\therefore$ The domain $= [-3, 2] \cup [3, \infty[$



* The y-coordinates of the points at the horizontal ray is $y = 4$

, the y-coordinates of the other points of the curve are on the interval $[-2, 3]$

$\therefore$ The range $= [-2, 3] \cup \{4\}$



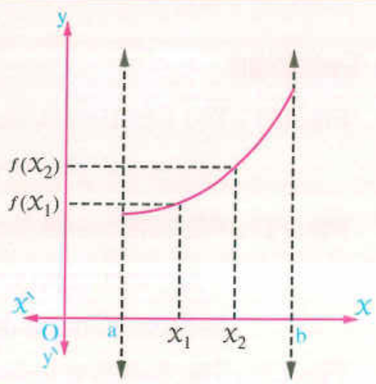
Discussing the monotony of a function from its graph

Discussion of the monotony (monotonicity) of a function means identifying the intervals on which the function is increasing , the intervals on which the function is decreasing , and the intervals on which the function is constant.

Definition (1) *(Increasing function) :*

The function f is said to be increasing on an interval $]a, b[$ if :

$x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$ for every $x_1, x_2 \in]a, b[$



Definition (2) ***(Decreasing function) :***

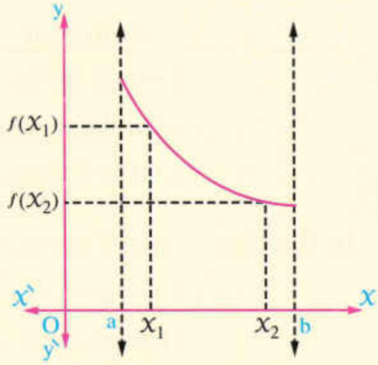
The function

f is said to be decreasing on

an interval $]a , b[$ if :

$x_2 > x_1 \Rightarrow f(x_2) < f(x_1)$ for every

$x_1 , x_2 \in]a , b[$



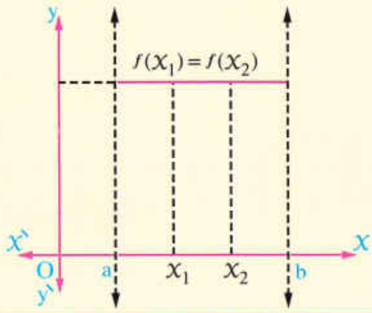
Definition (3) ***(Constant function) :***

The function

f is said to be constant on an interval $]a , b[$ if :

$x_2 > x_1 \Rightarrow f(x_2) = f(x_1)$ for every

$x_1 , x_2 \in]a , b[$



Example 7

Discuss the monotonicity of each of the functions represented by the following graphs :

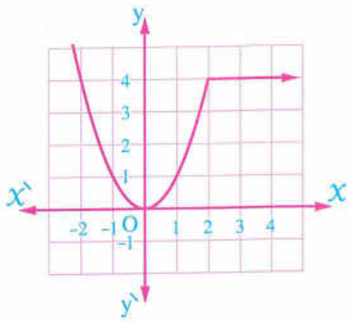


Fig. (1)

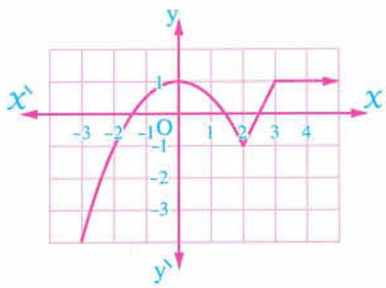


Fig. (2)

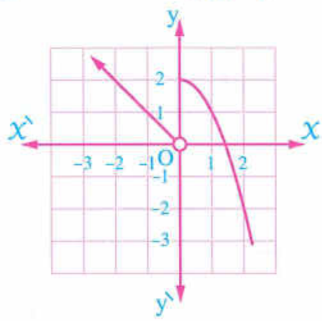


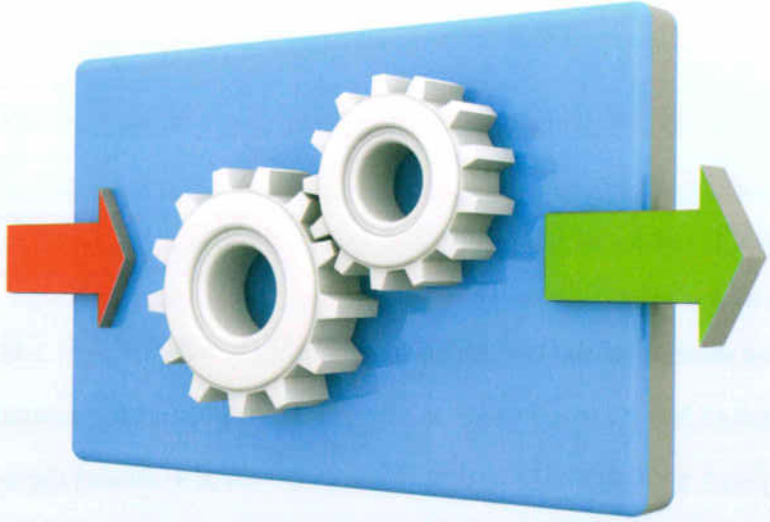
Fig. (3)

Solution

Fig. (1) : The function is decreasing on the interval $]-\infty , 0[$
, increasing on the interval $]0 , 2[$ and constant on the interval $]2 , \infty[$

Fig. (2) : The function is increasing on the interval $]-\infty , 0[$
, decreasing on the interval $]0 , 2[$, increasing on the interval $]2 , 3[$
and constant on the interval $]3 , \infty[$

Fig. (3) : The function is decreasing on each of the two intervals $]-\infty , 0[$ and $]0 , \infty[$



Lesson Two — Operations on functions – Composition of functions

If f_1 , f_2 are two functions whose domains are D_1 and D_2 respectively , then :

(1) $(f_1 \pm f_2)(X) = f_1(X) \pm f_2(X)$ and the domain of $(f_1 \pm f_2)$ is $D_1 \cap D_2$

(2) $(f_1 \times f_2)(X) = f_1(X) \times f_2(X)$ and the domain of $(f_1 \times f_2)$ is $D_1 \cap D_2$

(3) $\left(\frac{f_1}{f_2}\right)(X) = \frac{f_1(X)}{f_2(X)}$ such that $f_2(X) \neq$ zero

, the domain of $\left(\frac{f_1}{f_2}\right)$ is $(D_1 \cap D_2) - Z(f_2)$ where $Z(f_2)$ is the set of zeroes of f_2

Noticing that in all the operations on the functions , the domain of the resulting function equals the intersection of the domains of the two functions except the zeroes of the divisor in the division operation.

Example 1

If $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ where $f(X) = 2X^2 - 7X + 5$

and $g :]-\infty , 4] \longrightarrow \mathbb{R}$ where $g(X) = 2X - 5$

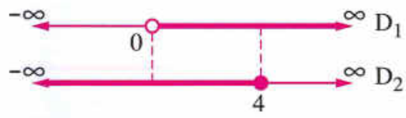
Find : (1) $(f + g)(X)$ **(2)** $(f - g)(X)$

(3) $(f \times g)(X)$ **(4)** $\left(\frac{f}{g}\right)(X)$

, then state the domain of each of them and calculate :

$(f + g)(3)$, $(f - g)(0)$, $(f \times g)(-3)$ and $\left(\frac{f}{g}\right)(1)$

Solution



- The domain of $f = D_1 = \mathbb{R}^+ =]0 , \infty[$
- The domain of $g = D_2 =]-\infty , 4]$

$\therefore$ The common domain of the two functions $= D_1 \cap D_2 = \mathbb{R}^+ \cap]-\infty , 4] =]0 , 4]$

(1) $(f + g)(X) = (2X^2 - 7X + 5) + (2X - 5) = 2X^2 - 5X$ and the domain $=]0 , 4]$

(2) $(f - g)(X) = (2X^2 - 7X + 5) - (2X - 5) = 2X^2 - 9X + 10$ and the domain $=]0 , 4]$

(3) $(f \times g)(X) = (2X^2 - 7X + 5)(2X - 5) = 4X^3 - 24X^2 + 45X - 25$ and the domain $=]0 , 4]$

(4) $\left(\frac{f}{g}\right)(X) = \frac{2X^2 - 7X + 5}{2X - 5} = \frac{(2X - 5)(X - 1)}{(2X - 5)} = X - 1$ and the domain $=]0 , 4] - \left\{\frac{5}{2}\right\}$

The numerical values :

- $(f + g)(3) = 2(9) - 5(3) = 3$
- $(f - g)(0)$ is undefined because $0 \notin]0 , 4]$
- $(f \times g)(-3)$ is undefined because $-3 \notin]0 , 4]$
- $\left(\frac{f}{g}\right)(1) = 0$

Example 2

If f and g are two functions where $f(X) = \frac{X}{X + 1}$ and $g(X) = \frac{X + 1}{X - 2}$, find :

(1) $(f + g)(X)$ and calculate $(f + g)(3)$

(2) $(f - g)(X)$ and calculate $(f - g)(2)$

(3) $(f \times g)(X)$ and calculate $(f \times g)(3)$

(4) $\left(\frac{f}{g}\right)(X)$ and calculate $\left(\frac{f}{g}\right)(-1) , \left(\frac{f}{g}\right)(1)$

Solution

The domain of $f = D_1 = \mathbb{R} - \{-1\}$ and the domain of $g = D_2 = \mathbb{R} - \{2\}$

$\therefore$ The common domain of the two functions is : $D_1 \cap D_2 = \mathbb{R} - \{-1 , 2\}$

(1) $(f + g)(X) = \frac{X}{X + 1} + \frac{X + 1}{X - 2} = \frac{X(X - 2) + (X + 1)^2}{(X + 1)(X - 2)}$

$$= \frac{X^2 - 2X + X^2 + 2X + 1}{(X + 1)(X - 2)} = \frac{2X^2 + 1}{(X + 1)(X - 2)}$$

and the domain is $\mathbb{R} - \{-1 , 2\}$, $(f + g)(3) = \frac{2(9) + 1}{(4)(1)} = \frac{19}{4}$

(2) $(f - g)(x) = \frac{x}{x+1} - \frac{x+1}{x-2} = \frac{x(x-2)-(x+1)^2}{(x+1)(x-2)} = \frac{-4x-1}{(x+1)(x-2)}$

and the domain is $\mathbb{R} - \{-1, 2\}$, $(f - g)(2)$ is undefined because $2 \notin$ the domain of $(f - g)$

(3) $(f \cdot g)(x) = \frac{x}{x+1} \times \frac{x+1}{x-2} = \frac{x}{x-2}$

and the domain $= \mathbb{R} - \{-1, 2\}$, $(f \cdot g)(3) = \frac{3}{1} = 3$

(4) $\left(\frac{f}{g}\right)(x) = \frac{x}{x+1} \div \frac{x+1}{x-2} = \frac{x}{x+1} \times \frac{x-2}{x+1} = \frac{x(x-2)}{(x+1)^2}$

and the domain is $\mathbb{R} - \{-1, 2\}$

$\left(\frac{f}{g}\right)(-1)$ is undefined because $-1 \notin$ the domain of $\left(\frac{f}{g}\right)$, $\left(\frac{f}{g}\right)(1) = \frac{1(-1)}{2^2} = -\frac{1}{4}$

Example 3

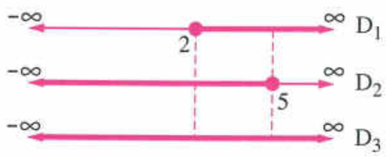
If : $f_1(x) = \sqrt{x-2}$, $f_2(x) = \sqrt{5-x}$, $f_3(x) = x - 3$

Find the rule and the domain of each of the following functions :

(1) $(f_1 + f_2)$ (2) $(f_2 - f_3)$ (3) $(f_1 \times f_2)$

(4) $\left(\frac{f_2}{f_3}\right)$ (5) $\left(\frac{f_3}{f_1}\right)$

Solution



$D_1 = [2, \infty[$, $D_2 =]-\infty, 5]$ and $D_3 = \mathbb{R}$

(1) $(f_1 + f_2)(x) = \sqrt{x-2} + \sqrt{5-x}$ and the domain $= [2, 5]$

(2) $(f_2 - f_3)(x) = \sqrt{5-x} + 3 - x$ and the domain $=]-\infty, 5]$

(3) $(f_1 \times f_2)(x) = \sqrt{x-2} \times \sqrt{5-x}$ and the domain $= [2, 5]$

(4) $\left(\frac{f_2}{f_3}\right)(x) = \frac{\sqrt{5-x}}{x-3}$ and the domain $=]-\infty, 5] - \{3\}$

(5) $\left(\frac{f_3}{f_1}\right)(x) = \frac{x-3}{\sqrt{x-2}}$ and the domain $=]2, \infty[$

Composition of functions

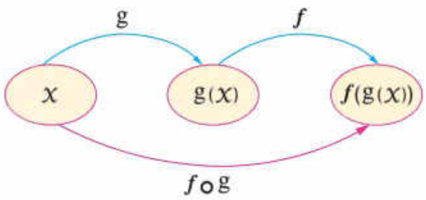
If f and g are two functions , and range of g intersection domain of f is not equal to $\varnothing$ then the composition of the function f with the function g gives a new function $(f \circ g)$ which is read as (f composed g) or (f after g)

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where $(f \circ g)(X) = f(g(X))$ such that the rule of the function g is applied at first , then the rule of the function f secondly where the domain of $(f \circ g)$ consists of values of X in domain of g which make $g(X)$ in the domain of f

i.e. Domain of $(f \circ g) = \{X : X \in \text{domain of } g , g(X) \in \text{domain of } f\}$

The opposite diagram shows the previous definition :



Illustrated Example

If g and f are two functions defined by a set of ordered pairs as the following :

$g = \{(1 , 5) , (2 , 10) , (3 , 15)\}$

$, f = \{(10 , 20) , (15 , 30) , (16 , 32)\}$, then

Domain of $g = \{1 , 2 , 3\}$, domain of $f = \{10 , 15 , 16\}$

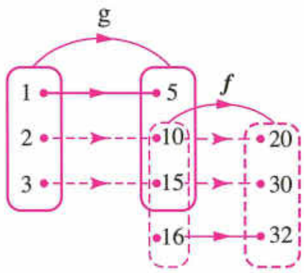
Domain of $(f \circ g)$

$= \{X : X \in \text{domain of } g , g(X) \in \text{domain of } f\}$

= the set of first projections in g whose second projections appear first projections in f

$\therefore$ Domain of $(f \circ g) = \{2 , 3\}$

, range of $(f \circ g) = \{20 , 30\}$, $(f \circ g) = \{(2 , 20) , (3 , 30)\}$



Example 4

If $f(X) = X^3$, $g(X) = X + 1$, then find :

(1) $(f \circ g)(1)$

(2) $(g \circ f)(1)$

(3) $(f \circ f)(-1)$

(4) $(g \circ g)(-1)$

Solution

(1) $\because (f \circ g)(X) = f\big(g(X)\big)$, $\because g(1) = 1 + 1 = 2$

$\therefore (f \circ g)(1) = f\big(g(1)\big)$

$\therefore (f \circ g)(1) = f(2) = 2^3 = 8$

(2) $\because (g \circ f)(X) = g\big(f(X)\big)$, $\because f(1) = 1^3 = 1$

$\therefore (g \circ f)(1) = g\big(f(1)\big)$

$\therefore (g \circ f)(1) = g(1) = 1 + 1 = 2$

(3) $\because (f \circ f)(x) = f\left(f(x)\right)$ $\therefore (f \circ f)(-1) = f\left(f(-1)\right)$

, $\because f(-1) = (-1)^3 = -1$ $\therefore (f \circ f)(-1) = f(-1) = -1$

(4) $(g \circ g)(x) = g\left(g(x)\right)$ $\therefore (g \circ g)(-1) = g\left(g(-1)\right)$

, $\because g(-1) = -1 + 1 = 0$ $\therefore (g \circ g)(-1) = g(0) = 0 + 1 = 1$

Example 5

If $f(x) = 2x + 1$, $g(x) = x^2 - 3$, then find each of the following compositions of functions :

(1) $(f \circ g)(x)$ (2) $(g \circ f)(x)$ (3) $(f \circ f)(x)$

Solution

(1) $(f \circ g)(x) = f\left(g(x)\right)$, putting $g(x)$ instead of x in the function f

$\therefore (f \circ g)(x) = 2\,g(x) + 1$, put $g(x) = x^2 - 3$

$\therefore (f \circ g)(x) = 2(x^2 - 3) + 1$, simplifying the resulted expression.

$\therefore (f \circ g)(x) = 2x^2 - 6 + 1 = 2x^2 - 5$

(2) $(g \circ f)(x) = g\left(f(x)\right)$, putting $f(x)$ instead of x in the function g

$\therefore (g \circ f)(x) = \left(f(x)\right)^2 - 3$, putting $f(x) = 2x + 1$

$\therefore (g \circ f)(x) = (2x + 1)^2 - 3$, simplifying the resulted expression.

$\therefore (g \circ f)(x) = 4x^2 + 4x + 1 - 3 = 4x^2 + 4x - 2$

(3) $(f \circ f)(x) = f\left(f(x)\right)$ $\therefore (f \circ f)(x) = 2\left(f(x)\right) + 1$

$\therefore (f \circ f)(x) = 2(2x + 1) + 1$ $\therefore (f \circ f)(x) = 4x + 2 + 1 = 4x + 3$

Remark

In the previous example notice that : $(f \circ g)(x) \neq (g \circ f)(x)$

, hence we can deduce that : $f \circ g \neq g \circ f$

i.e. Operation of composition of functions is not commutative.

Example 6

If f is a linear function and $(f \circ f)(x) = 4x + 3$ find $f(x)$:

Solution

Let $f(x) = ax + b$,

$\because (f \circ f)(x) = 4x + 3$ $\qquad \therefore f\left(f(x)\right) = 4x + 3$

$\therefore f(ax + b) = 4x + 3$ $\qquad \therefore a(ax + b) + b = 4x + 3$

$\therefore a^2 x + ab + b = 4x + 3$ $\qquad \therefore a^2 = 4$ and so $a = \pm 2$

, $ab + b = 3$ $\qquad \therefore b = \frac{3}{a+1}$

at $a = 2$ $\qquad \therefore b = 1$, at $a = -2$ $\qquad \therefore b = -3$

$\therefore f(x) = 2x + 1$ or $f(x) = -2x - 3$

Example 7

If $f(x) = 2x + 1$, $(f \circ g)(x) = x^2 + x - 1$ find $g(x)$

Solution

$\because (f \circ g)(x) = x^2 + x - 1$ $\qquad \therefore f\left(g(x)\right) = x^2 + x - 1$

$\therefore 2g(x) + 1 = x^2 + x - 1$ $\qquad \therefore 2g(x) = x^2 + x - 2$

$\therefore g(x) = \frac{1}{2}x^2 + \frac{1}{2}x - 1$

Example 8

If : $f\ \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = 5x - 2$, $g(x) = \begin{cases} x + 1 & , \quad x > 3 \\ 2x & , \quad x \leq 3 \end{cases}$

Find : (1) $(f \circ g)(4)$ (2) $(g \circ f)(1)$

Solution

(1) $(f \circ g)(4) = f\left(g(4)\right)$ (Notice that $g(4) = 4 + 1 = 5$)

$= f(5) = 5 \times 5 - 2 = 23$

(2) $(g \circ f)(1) = g\left(f(1)\right)$ (Notice that $f(1) = 5 \times 1 - 2 = 3$)

$= g(3) = 2 \times 3 = 6$

To determine the domain of the function $(f \circ g)$ do as the following :

(1) Find : D_1 = domain of g

(2) Find : D_2 = values of X that make g (X) in domain of f

(3) Find : $D_1 \cap D_2$ which is the domain of $(f \circ g)$

Example 9

Find the domain of $(f \circ g)$ if : $f(X) = \frac{3}{X-4}$, $g(X) = \frac{4}{X+2}$

Solution

To find the domain of $(f \circ g)$ do as the following :

Find D_1 = domain of g :

$\because g(X) = \frac{4}{X+2}$ $\qquad \therefore D_1 = \text{domain of } g = \mathbb{R} - \{-2\}$

$*$ Find D_2 = the set of values of X that makes g (X) in domain of f

$\because f(X) = \frac{3}{X-4}$ $\qquad \therefore f\left(g(X)\right) = \frac{3}{g(X) - 4}$

$\therefore g(X)$ in domain of f if $g(X) \neq 4$

, when $g(X) = 4$ $\qquad \therefore \frac{4}{X+2} = 4$ $\qquad \therefore X + 2 = 1$ $\qquad \therefore X = -1$

$\therefore D_2$ = set of values of X that makes g (X) in domain of $f = \mathbb{R} - \{-1\}$

$*$ Find domain of $(f \circ g) = D_1 \cap D_2 = \left(\mathbb{R} - \{-2\}\right) \cap \left(\mathbb{R} - \{-1\}\right) = \mathbb{R} - \{-2, -1\}$

We can summarize the previous steps in the following diagram

$$\therefore f[g(X)] = \frac{3}{g(X) - 4} = \frac{3}{\frac{4}{X+2} - 4}$$

$X + 2 \neq 0$

$\therefore X \neq -2$

$\frac{4}{X+2} - 4 \neq 0$

$\therefore \frac{4}{X+2} \neq 4$

$\therefore X + 2 \neq 1$

$\therefore X \neq -1$

$\therefore$ The domain of $(f \circ g) = \mathbb{R} - \{-1, -2\}$

Notice that :

In case of using this diagram to find the domain we write $f[(g(X)]$ without simplifying.

Example 10

If $f(x)=\sqrt{x-2}$, $g(x)=\sqrt{6-x}$, find the domain of each of the following functions :

(1) $f \circ g$ (2) $g \circ f$

Solution

(1) $(f \circ g)(x) = f\left(g(x)\right) = \sqrt{g(x)-2} = \sqrt{\sqrt{6-x}-2}$

, $\because g(x)=\sqrt{6-x}$, putting $6-x \geq 0$ $\therefore x \leq 6$

$\therefore D_1$ = domain of g = $]-\infty , 6]$

, putting $g(x)-2 \geq 0$ $\therefore \sqrt{6-x}-2 \geq 0$ $\therefore \sqrt{6-x} \geq 2$ (squaring the two sides)

$\therefore 6-x \geq 4$ $\therefore x \leq 2$

$\therefore D_2$ = The set of values of x that makes $g(x)$ in domain of f = $]-\infty , 2]$

$\therefore$ Domain of $(f \circ g) = D_1 \cap D_2$

$=]-\infty , 6] \cap]-\infty , 2]$

$=]-\infty , 2]$

(2) $(g \circ f)(x) = g\left(f(x)\right) = \sqrt{6-f(x)} = \sqrt{6-\sqrt{x-2}}$

, $\because f(x)=\sqrt{x-2}$, putting $x-2 \geq 0$

$\therefore x \geq 2$ $\therefore D_1$ = domain of f = $[2 , \infty[$

, putting $6-\sqrt{x-2} \geq 0$ $\therefore \sqrt{x-2} \leq 6$ (squaring the two sides)

$\therefore x-2 \leq 36$ $\therefore x \leq 38$

$\therefore D_2$ = the set of values of x that makes $f(x)$ in domain of g = $]-\infty , 38]$

$\therefore$ Domain of $(g \circ f) = D_1 \cap D_2$

$= [2 , \infty[\cap]-\infty , 38] = [2 , 38]$

Another solution to find the domain of $(g \circ f)$:

$\because$ Domain of $f = [2 , \infty[$, domain of $g =]-\infty , 6]$

, $\because$ domain of $(g \circ f) = \{x : x \in$ domain of f , $f(x) \in$ domain of g$\}$

$\because x \in$ domain of f $\therefore$ $\boxed{x \in [2 , \infty[}$ $\therefore f(x) \in$ domain of g

$\therefore f(x) \in]-\infty , 6]$

$\therefore \sqrt{x-2} \in]-\infty , 6]$ $\qquad$ $\therefore \sqrt{x-2} \le 6$

$\therefore x-2 \le 36$ $\qquad$ $\therefore x \le 38$

$\therefore \boxed{x \in]-\infty , 38]}$

i.e. Domain of $(g \circ f) = [2 , \infty[\cap]-\infty , 38] = [2 , 38]$

Example 11

If $f(x) = \sqrt{x-1}$, $g(x) = x^2 - 3$

, find $(f \circ g)(x)$ in the simplest form showing the domain , then find $(f \circ g)(2)$

Solution

$\because (f \circ g)(x) = f\left(g(x)\right) = \sqrt{g(x)-1} = \sqrt{x^2-3-1} = \sqrt{x^2-4}$

$\because$ Domain of $f = [1 , \infty[$, domain of $g = \mathbb{R}$

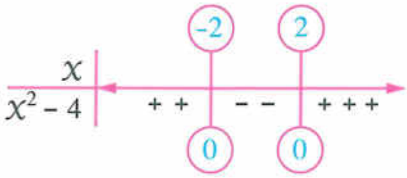
, $\because$ domain of $(f \circ g) = \{x : x \in \text{domain of } g , g(x) \in \text{domain of } f\}$

$\therefore x \in$ domain of g

$\therefore \boxed{x \in \mathbb{R}}$, $\because g(x) \in$ domain of f $\quad \therefore g(x) \in [1 , \infty[$

$\therefore x^2 - 3 \in [1 , \infty[$ $\qquad$ $\therefore x^2 - 3 \ge 1$

$\therefore x^2 - 4 \ge 0$ $\qquad$ $\therefore (x-2)(x+2) \ge 0$

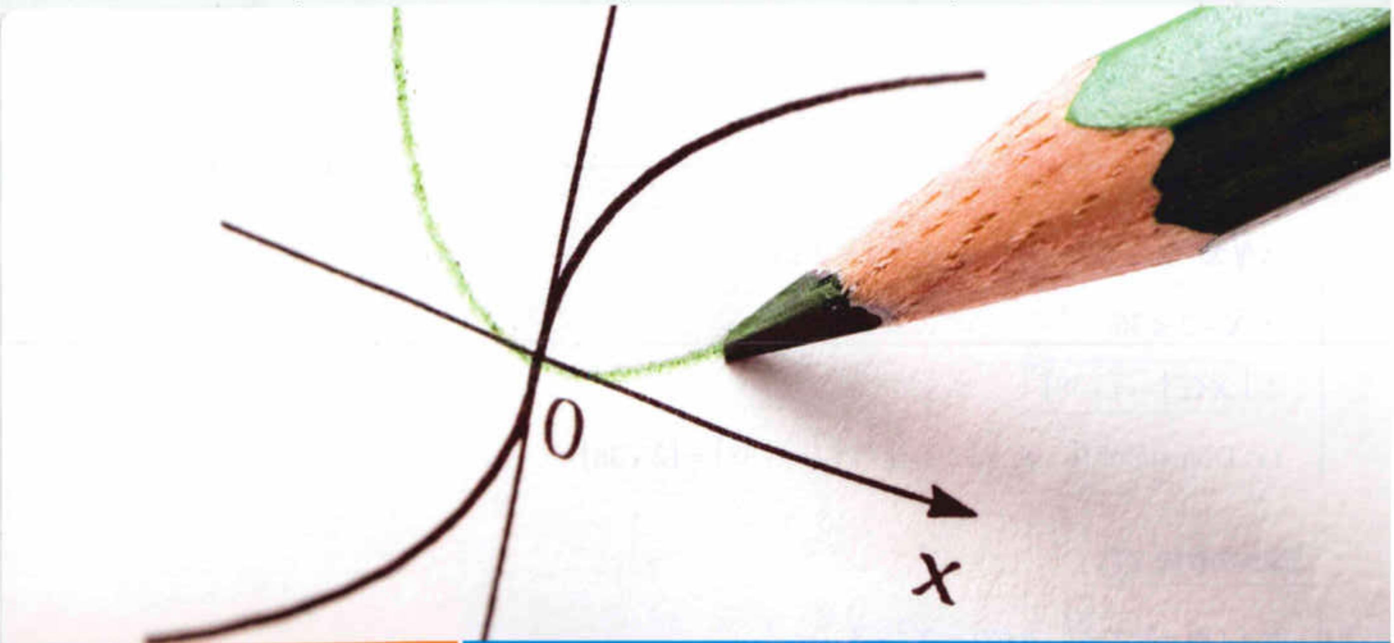


$\therefore x \ge 2$ or $x \le -2$

$\therefore \boxed{x \in \mathbb{R} -]-2 , 2[}$

i.e. Domain of $(f \circ g) = \mathbb{R} \cap \left(\mathbb{R} -]-2 , 2[\right) = \mathbb{R} -]-2 , 2[$

, $(f \circ g)(2) = \sqrt{4-4} =$ zero

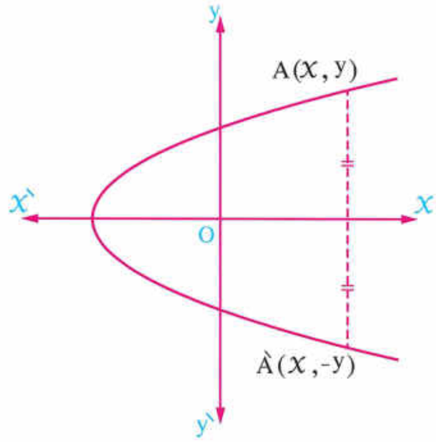


Lesson Three — Some properties of functions (even and odd functions / one - to - one functions)

Prelude

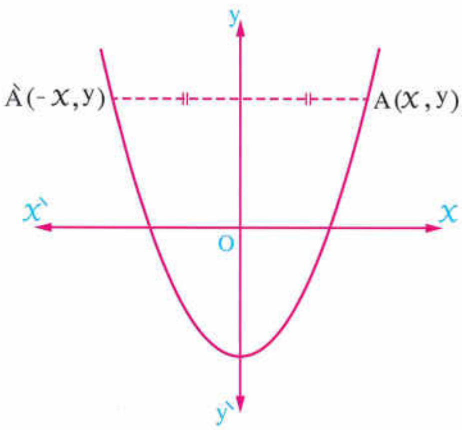
1 Symmetry about X-axis

The graph of a function is symmetric about X-axis if for each point A (X , y) lies on the graph there is a corresponding point Á (X , –y) lies on the same graph where Á is the image of A by reflection in X-axis.



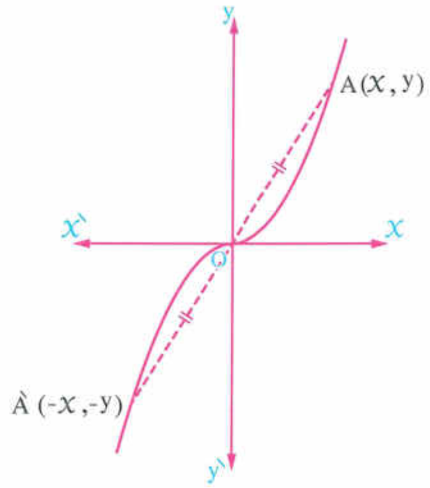
2 Symmetry about y-axis

The graph of a function is symmetric about y-axis if for each point A (X , y) lies on the graph there is a corresponding point Á (– X , y) lies on the same graph where Á is the image of A by reflection in y-axis.



3 Symmetry about the origin point "O"

The graph of a function is symmetric about the origin point (O) if for each point A (x , y) lies on the graph of the function there is a corresponding point Á (– x , – y) lies on the same graph where Á is the image of A by reflection on the origin point (O)



Even function and odd function

- **Even function :** The function f is said to be even if $f(-x) = f(x)$ for each x , $-x \in$ the domain of the function f

  The curve of the even function is symmetric about y-axis.

- **Odd function :** The function f is said to be odd if $f(-x) = -f(x)$ for each x , $-x \in$ the domain of the function f

  The curve of the odd function is symmetric about the origin point.

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Remarks

1. If $f(-x) \neq f(x)$, $f(-x) \neq -f(x)$, then the function f is neither even nor odd.
2. When we investigate whether the function f is even or odd , the two elements x , $-x$ must belong to the domain of the function. If this condition is not satisfied , then the function is neither even nor odd without getting $f(-x)$
3. If the domain of the function is $\mathbb{R} - \{a\}$, $a \neq 0$, then the function neither odd nor even.
4. If the function is even and its curve passes through (a , b) , then the curve must pass through (– a , b)
5. If the function is odd and its curve passes through (a , b) , then the curve must pass through (– a , – b)
6. The zero function $f : f(x) = 0$ is an even and odd function at the same time.

Example 1

Determine which of the functions defined by the following rules is even , odd or otherwise :

(1) $f(x) = x^2$

(2) $f(x) = 2x^3$

(3) $f(x) = \sqrt{x-1}$

(4) $f(x) = \cos x$

(5) $f(x) = x^2 - 5 , x \in [-2 , 2[$

Solution

(1) $\because$ f is polynomial $\quad \therefore$ The domain of $f = \mathbb{R}$

$\therefore$ For each $x , -x \in \mathbb{R}$, then $f(-x) = (-x)^2 = x^2 = f(x)$

$\therefore$ f is even.

(2) $\because$ f is polynomial

$\therefore$ The domain of $f = \mathbb{R}$

$\therefore$ For each $x , -x \in \mathbb{R}$, then $f(-x) = 2(-x)^3 = 2(-x^3) = -2x^3 = -f(x)$

$\therefore$ f is odd.

(3) $\because$ The domain of f is the set of values of x satisfying

$x - 1 \geq 0 \quad$ *i.e.* $x \geq 1$

$\therefore$ The domain of $f = [1 , \infty[$

$\therefore$ For each $x \in [1 , \infty[$

there is not $-x \in [1 , \infty[$

$\therefore$ f is neither even nor odd.

Notice that :

$3 \in [1 , \infty[$

while $-3 \notin [1 , \infty[$

(4) $\because$ The domain of $f : f(x) = \cos x$ is $\mathbb{R}$

$\therefore$ For each $x , -x \in \mathbb{R}$, then

$f(-x) = \cos(-x) = \cos x = f(x)$

$\therefore$ f is even.

Remember that

$\sin(-x) = -\sin x$

$\cos(-x) = \cos x$

$\tan(-x) = -\tan x$

(5) $-2 \in$ the domain of the function , $2 \notin$ the domain of the function

$\therefore$ f is neither odd nor even.

Remarks

1. The function $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(X) = a X^n$ where $a \neq 0$, $n \in \mathbb{Z}^+$ is called the power function , and it is :

* Even when n is an even number.
* Odd when n is an odd number.

2. $f(X) = \cos X$, $f(X) = \sec X$ are even functions but $f(X) = \sin X$, $f(X) = \csc X$, $f(X) = \tan X$ and $f(X) = \cot X$ are odd functions.

Example 2

If the function f is an even function where $f(X) = a X^2 + b X + 5$ and the curve of the function passes through the point (1 , 6) find the value of each of a and b

Solution

$\because$ The function is even and passes through (1 , 6)

$\therefore$ The curve passes through (– 1 , 6)

At the point (1 , 6) : $\therefore 6 = a + b + 5$ (1)

At the point (– 1 , 6) : $\therefore 6 = a - b + 5$ (2)

By adding (1), (2) : $\therefore 12 = 2a + 10$ $\therefore 2a = 2$ $\therefore a = 1$

By substituting in (1) : $\therefore 6 = 1 + b + 5$ $\therefore b = \text{zero}$

Important properties

If each of f_1 , f_2 is an even function , and each of g_1 , g_2 is an odd function , then :

(1) $f_1 \pm f_2$ is even.

(2) $g_1 \pm g_2$ is odd.

(3) $f_1 \pm g_1$ is neither even nor odd.

(4) each of $f_1 \times f_2$ and $\frac{f_1}{f_2}$ is even.

(5) each of $g_1 \times g_2$ and $\frac{g_1}{g_2}$ is even.

(6) each of $f_1 \times g_1$ and $\frac{f_1}{g_1}$ is odd.

Example 3

Determine which of the functions defined by the following rules is even , odd or otherwise :

(1) $f(X) = X^2 + \cos X$ (2) $f(X) = X^3 + \sin X$ (3) $f(X) = 3X^4 \tan X$

Solution

(1) $\because f(-X) = (-X)^2 + \cos(-X) = X^2 + \cos X = f(X)$

$\therefore f$ is even.

Another solution :

Let $f(X) = f_1(X) + f_2(X)$ where $f_1(X) = X^2$, $f_2(X) = \cos X$ two even functions.

$\therefore f_1 + f_2$ is even. $\therefore f$ is even.

(2) $\because f(-X) = (-X)^3 + \sin(-X) = -X^3 - \sin X = -(X^3 + \sin X) = -f(X)$

$\therefore f$ is odd

Note that :

The function resulted from adding two odd functions is odd.

(3) $\because f(-X) = 3(-X)^4 \tan(-X) = 3X^4(-\tan X) = -3X^4 \tan X = -f(X)$

$\therefore f$ is odd.

Note that :

The function resulted from multiplying an even function by an odd function is odd.

Example 4

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Each of the following graphs represents the curve of the function f , determine from the graph whether the function f is even , odd or otherwise verifying your answer algebraically :

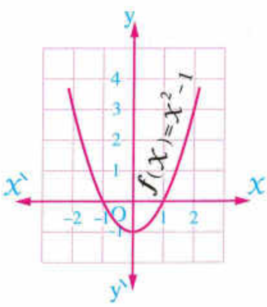


Fig. (1)

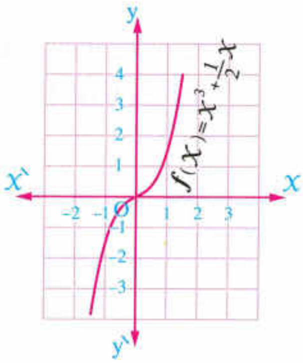


Fig. (2)

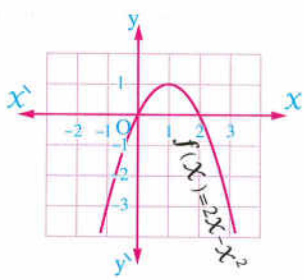


Fig. (3)

Solution

Fig. (1) : $f(x) = x^2 - 1$

$\because$ The domain of the function $f = \mathbb{R}$ and the curve is symmetric about y-axis

$\therefore f$ is even.

- *Algebraically satisfaction :*

$\because$ For each $x, -x \in \mathbb{R}$, then $f(-x) = (-x)^2 - 1 = x^2 - 1 = f(x)$ $\therefore f$ is even.

Fig. (2) : $f(x) = x^3 + \frac{1}{2}x$

$\because$ The domain of the function $f = \mathbb{R}$ and the curve is symmetric about origin point

$\therefore f$ is odd.

- *Algebraically satisfaction :*

$\because$ For each $x, -x \in \mathbb{R}$

, then $f(-x) = (-x)^3 + \frac{1}{2}(-x) = -x^3 - \frac{1}{2}x = -\left(x^3 + \frac{1}{2}x\right) = -f(x)$ $\therefore f$ is odd.

Fig. (3) : $f(x) = 2x - x^2$

$\because$ The domain of the function $f = \mathbb{R}$ and the curve is neither symmetric about y-axis nor about the origin point

$\therefore f$ is neither even nor odd.

- *Algebraically satisfaction :*

$\because$ For each $x, -x \in \mathbb{R}$, then $f(-x) = 2(-x) - (-x)^2 = -2x - x^2 = -(2x + x^2)$

$\therefore f(-x) \neq f(x)$, $f(-x) \neq -f(x)$ $\therefore f$ is neither even nor odd.

Example 5

Determine which of the functions defined by the following rules is even , odd or otherwise :

(1) $f(x) = 3x^4 - 5x^2 + 1$

(2) $f(x) = x^3 + 2x - 5$

(3) $f(x) = \frac{x - \sin 3x}{1 + x^2}$

(4) $f(x) = \frac{x - \tan x}{x^3 + x}$

Solution

(1) $\because f(-x) = 3(-x)^4 - 5(-x)^2 + 1 = 3x^4 - 5x^2 + 1 = f(x)$

$\therefore f$ is even.

(2) $\because f(-x) = (-x)^3 + 2(-x) - 5 = -x^3 - 2x - 5 = -(x^3 + 2x + 5)$

$\therefore f(-x) \neq f(x)$, $f(-x) \neq -f(x)$

$\therefore f$ is neither even nor odd.

(3) $\because f(-x) = \dfrac{(-x) - \sin 3(-x)}{1 + (-x)^2} = \dfrac{(-x) - (-\sin 3x)}{1 + x^2} = \dfrac{-(x - \sin 3x)}{1 + x^2} = -f(x)$

$\therefore f$ is odd.

(4) $\because f(-x) = \dfrac{(-x) - \tan(-x)}{(-x)^3 + (-x)}$

$$= \frac{-x - (-\tan x)}{-x^3 - x} = \frac{-x + \tan x}{-x^3 - x} = \frac{-(x - \tan x)}{-(x^3 + x)} = \frac{x - \tan x}{x^3 + x} = f(x)$$

$\therefore f$ is even.

Example 6

Determine which of the functions defined by the following rules is even , odd or otherwise :

(1) $f(x) = \left(x - \frac{1}{x}\right)^3 + \left(x + \frac{1}{x}\right)^3$

(2) $f(x) = \begin{cases} -\frac{1}{x} & , \quad x < 0 \\ \frac{1}{x} & , \quad x > 0 \end{cases}$

Solution

(1) $\because f(-x) = \left(-x + \frac{1}{x}\right)^3 + \left(-x - \frac{1}{x}\right)^3 = \left(-\left(x - \frac{1}{x}\right)\right)^3 + \left(-\left(x + \frac{1}{x}\right)\right)^3$

$$= -\left(x - \frac{1}{x}\right)^3 - \left(x + \frac{1}{x}\right)^3 = -\left(\left(x - \frac{1}{x}\right)^3 + \left(x + \frac{1}{x}\right)^3\right) = -f(x)$$

$\therefore f$ is odd.

(2) $\because f(-x) = \begin{cases} -\frac{1}{(-x)} & , \quad -x < 0 \\ \frac{1}{(-x)} & , \quad -x > 0 \end{cases} = \begin{cases} \frac{1}{x} & , \quad x > 0 \\ -\frac{1}{x} & , \quad x < 0 \end{cases} = \begin{cases} -\frac{1}{x} & , \quad x < 0 \\ \frac{1}{x} & , \quad x > 0 \end{cases} = f(x)$

$\therefore f$ is even.

One - to - one function

Definition

The function $f : X \longrightarrow Y$ is called one - to - one function if :

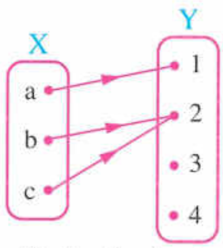
For each a , b $\in$ X , f (a) = f (b) , then a = b

or for each a , b $\in$ X , a $\neq$ b , then f (a) $\neq$ f (b)

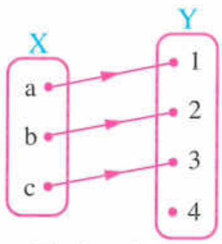
And this means there are not two elements in the domain of the one - to - one function having the same image.

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For example :



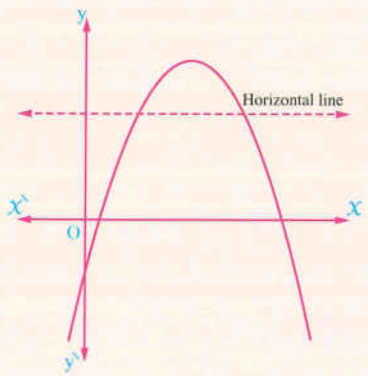
The function is not one-to-one from X to Y



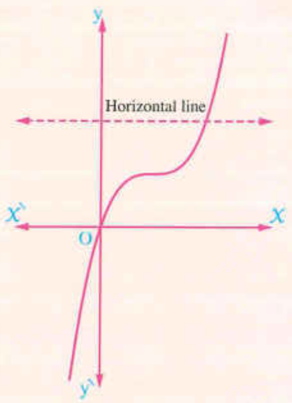
The function is one-to-one from X to Y

The horizontal line test

If there exist a horizontal line (parallel to X-axis) intersects the curve of the function at more than one point , then the curve represents a function not one - to - one.



The horizontal line intersects the curve at two points , so the function is not one - to - one.



Any horizontal line intersects the curve at one point at most , so the function is one - to - one.

Example 7

Prove that each of the functions defined by the following rules is one - to - one :

(1) $f(x) = x^3 + 2$ (2) $f(x) = \frac{2x-3}{3x+2}$

Solution

(1) Let a , b ∈ the domain of the function f

$\therefore f(a) = a^3 + 2$, $f(b) = b^3 + 2$ putting $f(a) = f(b)$

$\therefore a^3 + 2 = b^3 + 2$ $\therefore a^3 = b^3$

$\therefore a = b$ $\therefore f$ is one - to - one.

(2) Let a , b ∈ the domain of the function f

$\therefore f(a) = \frac{2a-3}{3a+2}$, $f(b) = \frac{2b-3}{3b+2}$ putting $f(a) = f(b)$

$\therefore \frac{2a-3}{3a+2} = \frac{2b-3}{3b+2}$ $\therefore (2a-3)(3b+2) = (3a+2)(2b-3)$

$\therefore 6ab + 4a - 9b - 6 = 6ab - 9a + 4b - 6$

$\therefore 4a + 9a = 4b + 9b$ $\therefore 13a = 13b$

$\therefore a = b$ $\therefore f$ is one - to - one.

Example 8

Prove that each of the functions defined by the following rules is not one - to - one :

(1) $f(x) = 3 - x^2$ (2) $f(x) = x^2 + x$

Solution

(1) Let a , b ∈ the domain of the function f

$\therefore f(a) = 3 - a^2$, $f(b) = 3 - b^2$ putting $f(a) = f(b)$

$\therefore 3 - a^2 = 3 - b^2$ $\therefore a^2 = b^2$

$\therefore a = \pm b$ $\therefore$ a has two values b , – b

$\therefore f$ is not one - to - one.

(2) Let a , b ∈ the domain of the function f

$\therefore f(a) = a^2 + a$, $f(b) = b^2 + b$, putting $f(a) = f(b)$

$\therefore a^2 + a = b^2 + b$ $\therefore a^2 - b^2 + a - b = 0$

$\therefore (a-b)(a+b) + (a-b) = 0$ $\therefore (a-b)(a+b+1) = 0$

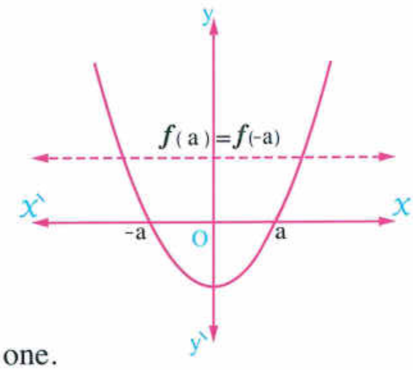
$\therefore a - b = 0$, then $a = b$ or $a + b + 1 = 0$, then $a = -b - 1$

$\therefore$ a has two values b , $-b-1$ $\therefore f$ is not one - to - one.

Remark

The even functions , in general , are not one - to - one , because for each two different values a , – a $\in$ the domain of the even function , then $f(-a) = f(a)$

i.e. The values a , – a for the variable x are corresponding to one value for the variable y , so the even function is not one - to - one , as shown using the horizontal line test in the opposite figure.



- The odd function could be one - to - one or not one - to - one.

Example 9

Graph the curve of an even function passing through the points (0 , – 2) , (– 1 , – 1) , (– 2 , 2) , then from the graph show that the function is not one - to - one.

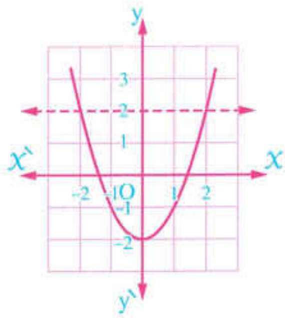
Solution

$\because$ The function is even.

$\therefore f(x) = f(-x)$

$\therefore f(1) = f(-1) = -1 \ , f(2) = f(-2) = 2$

i.e. The curve of the function passes through the two points (1 , –1) , (2 , 2)



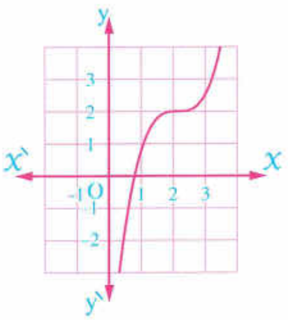
also from the graph :

The function is not one - to - one because there is a horizontal line intersects the curve of the function at two points.

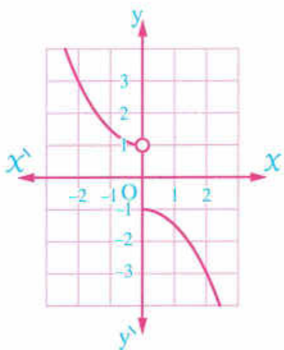
Remark

If the function f is continuously increasing or continuously decreasing for all values belonging to its domain , then the function f is one - to - one.

For example : In the following figures :



The function f is continuously increasing on its domain , so f is one - to - one.



The function f is continuously decreasing on its domain , so f is one - to - one.



Lesson Four — Graphical representation of basic functions and graphing piecewise functions

Representing the linear function

* The linear function $f : \mathbb{R} \longrightarrow \mathbb{R}$ where $f(X) = aX + b$ is represented graphically by a straight line passes through the point $(0 , b)$ and its slope $= a$

Example 1

Represent graphically the function f in each of the following and deduce from the graph the range of the function :

(1) $f : \mathbb{R} \longrightarrow \mathbb{R} , f(X) = -\frac{1}{3} X$

(2) $f : [-1 , 2[\longrightarrow \mathbb{R} , f(X) = 2X - 1$

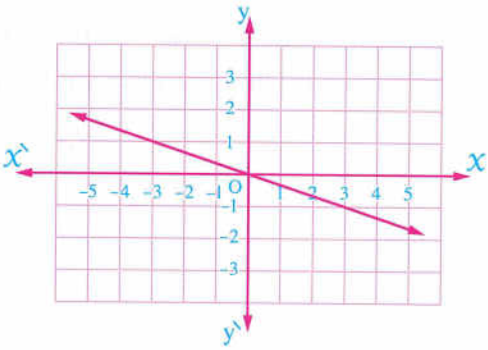
(3) $f :]-\infty , 1[\longrightarrow \mathbb{R} , f(X) = 2X - 1$

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Solution

(1) $\because$ The domain $= \mathbb{R}$

$\therefore$ The function is represented by a straight line passes through the point $(0 , 0)$ and its slope $= -\frac{1}{3}$

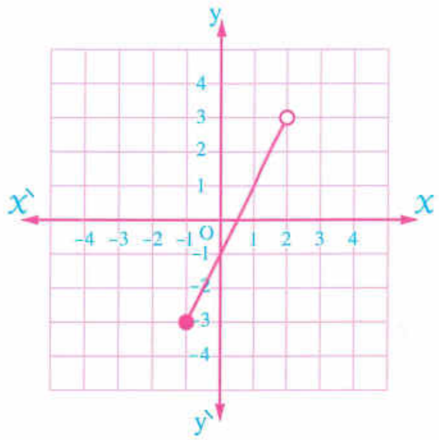


(2) $\because$ The domain $= [-1, 2[$

x	-1	0	2
$f(x)$	-3	-1	3

Notice that the point $(2, 3)$ $\notin$ the function so it is excluded from the graph by drawing unshaded circle at this point.

From the graph : The range $= [-3, 3[$

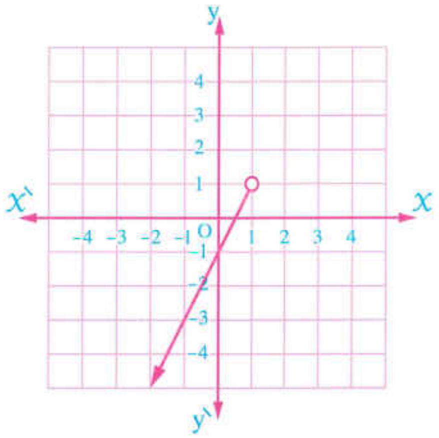


(3) $\because$ The domain $=]-\infty, 1[$

x	1	0	-1
$f(x)$	1	-1	-3

Notice that the point $(1, 1)$ $\notin$ the function so it is excluded from the graph by drawing unshaded circle at this point.

From the graph : The range $=]-\infty, 1[$



Example 2

Represent graphically the function $f : \mathbb{R} - \{0\} \longrightarrow \mathbb{R}$, $f(x) = \frac{x^2 - x}{x}$, from the graph deduce the range of the function.

Solution

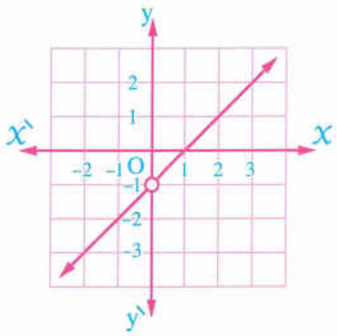
$\because$ Domain of the function $f = \mathbb{R} - \{0\}$

, $f(x) = \frac{x^2 - x}{x} = \frac{x(x-1)}{x} = x - 1$

, represented by a straight line

x	-1	0	1
$f(x)$	-2	-1	0

$\therefore$ The range $= \mathbb{R} - \{-1\}$



Notice that :

The unshaded circle at the point whose x-coordinate $= 0$ because it does not belong to the domain.

Example 3

If $f_1 :]-\infty , 2] \longrightarrow \mathbb{R}$ where $f_1(x) = 3x - 1$, $f_2 : [-1 , 5] \longrightarrow \mathbb{R}$ where $f_2(x) = 3 - 2x$, then graph the function $f_1 + f_2$ and from the graph deduce its range.

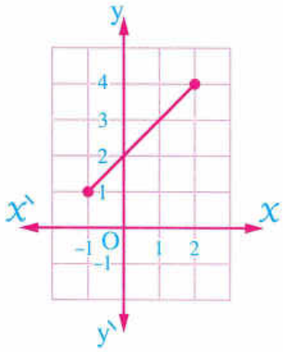
Solution

$\because$ Domain of $f_1 =]-\infty , 2]$

, domain of $f_2 = [-1 , 5]$

$\therefore$ Domain of $(f_1 + f_2) =]-\infty , 2] \cap [-1 , 5] = [-1 , 2]$

, $\because (f_1 + f_2)(x) = 3x - 1 + 3 - 2x = x + 2$



x	-1	0	2
$(f_1 + f_2)(x)$	1	2	4

From the graph : Range of $(f_1 + f_2) = [1 , 4]$

Graphing the quadratic function

Example 4

Graph the function f :
$f(x) = x^2 - x$ where $x \in]-1 , 3]$

Remember that

The point of the curve vertex of the quadratic function $f(x) = a x^2 + b x + c$ is $\left(\frac{-b}{2a} , f\left(\frac{-b}{2a}\right)\right)$

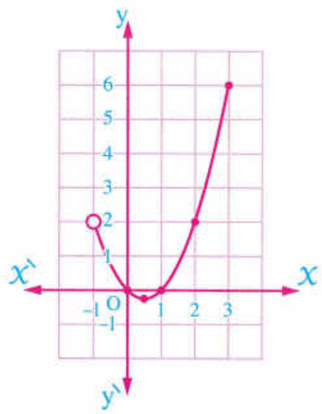
Solution

$\because f(x) = x^2 - x$, the domain $=]-1 , 3]$

$\therefore$ The x-coordinate of the vertex $= \frac{-b}{2a} = \frac{1}{2}$

, $f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{2} = -\frac{1}{4}$

$\therefore$ The curve vertex is $\left(\frac{1}{2} , -\frac{1}{4}\right)$



x	(-1)	0	$\frac{1}{2}$	1	2	3
$f(x)$	(2)	0	$-\frac{1}{4}$	0	2	6

From the graph :

* The range $= [-\frac{1}{4} , 6]$
* The function is decreasing on the interval $]-1 , \frac{1}{2}[$ and is increasing on the interval $]\frac{1}{2} , 3[$

Graphing the piecewise function

Example 5

Graph the function $f : f(x) = \begin{cases} 2 - x & , -1 \le x < 2 \\ x - 2 & , 2 \le x < 5 \end{cases}$, then from the graph :

(1) Determine the domain and the range of f **(2)** Discuss the monotonicity of f

(3) Determine whether f is even , odd or otherwise , giving reason.

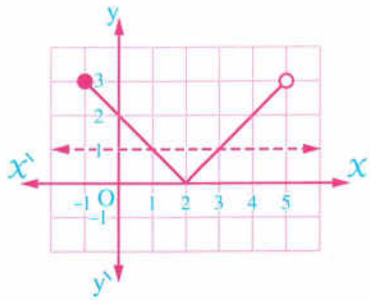
(4) Mention whether f is one - to - one or not , giving reason.

Solution

The function f is defined by two rules

- $f_1(x) = 2 - x$, $x \in [-1 , 2[$

x	−1	0	(2)
$f_1(x)$	3	2	(0)



- $f_2(x) = x - 2$, $x \in [2 , 5[$

x	2	3	(5)
$f_2(x)$	0	1	(3)

(1) The domain of $f = [-1 , 2[\cup [2 , 5[= [-1 , 5[$
, the range of $f = [0 , 3]$

(2) The function f is decreasing on $]-1 , 2[$
and increasing on $]2 , 5[$

(3) The function f is neither even nor odd
because it is not symmetric about y-axis
nor the origin point O

Notice that :

$2 \notin [-1 , 2[$, while $2 \in [2 , 5[$
so $(2 , 5) \in f$
i.e. We don't put unshaded circle on the point (2 , 5) in the graph.

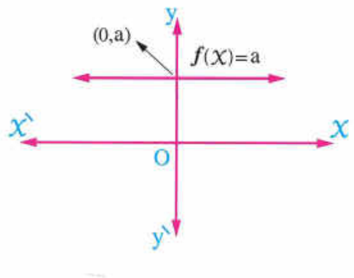
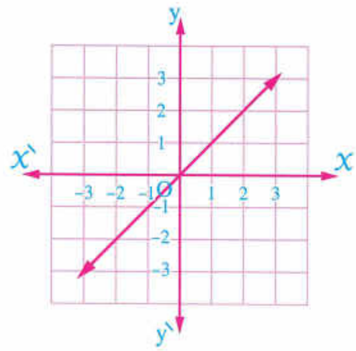
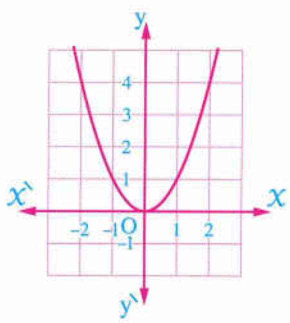
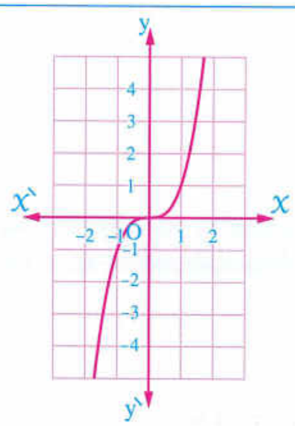
(4) The function f is not one - to - one because there exist a horizontal line intersects its curve at two points.

The basic forms of some functions

Now we will recognize the graph of simple forms (basic forms), (standard forms) for the real functions and this is preface to use it in representing the real functions in their different forms next lesson.

UNIT 1

1 The simplest forms of some polynomial functions

	The constant function	The first degree (linear) function
The simplest form	$f:\mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = a$ where $a \in \mathbb{R}$	$f:\mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x$
The graph		
The range, monotony and some properties	• Range of the function = $\{a\}$ • The function is constant on its domain. • The function is even (symmetric about y-axis) • The function is not one - to - one function.	• Range of the function = $\mathbb{R}$ • The function is increasing on its domain $\mathbb{R}$ • The function is odd (symmetric about the origin point) • The function is one - to - one function.
	The second degree (quadratic) function	**The third degree (cube) function**
The simplest form	$f:\mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x^2$	$f:\mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x^3$
The graph		

The range, monotony and some properties	• Range of the function = $[0, \infty[$ • The function is decreasing on $]-\infty, 0[$ and increasing on $]0, \infty[$ • The function is even (symmetric about y-axis) • The function is not one - to - one function.	• Range of the function = $\mathbb{R}$ • The function is increasing on its domain $\mathbb{R}$ • The function is odd (symmetric about the origin point) • The function is one - to - one function.

2 The basic form of the absolute function

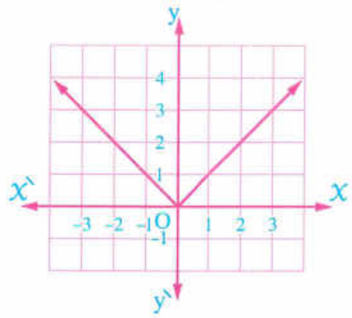
• **Simplest form**

$f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = |x|$ and

it is redefined as follows :

$$f(x) = \begin{cases} x & , x \geq 0 \\ -x & , x < 0 \end{cases}$$

The graph



• **Range , monotony and some properties :**

* The range of the function = $[0, \infty[$
* The function is decreasing on $]-\infty, 0[$ and increasing on $]0, \infty[$
* The function is even (symmetric about y-axis)
* The function is not one - to - one function.

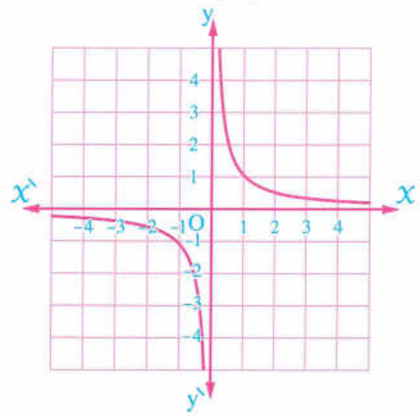
3 The basic form of the rational function

• **Simplest form**

$f : \mathbb{R} - \{0\} \longrightarrow \mathbb{R}, f(x) = \frac{1}{x}$

"approaching each of the two parts of the curve to the two axes without intersection with them , then the two axes $\overleftrightarrow{xx'}$ and $\overleftrightarrow{yy'}$ are called asymptotical lines of the curve"

The graph



• **Range , monotony and some properties :**

* Range of the function = $\mathbb{R} - \{0\}$
* The function is decreasing on $]-\infty, 0[$ and decreasing on $]0, \infty[$
* The function is odd (symmetric about the origin point)
* The function is one - to - one function.

Example 6

Graph each of the functions which are defined by the following rules and from the graph find the domain , the range of the function and deduce its monotony and state whether the function is even , odd or otherwise :

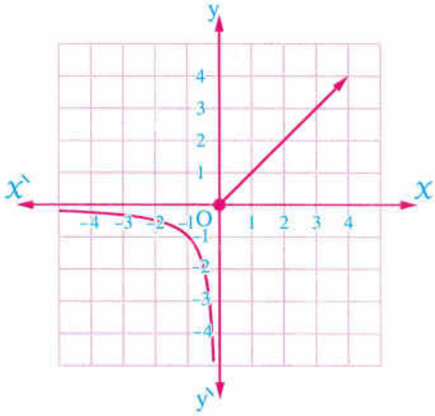
(1) $f(x) = \begin{cases} \frac{1}{x} & , \quad x < 0 \\ |x| & , \quad x \geq 0 \end{cases}$

(2) $f(x) = \begin{cases} x^3 & , \quad x < 0 \\ x & , \quad x > 0 \end{cases}$

Solution

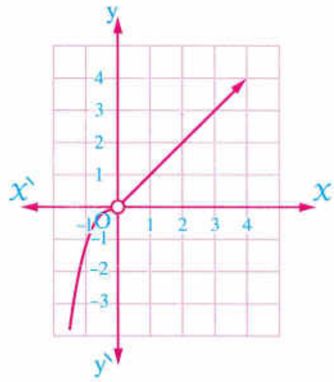
(1) * The domain = $\mathbb{R}$

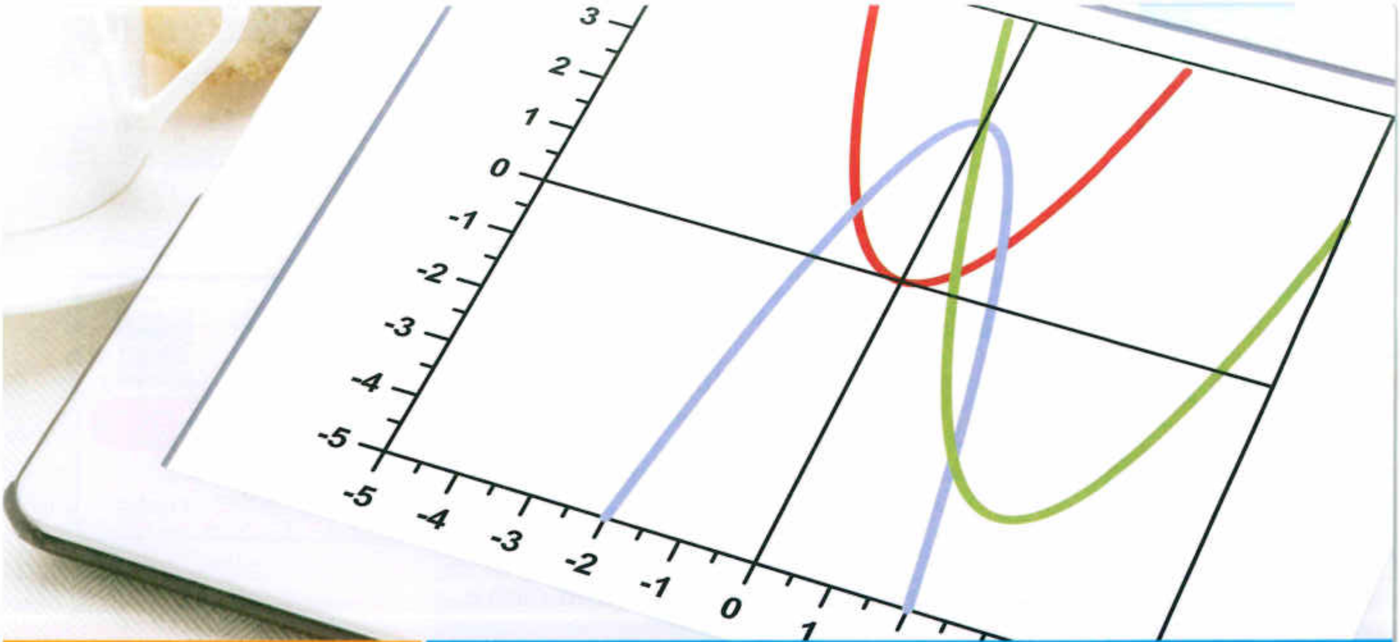
* The range = $\mathbb{R}$
* The function is decreasing on $]-\infty , 0[$ and is increasing on $]0 , \infty[$
* The function is neither odd nor even.



(2) * The domain = $\mathbb{R} - \{0\}$

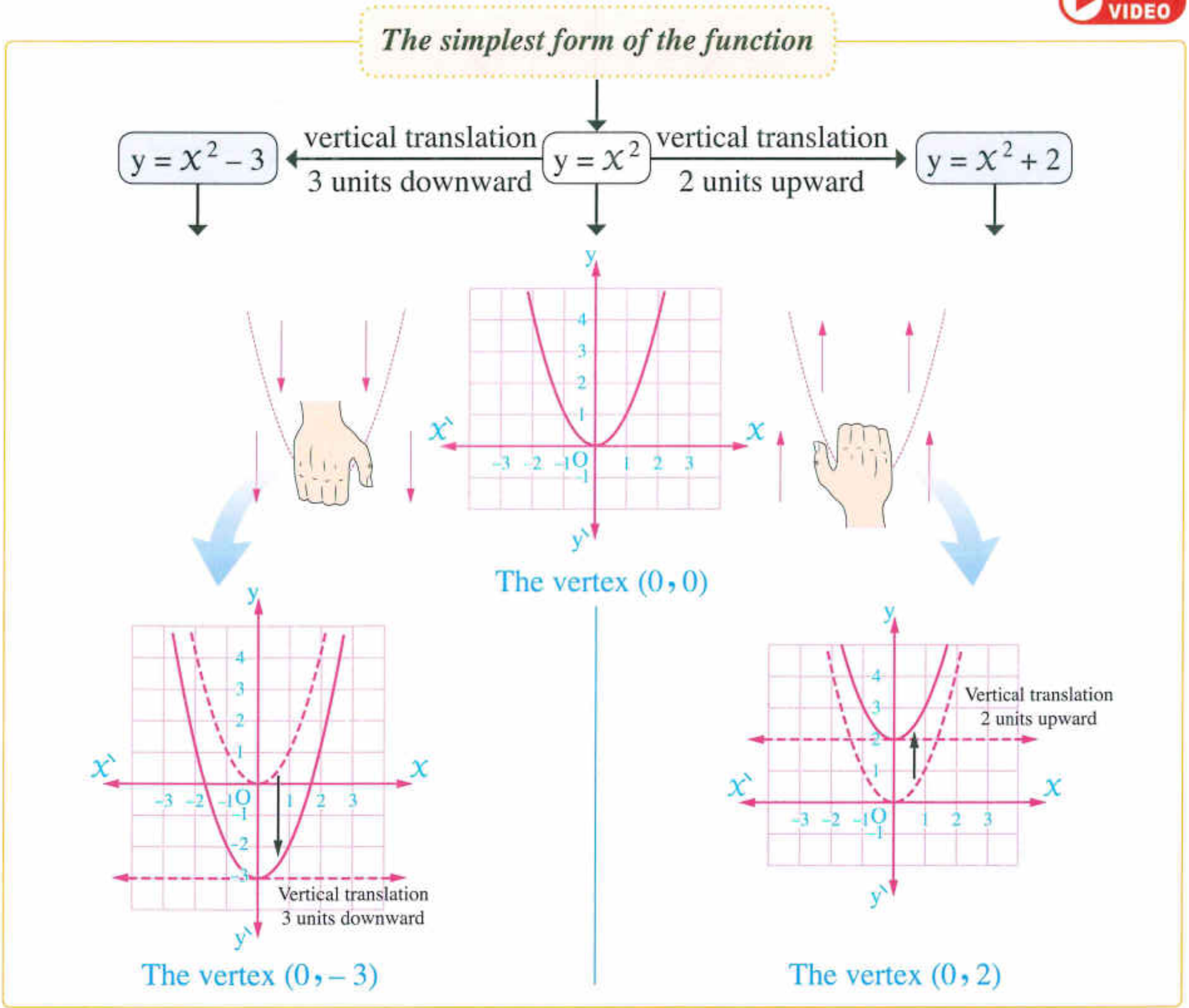
* The range = $\mathbb{R} - \{0\}$
* The function is increasing on its domain.
* The function is neither odd nor even.





Lesson Five — Geometrical transformations of basic function curves

First: Vertical translation of the function curve



UNIT 1

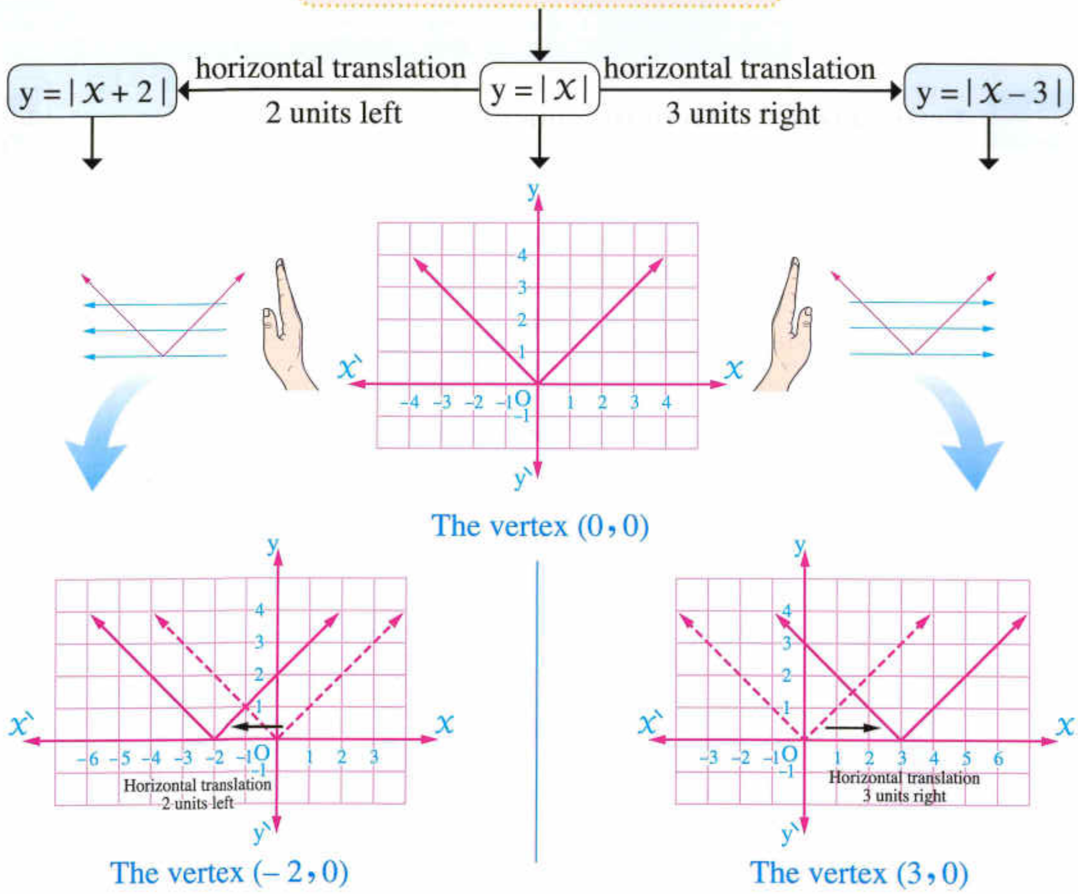
In general

For any function f , the curve of $y = f(x) + a$, $a \in \mathbb{R} - \{0\}$
is the same curve of $y = f(x)$ by a vertical translation
, its value is $|a|$ length unit in the direction : $\begin{cases} \overrightarrow{Oy} & (\textit{i.e.}\text{ Upward}) \quad \text{at } a > 0 \\ \overleftarrow{Oy} & (\textit{i.e.}\text{ Downward}) \quad \text{at } a < 0 \end{cases}$

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Second Horizontal translation of the function curve

The simplest form of the function



In general

For any function f , the curve of $y = f(x + a)$, $a \in \mathbb{R} - \{0\}$
is the same curve of $y = f(x)$ by a horizontal translation
, its value is $|a|$ length unit in the direction : $\begin{cases} \overrightarrow{Ox} & (\textit{i.e.}\text{ To the right}) \quad \text{at } a < 0 \\ \overleftarrow{Ox} & (\textit{i.e.}\text{ To the left}) \quad \text{at } a > 0 \end{cases}$

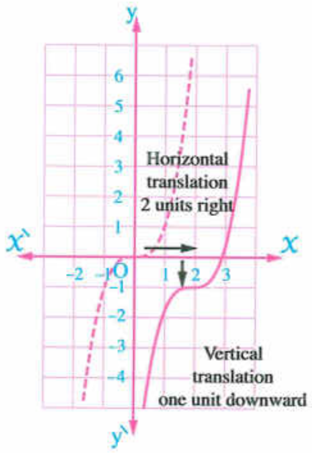
Third Horizontal translation followed by vertical translation of the function curve

The simplest form of the function

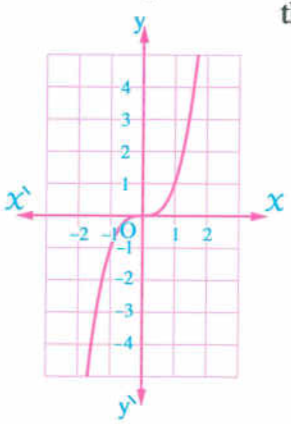
$y = x^3$

horizontal translation two units to the right followed by vertical translation one unit downward

$y = (x - 2)^3 - 1$



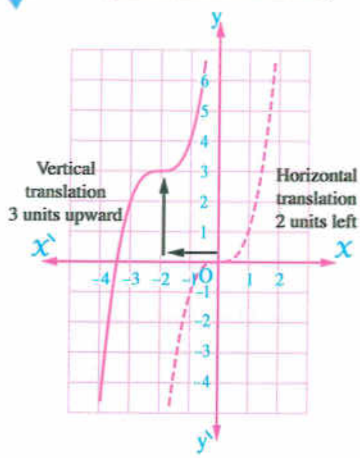
Symmetric point $(2, -1)$



Symmetric point $(0, 0)$

horizontal translation two units to the left followed by vertical translation three units upward

$y = (x + 2)^3 + 3$



Symmetric point $(-2, 3)$

In general

For any function f, the curve of $y = f(x + a) + b$ where $a, b \in \mathbb{R} - \{0\}$ is the same curve of $y = f(x)$ by a horizontal translation, its value $|a|$ length unit in the direction $\overrightarrow{Ox}$ if $a < 0$ or in the direction $\overrightarrow{Ox`}$ if $a > 0$, then a vertical translation, its value is $|b|$ length unit in the direction $\overrightarrow{Oy}$ if $b > 0$ or in the direction $\overrightarrow{Oy`}$ if $b < 0$

Example 1

Use the curves of the basic functions to graph the curves of the functions which are defined by the following rules, then from the graph determine the domain and the range of each function and discuss its monotony and state whether the function is even, odd or otherwise :

(1) $g(x) = |x - 2| - 1$ (2) $g(x) = (2 - x)^2 + 1$

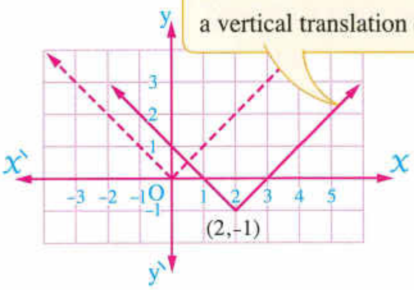
Solution

(1) • The domain of g = $\mathbb{R}$, the range of g = $[-1 , \infty[$

• The function g is decreasing on $]-\infty , 2[$ and is increasing on $]2 , \infty[$

• The function g is neither even nor odd.

The curve of the function g is the same curve of the function $f : f(x) = |x|$ by a horizontal translation two units in the direction $\overrightarrow{Ox}$, then a vertical translation one unit in the direction $\overrightarrow{Oy}$



(2) $\because (2 - x)^2 = (x - 2)^2$

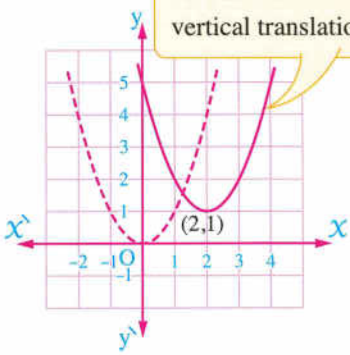
$\therefore g(x) = (x - 2)^2 + 1$

• The domain of g = $\mathbb{R}$, the range of g = $[1 , \infty[$

• The function g is decreasing on $]-\infty , 2[$ and is increasing on $]2 , \infty[$

• The function g is neither even nor odd.

The curve of the function g is the same curve of the function $f : f(x) = x^2$ by a horizontal translation 2 units in the direction $\overrightarrow{Ox}$, then a vertical translation one unit in the direction $\overrightarrow{Oy}$



Example 2

Use the curve of the function $f : f(x) = \frac{1}{x}$ to represent the functions g , h and k where :

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(1) $g(x) = \frac{1}{x - 2} + 1$ (2) $h(x) = \frac{1}{x} + 3$

(3) $k(x) = \frac{2x - 1}{x - 1}$

From the graph , determine the domain and the range of each function , then discuss its monotony.

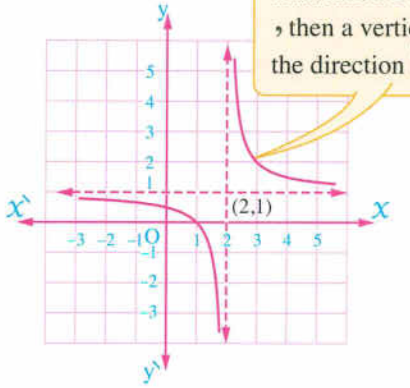
Solution

(1) • The domain of g = $\mathbb{R} - \{2\}$

• The range of g = $\mathbb{R} - \{1\}$

• The function is decreasing on $]-\infty, 2[$ and also decreasing on $]2, \infty[$

The curve of the function g is the same curve of the function f by a horizontal translation 2 units in the direction $\overrightarrow{OX}$, then a vertical translation 1 unit in the direction $\overrightarrow{Oy}$



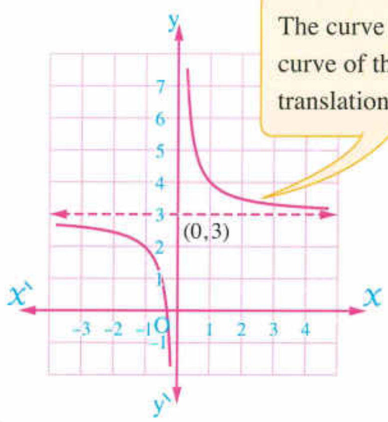
(2) • The domain of h = $\mathbb{R} - \{0\}$

• The range of h = $\mathbb{R} - \{3\}$

• The function is decreasing on $]-\infty, 0[$

and also decreasing on $]0, \infty[$

The curve of the function h is the same curve of the function f by a vertical translation 3 units in the direction $\overrightarrow{Oy}$



(3) $k(X) = \frac{2X - 1}{X - 1} = \frac{2X - 2 + 1}{X - 1}$

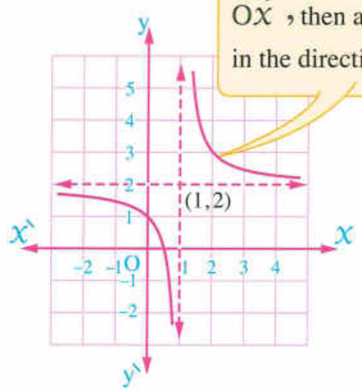
$$= \frac{2(X-1)+1}{X-1} = 2 + \frac{1}{X-1}$$

• The domain of k = $\mathbb{R} - \{1\}$

• The range of k = $\mathbb{R} - \{2\}$

• The function is decreasing on $]-\infty, 1[$ and also decreasing on $]1, \infty[$

The curve of the function k is the same curve of the function f by a horizontal translation one unit in the direction $\overrightarrow{OX}$, then a vertical translation 2 units in the direction $\overrightarrow{Oy}$



Example 3

Graph each of the functions which are defined by the following rules and from the graph , find the domain , the range of the function and deduce its monotony and state whether the function is even , odd or otherwise :

(1) $f(X) = \begin{cases} X^2 + 1 & , \quad X < 1 \\ 2 & , \quad X > 1 \end{cases}$

(2) $f(X) = \begin{cases} |X - 2| + 1 & , \quad X < 4 \\ \frac{1}{X - 4} + 2 & , \quad X > 4 \end{cases}$

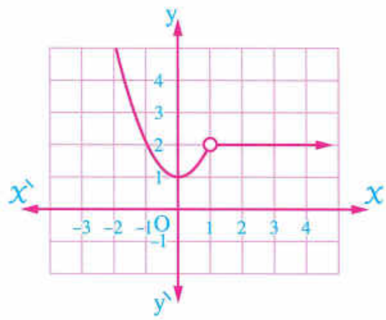
Solution

(1) • The domain $= \mathbb{R} - \{1\}$

• The range $= [1 , \infty[$

• The function is decreasing on $]-\infty , 0[$ and increasing on $]0 , 1[$, constant on $]1 , \infty[$

• The function is neither odd nor even.

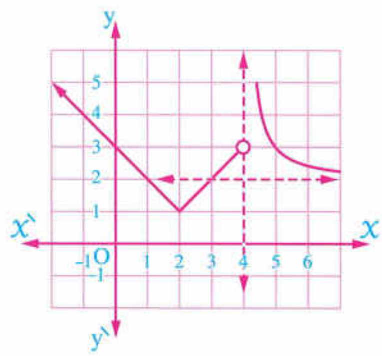


(2) • The domain $= \mathbb{R} - \{4\}$

• The range $= [1 , \infty[$

• The function is decreasing on each $]-\infty , 2[$, $]4 , \infty[$ and increasing on $]2 , 4[$

• The function is neither odd nor even.



Example 4

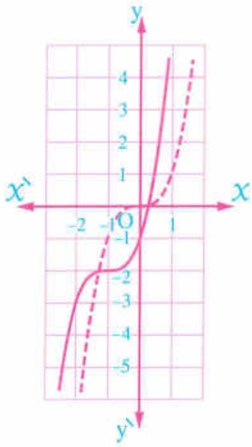
If $f : \mathbb{R} \longrightarrow \mathbb{R}$ where $f(X) = X^3$, graph the function g where $g(X) = f(X + 1) - 2$, from the graph , determine the domain of g and its range , then discuss its monotony and state whether the function g is even , odd or otherwise and write whether the function g is one - to - one or not :

Solution

$\because g(x) = f(x+1) - 2$

$\therefore g(x) = (x+1)^3 - 2$

$\therefore$ The curve of the function g is the same curve of the function f by a horizontal translation one unit in the direction $\overleftarrow{Ox}$, then a vertical translation 2 units in the direction $\overleftarrow{Oy}$

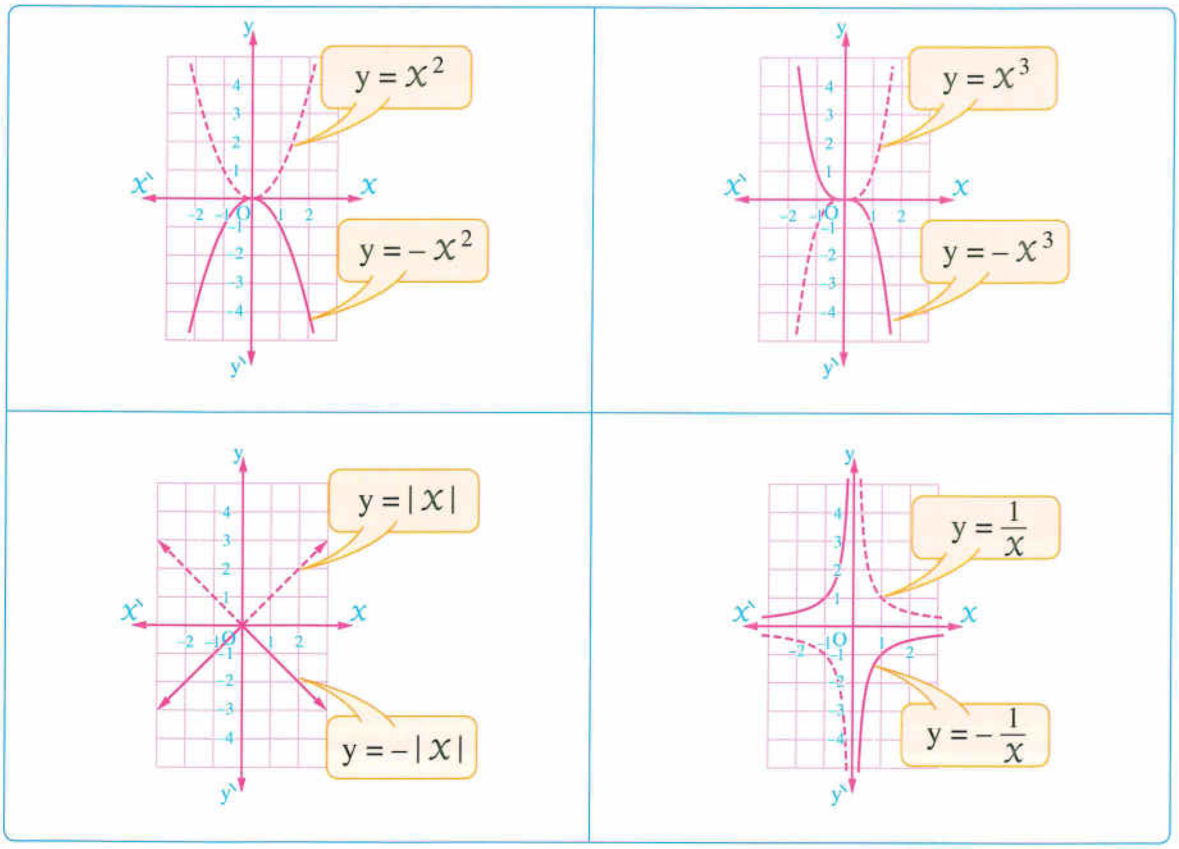


- The domain of $g = \mathbb{R}$, the range of $g = \mathbb{R}$
- The function g is increasing on its domain $\mathbb{R}$
- The function g is neither even nor odd.
- The function g is one - to - one function.

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Fourth Reflection of the function curve in x-axis

For any function f , the curve of $y = -f(x)$ is the same curve $y = f(x)$ by reflection in x-axis



Important remark

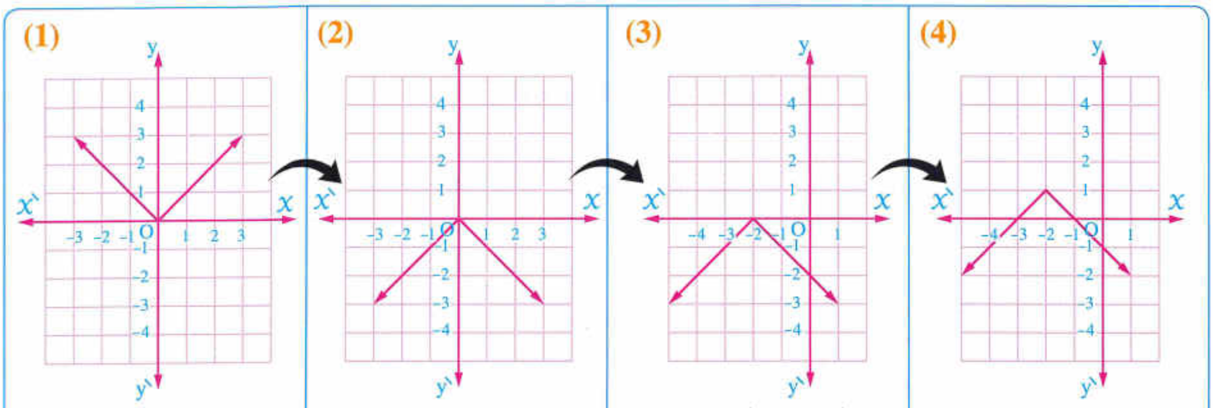
It is necessary that ordering the performing of transformations on the curve $y = f(x)$ to get from it the curve $y = -f(x + a) + b$ as follows :

1. Reflection in x-axis.
2. Horizontal translation.
3. Vertical translation.

If we reverse the order of performing the vertical translation before performing the reflection in x-axis , then we get another curve not the required curve.

For example :

From the curve of the simplest form of the function $y = |x|$, we can get the curve of the function $y = -|x + 2| + 1$ as follows :



(1) $y = \lvert x \rvert$	(2) $y = -\lvert x \rvert$	(3) $y = -\lvert x + 2 \rvert$	(4) $y = -\lvert x + 2 \rvert + 1$
The simplest form of the function (then)	Reflect in x-axis (then)	Horizontal translation 2 units left (then)	Vertical translation 1 unit upward

Example 5

Using the curves of the basic functions , graph the curves of the functions g , k and z where :

(1) $g(x) = -(x - 2)^3$

(2) $k(x) = \frac{1}{2 - x} + 3$

(3) $z(x) = 4x - x^2 - 3$

From the graph , determine the range of each function , discuss its monotony and its symmetry , and state whether the function is even , odd or otherwise.

Solution

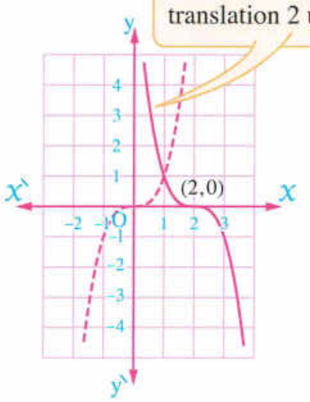
(1) • The range of g = $\mathbb{R}$

• The function g is decreasing on its domain $\mathbb{R}$

• The function g is symmetric about the point (2 , 0)

• The function g is neither even nor odd.

The curve of the function g is the same curve of the function $f : f(x) = x^3$ by reflection in x-axis , then a horizontal translation 2 units in the direction $\overrightarrow{Ox}$



The curve of the function k is the same curve of the function $f : f(x) = \frac{1}{x}$ by reflection in x-axis followed by a horizontal translation 2 units in the direction $\overrightarrow{Ox}$, then a vertical translation 3 units in the direction $\overrightarrow{Oy}$

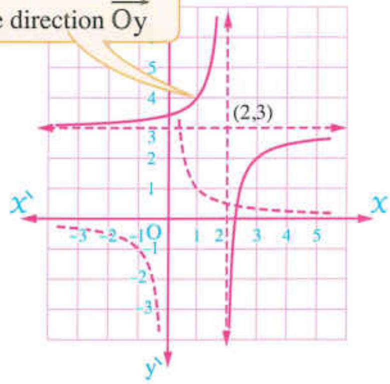
(2) $k(x) = \frac{1}{-x+2} + 3 = \frac{1}{-(x-2)} + 3$
$= \frac{-1}{x-2} + 3$

• The range of k = $\mathbb{R} - \{3\}$

• The function k is increasing on the interval $]-\infty , 2[$ and also is increasing on the interval $]2 , \infty[$

• The function k is symmetric about the point (2 , 3)

• The function k is neither even nor odd.



(3) $z(x) = -x^2 + 4x - 3 = -(x^2 - 4x + 3)$
$= -(x^2 - 4x + 4 - 1)$
$= -[(x-2)^2 - 1]$
$= -(x-2)^2 + 1$

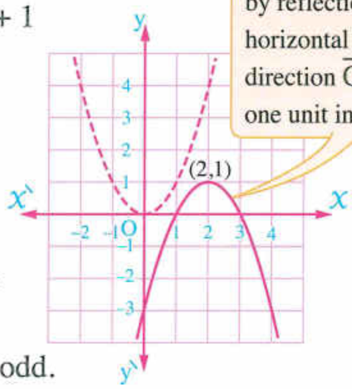
• The range of z = $]-\infty , 1]$

• The function z is increasing on the interval $]-\infty , 2[$ and is decreasing on the interval $]2 , \infty[$

• The function z is symmetric about the line $x = 2$

• The function z is neither even nor odd.

The curve of the function z is the same curve of the function $f : f(x) = x^2$ by reflection in x-axis followed by a horizontal translation two units in the direction $\overrightarrow{Ox}$, then a vertical translation one unit in the direction $\overrightarrow{Oy}$



Notice that :

The vertex of the curve of the function z is (2 , 1) we can get it from the law :

The vertex of the curve = $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$

for the functions whose rules are in the form :

$f(x) = a x^2 + b x + c$

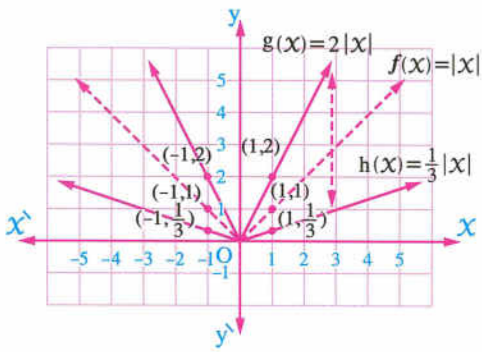
Fifth Stretching of the function curve

For any function f , the curve of $y = a f(x)$ where $a \in \mathbb{R}^*$

- Vertical stretch for the curve $y = f(x)$ if $a > 1$
- Vertical shrinking for the curve $y = f(x)$ if $0 < a < 1$

For example :

In the opposite figure :



- The curve of the function g :

  $g(x) = 2|x|$ is vertical stretch for the curve of the function $f : f(x) = |x|$

  because : $a > 1$

  i.e. For each $(x, y) \in f$, then $(x, 2y) \in g$

- The curve of the function h :

  $h(x) = \frac{1}{3}|x|$ is vertical shrinking for the curve of the function $f : f(x) = |x|$

  because : $0 < a < 1$

  i.e. For each $(x, y) \in f$, then $\left(x, \frac{1}{3} y\right) \in h$

Example 6

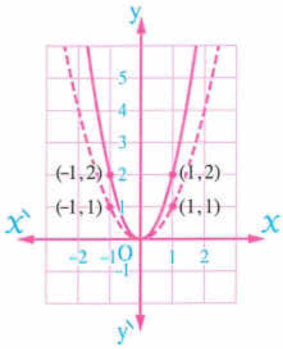
Use the curve of the function $f : f(x) = x^2$ to represent each of the following curves :

(1) $g(x) = 2x^2$ (2) $h(x) = -\frac{1}{2}x^2$ (3) $k(x) = 2(x-1)^2 - 3$

From the graph , determine the range of each one , discuss its monotony and state whether the function is even , odd or otherwise.

Solution

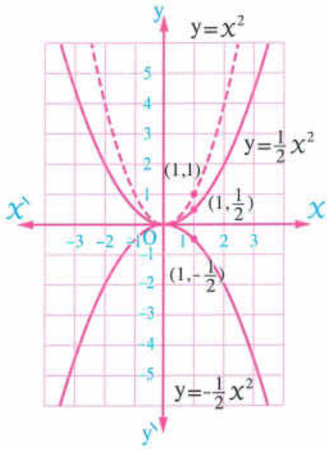
(1) The curve of the function g is vertical stretch for the curve of the function f where $a = 2 > 1$



i.e. For each $(x , y) \in f$, then $(x , 2y) \in g$

- Range of g = $[0 , \infty[$
- The function g is decreasing on $]-\infty , 0[$ and is increasing on $]0 , \infty[$
- The function g is even.

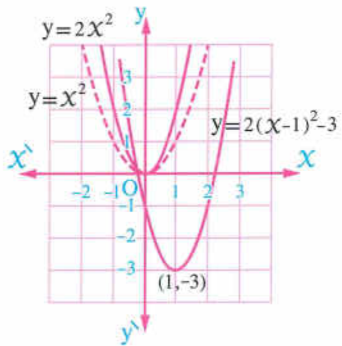
(2) The curve of the function h is vertical shrinking for the curve of the function f where $a = \frac{1}{2} < 1$, then reflection in x-axis



i.e. For each $(x , y) \in f$, then $\left(x , -\frac{1}{2} y\right) \in h$

- Range of h = $]-\infty , 0]$
- The function h is increasing on $]-\infty , 0[$ and is decreasing on $]0 , \infty[$
- The function h is even.

(3) The curve of the function k is vertical stretch for the curve of the function f where $a = 2 > 1$, then a horizontal translation one unit in the direction $\overrightarrow{Ox}$ followed by a vertical translation three units in the direction $\overrightarrow{Oy}$



- Range of k = $[-3 , \infty[$
- The function k is decreasing on $]-\infty , 1[$ and is increasing on $]1 , \infty[$
- The function k is neither even nor odd.

Remarks

For any polynomial function f :

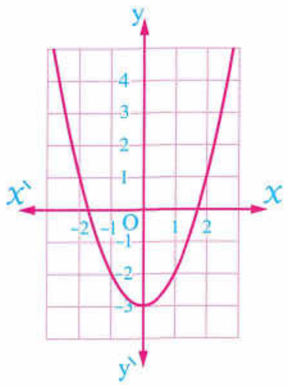
(1) The curve of $y = |f(X)|$ *i.e.* $y = \begin{cases} f(X) & , \quad f(X) \geq 0 \\ -f(X) & , \quad f(X) < 0 \end{cases}$

is represented graphically by the curve of $y = f(X)$ with replacing the part of the curve which is under the X-axis by its image by reflection in the X-axis.

For example :

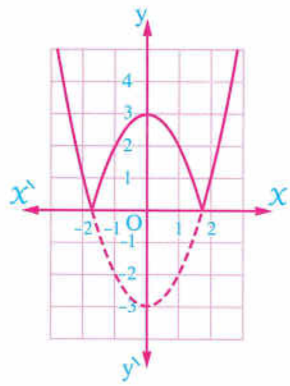
$y = X^2 - 3$

represented graphically as follows



$y = |X^2 - 3|$

represented graphically as follows



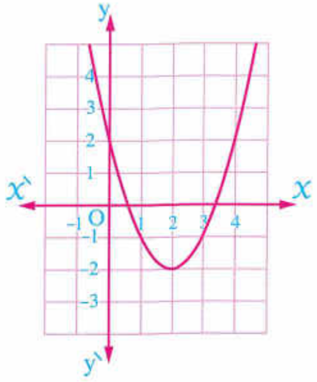
(2) The curve of $y = -|f(X)|$ *i.e.* $y = \begin{cases} -f(X) & , \quad f(X) \geq 0 \\ f(X) & , \quad f(X) < 0 \end{cases}$

is represented graphically by the curve of $y = f(X)$ with replacing the part of the curve which is up the X-axis by its image by reflection in the X-axis.

For example :

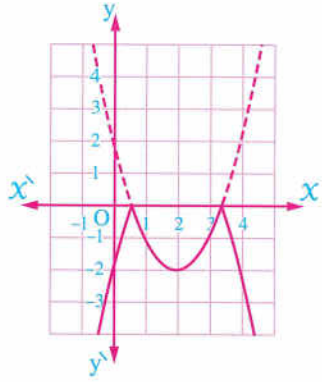
$y = (X - 2)^2 - 2$

represented graphically as follows



$y = -|(X - 2)^2 - 2|$

represented graphically as follows



Example 7

Graph each of the functions which are defined by the following rules , and from the graph find its range and state whether the function is even , odd or otherwise :

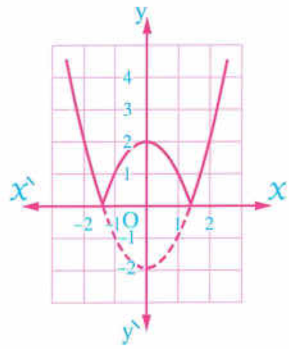
(1) $f(X) = |X^2 - 2|$ (2) $f(X) = -|X^3|$ (3) $f(X) = |X^3| + 1$

Solution

(1) Let $g(X) = X^2 - 2$

∴ The curve of the function f is the same curve of the function g with replacing the part of the curve which is under the X-axis by its image by reflection in the X-axis

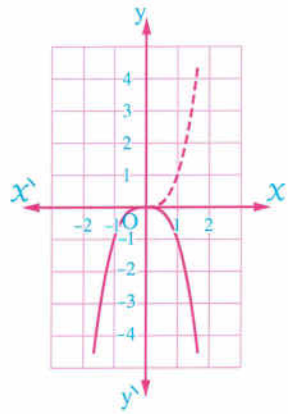
- Range of $f = [0 , \infty[$
- The function is even.



(2) Let $g(X) = X^3$

∴ The curve of the function f is the same curve of the function g with replacing the part of the curve which is up the X-axis by its image by reflection in the X-axis.

- Range of $f =]-\infty , 0]$
- The function is even.

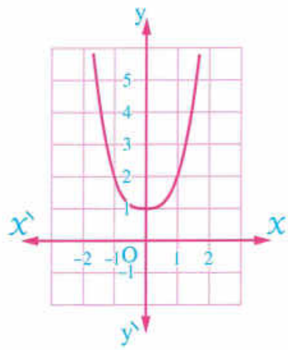


(3) Let $g(X) = X^3$

∴ $f(X) = |g(X)| + 1$

∴ The curve of the function f is the same curve of the function g with replacing the part of the curve which is under the X-axis by its image by reflection in the X-axis , then a vertical translation one unit in the direction $\overrightarrow{Oy}$

- Range of $f = [1 , \infty[$
- The function is even.

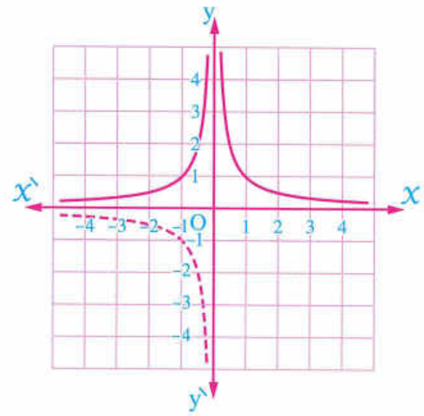


Example 8

Graph the function $f : f(X) = \frac{1}{|X|}$, then use it for the graphing of the functions which are defined by the following rules :

(1) $g(X) = f(X+1)$ (2) $h(X) = f(X) + 1$ (3) $k(X) = -f(X) + 2$

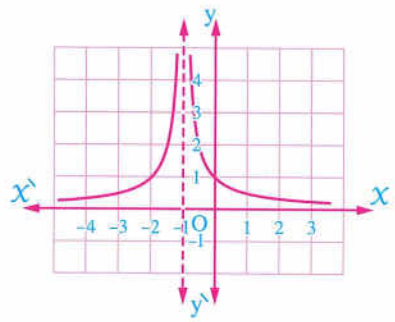
Solution



- Let $n(X) = \frac{1}{X}$

$\therefore f(X) = |n(X)|$

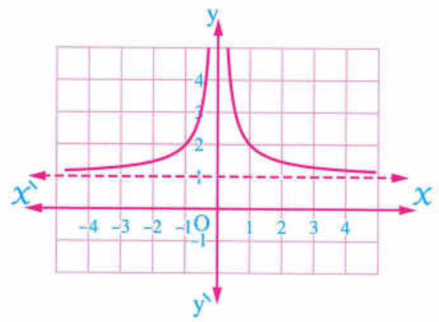
i.e. The curve of the function f is the same curve of the function n with replacing the part of the curve which is under the X-axis by its image by reflection in the X-axis



(1) $\because g(X) = f(X+1)$

$\therefore g(X) = \frac{1}{|X+1|}$

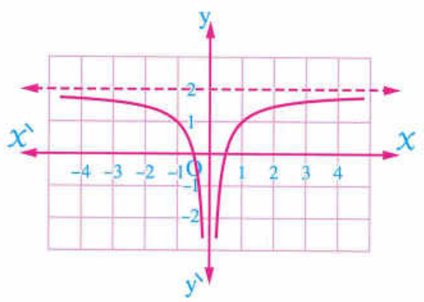
$\therefore$ The function g is represented graphically by the curve of the function f with a horizontal translation one unit in the direction $\overrightarrow{OX}$



(2) $\because h(X) = f(X) + 1$

$\therefore h(X) = \frac{1}{|X|} + 1$

$\therefore$ The function h is represented graphically by the curve of the function f with a vertical translation one unit in the direction $\overrightarrow{Oy}$



(3) $\because k(X) = -f(X) + 2$

$\therefore k(X) = \frac{-1}{|X|} + 2$

$\therefore$ The function k is represented graphically by the image of the curve of the function f by reflection in the X-axis , and a vertical translation two units in the direction $\overrightarrow{Oy}$

Example 9

Graph each of the functions which are defined by the following rules , from the graph , find the domain and the range of each function , then discuss its monotony and its symmetry , and state whether the function is even , odd or otherwise :

(1) $f(x) = x|x| + 2$

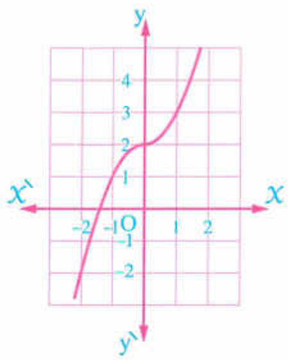
(2) $f(x) = x^2|x| - 3$

(3) $f(x) = \sqrt{x^2 - 6x + 9}$

Solution

(1) $f(x) = \begin{cases} x(x) + 2 & , x \geq 0 \\ x(-x) + 2 & , x < 0 \end{cases}$

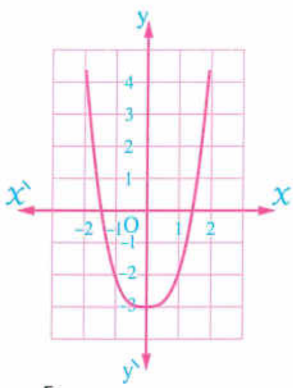
$= \begin{cases} x^2 + 2 & , x \geq 0 \\ -x^2 + 2 & , x < 0 \end{cases}$



- Domain of $f = \mathbb{R}$, range of $f = \mathbb{R}$
- The function is increasing on its domain $\mathbb{R}$
- The function is symmetric about the point (0 , 2)
- The function is neither even nor odd.

(2) $f(x) = \begin{cases} x^2(x) - 3 & , x \geq 0 \\ x^2(-x) - 3 & , x < 0 \end{cases}$

$= \begin{cases} x^3 - 3 & , x \geq 0 \\ -x^3 - 3 & , x < 0 \end{cases}$

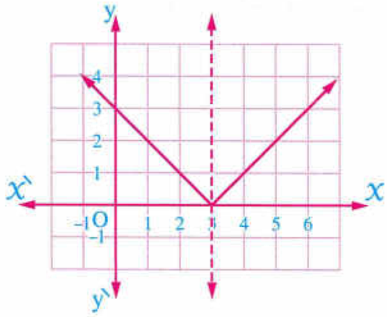


- Domain of $f = \mathbb{R}$, range of $f = [-3 , \infty[$
- The function is decreasing on $]-\infty , 0[$ and is increasing on $]0 , \infty[$
- The function is symmetric about y-axis.
- The function is even.

(3) $f(x) = \sqrt{(x-3)^2}$

$\therefore f(x) = |x - 3|$

$\therefore$ The function f is represented graphically by the curve of $y = |x|$ with a horizontal translation 3 units in the direction $\overrightarrow{Ox}$



- Domain of $f = \mathbb{R}$, range of $f = [0, \infty[$
- The function is decreasing on $]-\infty, 3[$ and is increasing on $]3, \infty[$
- The function is symmetric about the line $x = 3$
- The function is neither even nor odd.

Notice that :

$$\sqrt{x^2} = |x| = \begin{cases} x & , x \geq 0 \\ -x & , x < 0 \end{cases}$$

Enrichment your knowledge

if $f(x)$ is a real function , then :

The image of the curve $y = f(x)$ by reflection
- In y-axis → is the curve of the function → $y = f(-x)$
- In x-axis → is the curve of the function → $y = -f(x)$
- In the origin → is the curve of the function → $y = -f(-x)$



Lesson Six Solving absolute value equations and inequalities

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First Solving absolute value equations

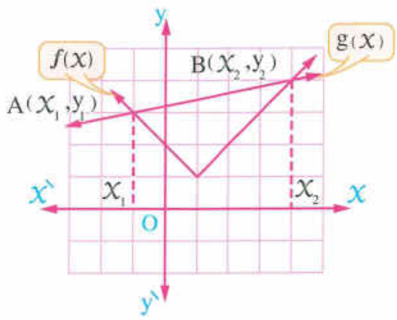
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There are two methods for solving absolute value equations :

1 Graphical method

In this method , we use graphing the real functions in solving equations , noticing that for any two functions f and g the solutions set of the equation $f(x) = g(x)$ is the set of x-coordinates of the intersecting points of the curves of the two functions f and g

In the opposite figure :



If the two curves of the two functions f and g intersecting at the two points A (x_1 , y_1) and B (x_2 , y_2) , then the solution set of the equation $f(x) = g(x)$ in $\mathbb{R}$ is $\{x_1 , x_2\}$

2 Algebraic method

In this method , we use the definition of the absolute value and some properties of the absolute value of the real number in solving the equations.

Definition of the absolute value

If $x \in \mathbb{R}$, a , b are real numbers , then $|x| = \begin{cases} x & , \quad x \geq 0 \\ -x & , \quad x < 0 \end{cases}$

and so $|x + a| = \begin{cases} x + a & , \quad x \geq -a \\ -x - a & , \quad x < -a \end{cases}$, $|a x + b| = \begin{cases} a x + b & , \quad x \geq \frac{-b}{a} \\ -a x - b & , \quad x < \frac{-b}{a} \end{cases}$

UNIT 1

Properties of the absolute value of the real number

1 $|a| \geq 0$

2 $|a\,b| = |a| \times |b|$

3 $|a + b| \leq |a| + |b|$

i.e. The absolute value of the sum of two numbers is smaller than or equal to the sum of their absolute values and the equality is happened if a , b are negative together , positive together or each of them equals zero.

For example :

i.e. $|4 + (-7)| < |4| + |-7|$

$, |-4 + (-7)| = |-4| + |-7|$

Remarks

1. For any real number a , then : $|a| = |-a|$

For example : $|3| = |-3|$

2. $|a - x| = |x - a|$

For example : $|2 - x| = |x - 2|$

3. $|x| = c \,, c > 0 \Leftrightarrow x = \pm c$

For example : If $|x| = 3$, then : $x = \pm 3$ and if $a = \pm 5$, then : $|a| = 5$

4. If a and b are two real numbers , then : $|a| = |b| \Leftrightarrow a = \pm b$

5. For any real number a , then : $\left(|a|\right)^2 = a^2$

For example : $\left(|-2|\right)^2 = 4 \,, \left(|-\frac{1}{2}|\right)^2 = \frac{1}{4}$

6. For any real number a , then : $\sqrt{a^2} = |a|$

For example : $\sqrt{(5)^2} = |5| = 5 \,, \sqrt{(-3)^2} = |-3| = 3$

7. If $|x| = x$, then : $x \in [0 \,, \infty[$

8. If $|x| = -x$, then : $x \in]-\infty \,, 0]$

1 Solving the equation in the form : $|a\mathcal{X} + b| = c$, $c \in [0, \infty[$

"i.e. absolute of first degree expression = non negative real number"

The algebraic solution

(1) Using the definition
(2) Using the property value inside the absolute sign $= \pm$ the real number

The graphical solution

The $\mathcal{X}$-coordinates of the intersection points of the two curves $f(\mathcal{X}) = |a\mathcal{X} + b|$, $g(\mathcal{X}) = c$

Remark

If $|a\mathcal{X} + b| = c$, $c \in]-\infty, 0[$, then the solution set in $\mathbb{R} = \varnothing$

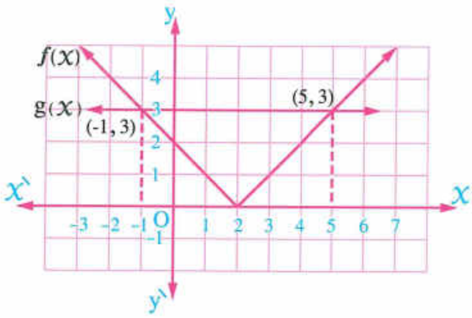
For example the solution set of the equation $|3\mathcal{X} - 4| = -5$ in $\mathbb{R}$ is $\varnothing$

Example 1

Find graphically , then perform algebraically the solution set in $\mathbb{R}$ for the equation :
$|\mathcal{X} - 2| = 3$

Solution

Graphical solution :



Putting $f(\mathcal{X}) = |\mathcal{X} - 2|$, $g(\mathcal{X}) = 3$

- We draw the curve of the function $f : f(\mathcal{X}) = |\mathcal{X} - 2|$ and it is the same curve of $y = |\mathcal{X}|$ with a horizontal translation 2 units in the direction $\overrightarrow{O\mathcal{X}}$
- We draw the curve of the function $g : g(\mathcal{X}) = 3$ and it is a constant function represented by a straight line parallel to the $\mathcal{X}$-axis and intersects the y-axis at the point $(0, 3)$
- We find the intersection points of the two curves are $(-1, 3)$ and $(5, 3)$

$\therefore$ The solution set $= \{-1, 5\}$

Algebraic solution :

First : Using the definition of the absolute value function

$$f(X) = |X - 2| = \begin{cases} X - 2 & , \quad X - 2 \geq 0 \\ -X + 2 & , \quad X - 2 < 0 \end{cases}$$

$$\therefore f(X) = \begin{cases} X - 2 & , \quad X \geq 2 \\ -X + 2 & , \quad X < 2 \end{cases}$$

At $X \geq 2 : X - 2 = 3$ $\qquad \therefore X = 5 \in [2, \infty[$

At $X < 2 : -X + 2 = 3$ $\qquad \therefore X = -1 \in]-\infty, 2[$

$\therefore$ The solution set $= \{-1, 5\}$

Second : Using the property "what inside the absolute sign = ± the real number"

We can summarize the steps of algebraic solution as the following :

$\because |X - 2| = 3$ $\qquad \therefore X - 2 = \pm 3$

$\therefore X - 2 = 3$ $\qquad$ *i.e.* $X = 5$

or $\quad X - 2 = -3$ $\qquad$ *i.e.* $X = -1$

$\therefore$ The solution set $= \{-1, 5\}$

2 Solving the equation in the form : $|a X + b| = |c X + d|$

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"***i.e.* absolute of first degree expression in X = absolute of first degree expression in X**"

The algebraic solution

(1) One of the two expressions = ± the other expression.

(2) By squaring the two sides of the equation.

The graphical solution

The X-coordinates of the intersection points of the two curves $f(X) = |a X + b|$, $g(X) = |c X + d|$

Example 2

Find graphically , then perform algebraically the solution set of the equation :

$|x-4| = |2x-5|$ in $\mathbb{R}$

Solution

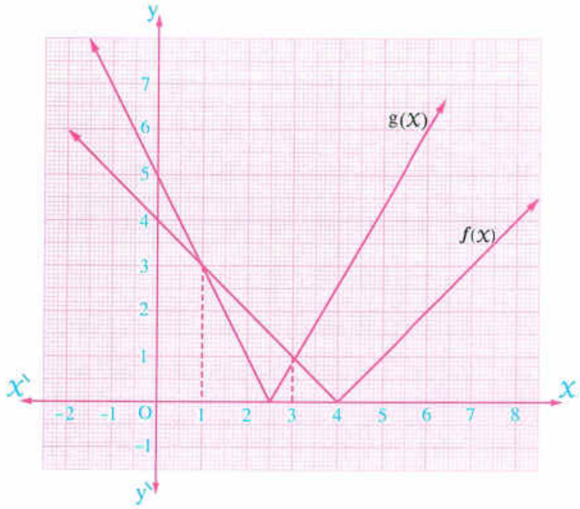
Put $f(x) = |x-4|$, $g(x) = |2x-5| = 2\left|x - 2\frac{1}{2}\right|$

Graphical solution :

The function f is represented graphically by the curve $y = |x|$ with horizontal translation 4 units in $\overrightarrow{Ox}$ directions , the function g is represented graphically by the graph $y = 2|x|$ with horizontal translation $2\frac{1}{2}$ units in $\overrightarrow{Ox}$ direction.

, $\because$ the two curves are intersecting at the two points (1 , 3) , (3 , 1)

$\therefore$ The solution set = $\{1 , 3\}$



Algebraic solution :

First : By using the property "one of the two expressions = ± the other expression"

$\because |x-4| = |2x-5|$

$\therefore x-4 = \pm(2x-5)$ (from absolute property)

$\therefore x-4 = 2x-5$ and so $x = 1$

or $x-4 = -2x+5$ $\therefore 3x = 9$ and so $x = 3$

$\therefore$ The solution set = $\{1 , 3\}$

Second : By squaring both sides

$\therefore (x-4)^2 = (2x-5)^2$ $\therefore x^2 - 8x + 16 = 4x^2 - 20x + 25$

$\therefore 3x^2 - 12x + 9 = 0$ $\therefore x^2 - 4x + 3 = 0$

$\therefore (x-3)(x-1) = 0$ $\therefore x = 3$ or $x = 1$

$\therefore$ The solution set = $\{1 , 3\}$

3 Solving the equation in the form : $|a\mathcal{X}+b| = c\mathcal{X}+d$

"i.e. absolute of first degree expression in x = first degree expression in x"

The algebraic solution

Using the defintion to redefine the absolute to get the two equation

$a\mathcal{X}+b = c\mathcal{X}+d$ at $\mathcal{X} \geq \frac{-b}{a}$

, $a\mathcal{X}+b = -c\mathcal{X}-d$ at $\mathcal{X} < \frac{-b}{a}$

The graphical solution

The $\mathcal{X}$-coordinates of the intersection points of the two curves

$f(\mathcal{X}) = |a\mathcal{X}+b|$

, $g(\mathcal{X}) = c\mathcal{X}+d$

Example 3

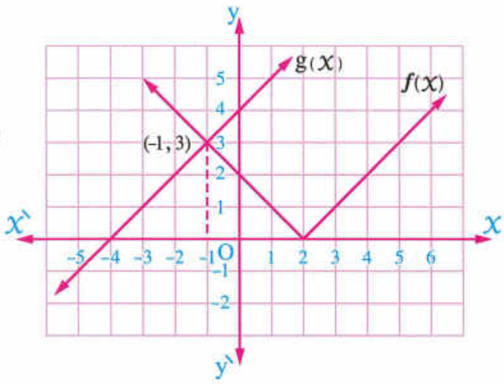
Find graphically , then perform algebraically the solution set in $\mathbb{R}$ for each of the following equations :

(1) $|\mathcal{X}-2| = \mathcal{X}+4$ (2) $|\mathcal{X}+3| - \frac{1}{2}\mathcal{X} = 3$ (3) $|\mathcal{X}-3| = 3-\mathcal{X}$

Solution

(1) Putting $f(\mathcal{X}) = |\mathcal{X}-2|$, $g(\mathcal{X}) = \mathcal{X}+4$

Graphical solution :



$\because$ The function f is represented by the curve of $y = |\mathcal{X}|$ with a horizontal translation 2 units in the direction of $\overrightarrow{O\mathcal{X}}$, the function g is represented by the curve of $y = \mathcal{X}$ with a vertical translation 4 units in the direction $\overrightarrow{Oy}$

$\therefore$ The two curves intersect at one point $(-1 , 3)$

$\therefore$ The solution set $= \{-1\}$

Algebraic solution :

$$f(\mathcal{X}) = |\mathcal{X}-2| = \begin{cases} \mathcal{X}-2 & , \quad \mathcal{X} \geq 2 \\ -\mathcal{X}+2 & , \quad \mathcal{X} < 2 \end{cases}$$

At $\mathcal{X} \geq 2 : \mathcal{X}-2 = \mathcal{X}+4$, then $-2 = 4$ (not possible)

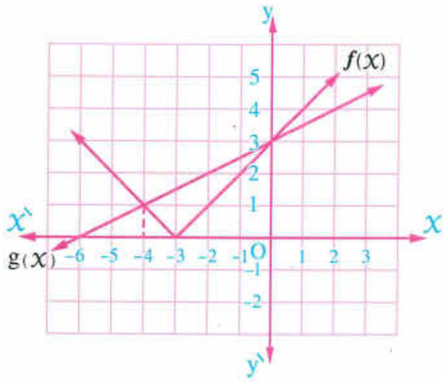
At $\mathcal{X} < 2 : -\mathcal{X}+2 = \mathcal{X}+4$, then $\mathcal{X} = -1 \in]-\infty , 2[$

$\therefore$ The solution set $= \{-1\}$

(2) $|x+3| = \frac{1}{2}x + 3$

Putting $f(x) = |x+3|$, $g(x) = \frac{1}{2}x + 3$

Graphical solution :



f is represented by the curve of $y = |x|$ with a horizontal translation 3 units in the direction of $\overrightarrow{Ox}$, g is represented by the curve of $y = x$ with a vertical shrinking in which : $a = \frac{1}{2}$ and a vertical translation 3 units in the direction $\overrightarrow{Oy}$ (*i.e.* A straight line of slope $= \frac{1}{2}$ and passes through the point (0 , 3))

$\therefore$ The two curves intersect at (0 , 3) and (– 4 , 1)

$\therefore$ The solution set $= \{0 , -4\}$

Algebraic solution :

$$f(x) = |x+3| = \begin{cases} x+3 & , \quad x \geq -3 \\ -x-3 & , \quad x < -3 \end{cases}$$

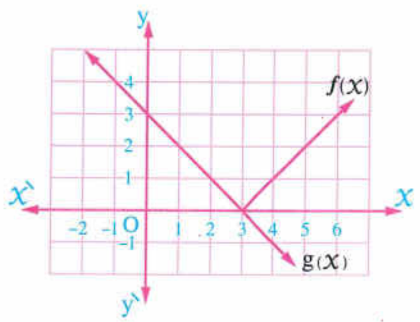
At $x \geq -3 : x + 3 = \frac{1}{2}x + 3$, then : $x = 0 \in [-3 , \infty[$

At $x < -3 : -x - 3 = \frac{1}{2}x + 3$, then : $x = -4 \in]-\infty , -3[$

$\therefore$ The solution set $= \{0 , -4\}$

(3) Let $f(x) = |x-3|$, $g(x) = 3 - x = -x + 3$

Graphical solution :



f is represented graphically by the curve of $y = |x|$ with a horizontal translation 3 units in the direction $\overrightarrow{Ox}$, g is represented graphically by the curve of $y = x$ by reflection in the x-axis , then a vertical translation 3 units in the direction $\overrightarrow{Oy}$

$\therefore$ The two curves intersect at an infinite number of points (x , y) where $x \in]-\infty , 3]$

$\therefore$ The solution set $=]-\infty , 3]$

Algebraic solution :

$f(x) = |x-3| = \begin{cases} x-3 & , \quad x \geq 3 \\ -x+3 & , \quad x < 3 \end{cases}$

At $x \geq 3$: $\therefore x - 3 = 3 - x$

$\therefore 2x = 6$ $\qquad \therefore x = 3 \in [3 , \infty[$

At $x < 3$: $\therefore -x + 3 = 3 - x$

and this relation is performed for all values of $x \in]-\infty , 3[$

$\therefore$ The solution set $= \{3\} \cup]-\infty , 3[=]-\infty , 3]$

Example 4

Find algebraically the solution set in $\mathbb{R}$ for each of the following equations :

(1) $|3x-9| - |3-x| = 10$

(2) $\sqrt{x^2-4x+4} + x = 10$

(3) $x^2 - 3|x| - 28 = 0$

Solution

(1) $\because |3x-9| - |3-x| = 10$

$\therefore 3|x-3| - |x-3| = 10$ $\qquad$ (Notice that : $|x-3| = |3-x|$)

$\therefore 2|x-3| = 10$ $\qquad \therefore |x-3| = 5$

$\therefore x - 3 = \pm 5$ $\qquad \therefore x - 3 = 5$, then $x = 8$

or $x - 3 = -5$, then $x = -2$

$\therefore$ The solution set $= \{8 , -2\}$

(2) $\because \sqrt{x^2-4x+4} + x = 10$ $\qquad \therefore \sqrt{(x-2)^2} = 10 - x$

$\therefore |x-2| = 10 - x$

, $\because |x-2| = \begin{cases} x-2 & , \quad x \geq 2 \\ -x+2 & , \quad x < 2 \end{cases}$

$\therefore$ When $x \geq 2$: $x - 2 = 10 - x$, then $x = 6 \in [2 , \infty[$

, when $x < 2$: $-x + 2 = 10 - x$

$\therefore 2 = 10$ (this is impossible) $\qquad \therefore$ The solution set $= \{6\}$

(3) $\because |x| = \begin{cases} x & , \quad x \geq 0 \\ -x & , \quad x < 0 \end{cases}$

$\therefore x^2 - 3|x| - 28 = \begin{cases} x^2 - 3x - 28 & , \quad x \geq 0 \\ x^2 + 3x - 28 & , \quad x < 0 \end{cases}$

$\therefore$ When $x \geq 0$: $x^2 - 3x - 28 = 0$

$\therefore$ $(x-7)(x+4) = 0$, then : $x = 7 \in [0, \infty[$ or $x = -4 \notin [0, \infty[$

$\therefore$ When $x < 0$: $x^2 + 3x - 28 = 0$

$\therefore$ $(x+7)(x-4) = 0$, then : $x = -7 \in]-\infty, 0[$ or $x = 4 \notin]-\infty, 0[$

$\therefore$ The solution set $= \{7, -7\}$

Another solution :

$\because |x|^2 = |x^2| = x^2$

$\therefore |x|^2 - 3|x| - 28 = 0$ $\qquad$ $\therefore (|x| - 7)(|x| + 4) = 0$

$\therefore |x| = 7$, then $x = \pm 7$ or $|x| = -4$ (refused)

$\therefore$ The solution set $= \{7, -7\}$

Example 5

Find the solution set of each of the following equations graphically in $\mathbb{R}$:

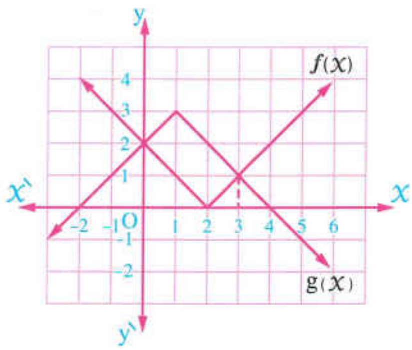
(1) $|x-2| + |x-1| = 3$ $\qquad$ **(2)** $|x-3| = 3 - |x|$

Solution

(1) $|x-2| = -|x-1| + 3$

Putting $f(x) = |x-2|$, $g(x) = -|x-1| + 3$

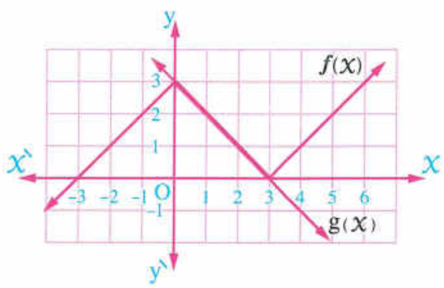
$\because f$ is represented by the curve of $y = |x|$ with a horizontal translation 2 units in the direction $\overrightarrow{Ox}$, g is represented by the curve of $y = |x|$ with reflection in the x-axis , then a horizontal translation one unit in the direction $\overrightarrow{Ox}$ and a vertical translation 3 units in the direction $\overrightarrow{Oy}$, $\because$ the two curves intersect at (0 , 2) and (3 , 1)



$\therefore$ The solution set $= \{0, 3\}$

(2) Putting $f(x) = |x-3|$, $g(x) = 3 - |x|$

$\therefore f$ is represented by the curve of $y = |x|$ with a horizontal translation 3 units in the direction $\overrightarrow{Ox}$, g is represented by the curve of $y = |x|$ with reflection in the x-axis , then a vertical translation 3 units in the direction $\overrightarrow{Oy}$



$\therefore$ The two curves intersect at an infinite number of points (x, y) where $x \in [0, 3]$

$\therefore$ The solution set $= [0, 3]$

Example 6

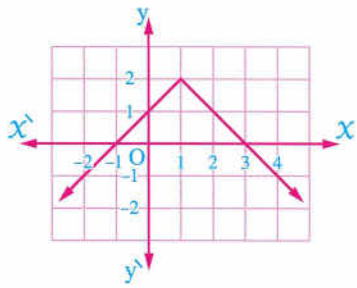
Graph the function $f : f(X) = 2 - |X - 1|$

and from the graph deduce in $\mathbb{R}$ the solution set of the equation : $f(X) = 0$

Solution

$\because f(X) = -|X - 1| + 2$

$\therefore f$ is represented graphically by the image of the curve of $y = |X|$ with the reflection in the X-axis , then a horizontal translation one unit in the direction $\overrightarrow{OX}$ and a vertical translation 2 units in the direction $\overrightarrow{Oy}$



$\therefore$ The solution set of the equation $f(X) = 0$ is the set of the X-coordinates for the intersecting points of the curve of the function f with the X-axis

i.e. With the line $y = 0$ and they are 3 and -1

$\therefore$ The solution set of the equation $= \{3 , -1\}$

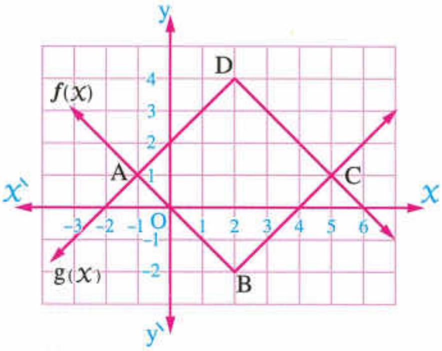
Example 7

Find in the square units , the included area between the two curves of f and g where :

$f(X) = |X - 2| - 2$, $g(X) = 4 - |X - 2|$

Solution

f is represented by the curve of $y = |X|$ with a horizontal translation 2 units in the direction $\overrightarrow{OX}$ and a vertical translation 2 units in the direction $\overrightarrow{Oy}$, g is represented by the image of the curve of $y = |X|$ with the reflection in the X-axis , then a horizontal translation 2 units in the direction $\overrightarrow{OX}$ and a vertical translation 4 units in the direction $\overrightarrow{Oy}$



From the graph :

The included area between the two curves of the two functions f and g is a surface of a square whose vertices are the points A $(-1 , 1)$, B $(2 , -2)$, C $(5 , 1)$, D $(2 , 4)$ and the length of its diagonal $\overline{AC} = 5 - (-1) = 6$ length units.

$\therefore$ The area of the included area between the two curves f and g = area of the square ABCD

$= \frac{1}{2} (AC)^2 = \frac{1}{2} \times 36 = 18$ square units.

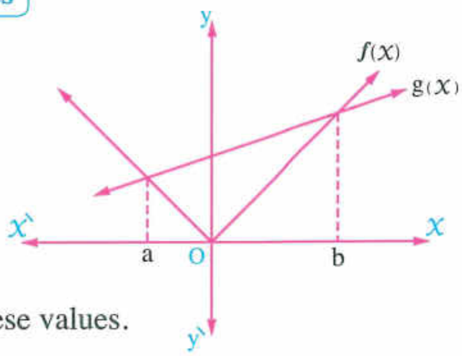
Notice that :

The two diagonals $\overline{AC}$ and $\overline{BD}$ bisect each other and perpendicular and equal in length.

$\therefore$ The figure represents a square.

Second Solving absolute value inequalities

1 Graphical solution of the absolute value inequalities



In the opposite figure :

For any two functions f and g :

- The solution set of the inequality : $f(x) < g(x)$ is $]a , b[$

and this is the set of values of x where the curve of the function f is under the curve of the function g at these values.

- The solution set of the inequality : $f(x) > g(x)$ is $]-\infty , a[\cup]b , \infty[= \mathbb{R} - [a , b]$

and this is the set of values of x where the curve of the function f is up the curve of the function g at these values.

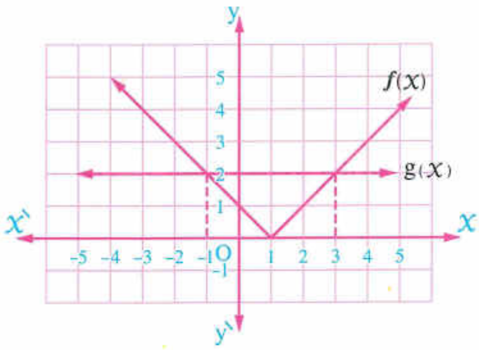
From the graph , notice that :

The solution set of the equation $f(x) = g(x)$ is $\{a , b\}$, then :

- The solution set of the inequality $f(x) \leq g(x)$ is $[a , b]$
- The solution set of the inequality $f(x) \geq g(x)$ is $]-\infty , a] \cup [b , \infty[= \mathbb{R} -]a , b[$

For example :

In Fig. (1)



The solution set of the inequality :

$f(x) < g(x)$ is $]-1 , 3[$

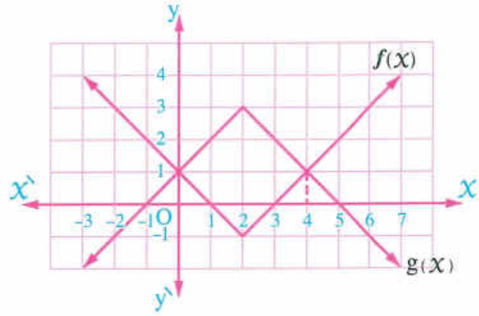
The solution set of the inequality :

$f(x) > g(x)$ is $\mathbb{R} - [-1 , 3]$

The solution set of the equation :

$f(x) = g(x)$ is $\{-1 , 3\}$

In Fig. (2)



The solution set of the inequality :

$f(x) < g(x)$ is $]0 , 4[$

The solution set of the inequality :

$f(x) > g(x)$ is $\mathbb{R} - [0 , 4]$

The solution set of the equation :

$f(x) = g(x)$ is $\{0 , 4\}$

Example 8

Find graphically in $\mathbb{R}$ the solution set of each of the following inequalities :

(1) $|X+3|<2$

(2) $|2X-8|\le 6$

(3) $|X-3|>\frac{1}{2}X$

(4) $|X-2|\ge 3-|X-3|$

Solution

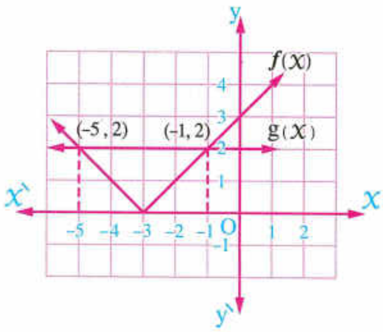
(1) Putting $f(X)=|X+3|$, $g(X)=2$

From the graph of the two functions f and g in the opposite figure

, we get that : $f(X)<g(X)$

i.e. : $|X+3|<2$ on the interval $]-5,-1[$

$\therefore$ The solution set of the inequality $=]-5,-1[$



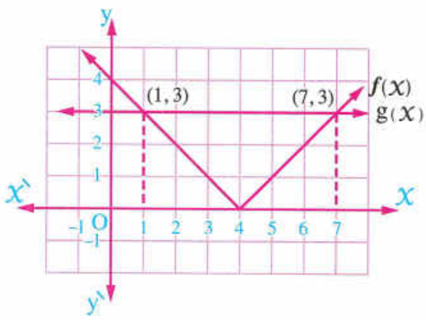
(2) $\because |2(X-4)|\le 6$ $\therefore 2|X-4|\le 6$ $\therefore |X-4|\le 3$

Putting $f(X)=|X-4|$, $g(X)=3$

From the graph of the two functions f and g in the opposite figure , we get that : $f(X)\le g(X)$

i.e. $|X-4|\le 3$ on the interval $[1,7]$

$\therefore$ The solution set of the inequality $=[1,7]$



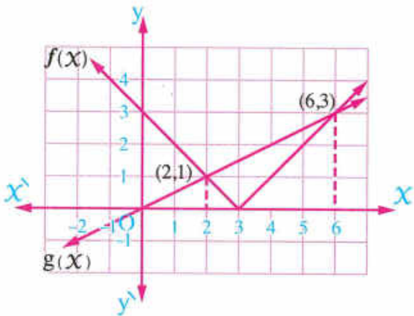
(3) Putting $f(X)=|X-3|$, $g(X)=\frac{1}{2}X$

From the graph of the two functions f and g in the opposite figure , we get that : $f(X)>g(X)$

i.e. $|X-3|>\frac{1}{2}X$ on the interval

$]-\infty,2[\cup]6,\infty[=\mathbb{R}-[2,6]$

$\therefore$ The solution set of the inequality $=\mathbb{R}-[2,6]$



(4) Putting $f(X)=|X-2|$

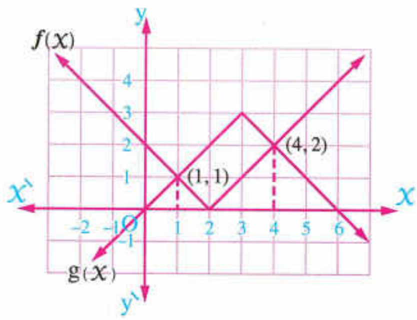
, $g(X)=3-|X-3|=-|X-3|+3$

From the graph of the two functions f and g in the opposite figure , we get that : $f(X)\ge g(X)$

i.e. $|X-2|\ge 3-|X-3|$ on the interval

$]-\infty,1]\cup[4,\infty[=\mathbb{R}-]1,4[$

$\therefore$ The solution set of the inequality $=\mathbb{R}-]1,4[$



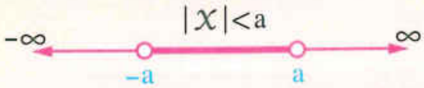
2 Algebraic solution of the absolute value inequalities

Corollaries

* For each $a \in \mathbb{R}^+$

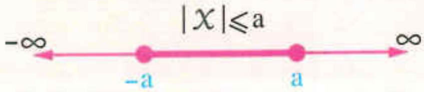
(1) If $|X| < a$, then $-a < X < a$

i.e. $X \in]-a , a[$



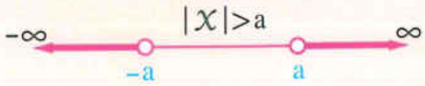
(2) If $|X| \leq a$, then $-a \leq X \leq a$

i.e. $X \in [-a , a]$



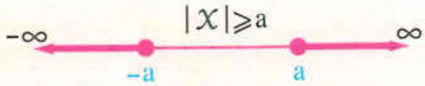
(3) If $|X| > a$, then $X > a$ or $X < -a$

i.e. $X \in \mathbb{R} - [-a , a]$



(4) If $|X| \geq a$, then $X \geq a$ or $X \leq -a$

i.e. $X \in \mathbb{R} -]-a , a[$



* For every $a \in \mathbb{R}^-$

(1) The solution set of the inequality $|X| < a$ or $|X| \leq a$ in $\mathbb{R}$ equals $\varnothing$

(2) The solution set of the inequality $|X| > a$ or $|X| \geq a$ in $\mathbb{R}$ equals $\mathbb{R}$

Example 9

Find in $\mathbb{R}$ the solution set for each of the following inequalities :

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(1) $|2X - 5| < 1$

(2) $|X + 2| > 3$

(3) $\sqrt{4X^2 + 12X + 9} \leq 1$

(4) $\frac{1}{|3X - 1|} \geq 5$

(5) $|2X - 5| + |5 - 2X| < 14$

Solution

(1) $\because |2X - 5| < 1$

$\therefore -1 < 2X - 5 < 1$

$\therefore -1 + 5 < 2X < 1 + 5$

$\therefore 4 < 2X < 6$ (dividing by 2)

$\therefore 2 < X < 3$

$\therefore$ The solution set $=]2 , 3[$

(2) $\because |x+2|>3$

$\therefore x+2>3$ or $x+2<-3$

$\therefore x>1$ or $x<-5$

$\therefore$ The solution set $= \mathbb{R} - [-5 , 1]$

Notice that :

Instead of solving the inequality $|x+2|>3$, first you can solve the inequality $|x+2| \le 3$ as the following

$\because |x+2| \le 3$ $\therefore -3 \le x+2 \le 3$

$\therefore -5 \le x \le 1$

$\therefore$ The solution set $= [-5 , 1]$

$\therefore$ The solution set of the required inequality $|x+2|>3$ is $\mathbb{R} - [-5 , 1]$

(3) $\because \sqrt{4x^2+12x+9} \le 1$

$\therefore \sqrt{(2x+3)^2} \le 1$

$\therefore |2x+3| \le 1$

$\therefore -1 \le 2x+3 \le 1$

$\therefore -4 \le 2x \le -2$

$\therefore -2 \le x \le -1$

$\therefore$ The solution set $= [-2 , -1]$

(4) $\because \frac{1}{|3x-1|} \ge 5$

$\therefore |3x-1| \le \frac{1}{5}$

$\therefore -\frac{1}{5} \le 3x-1 \le \frac{1}{5}$

$\therefore \frac{4}{5} \le 3x \le \frac{6}{5}$

$\therefore \frac{4}{15} \le x \le \frac{2}{5}$

, $\because |3x-1| = 0$, when $x = \frac{1}{3}$

$\therefore$ The solution set $= [\frac{4}{15} , \frac{2}{5}] - \{\frac{1}{3}\}$

Notice that :

- If a , b $\in \mathbb{R}^+$, a < b , then $\frac{1}{a} > \frac{1}{b}$
- At finding the solution set of the inequality , cancel the set of zeroes of the denominator from the solution set.

(5) $\because |2x-5|+|5-2x|<14$

$\therefore |2x-5|+|2x-5|<14$

$\therefore 2|2x-5|<14$ (dividing by 2)

$\therefore |2x-5|<7$

$\therefore -7<2x-5<7$

$\therefore -2<2x<12$

$\therefore -1<x<6$

$\therefore$ The solution set $=]-1 , 6[$

Third Applications on properties of equations and inequalities of absolute values

Example (10)

Determine the domain of each of the functions defined by the following rules :

(1) $f(x) = \frac{2x}{|x| - 3}$

(2) $f(x) = \frac{x^2}{|x - 1| + 2}$

(3) $f(x) = \sqrt{4 - |x|}$

(4) $f(x) = \sqrt{|x| - 2}$

(5) $f(x) = \frac{2x}{\sqrt{|x - 3| - 2}}$

(6) $f(x) = \frac{x^2}{\sqrt{3 - |x + 2|}}$

Solution

(1) We find the zeroes of the denominator : $|x| - 3 = 0$

$\therefore |x| = 3$ $\qquad \therefore x = \pm 3$

$\therefore$ The domain $= \mathbb{R} - \{3 , -3\}$

(2) We find the zeroes of the denominator : $|x - 1| + 2 = 0$

$\therefore |x - 1| = -2$

, $\because -2 < 0$ and this conflicts with the definition of the absolute value

$\therefore$ There are no zeroes for the denominator

$\therefore$ The domain $= \mathbb{R}$

(3) The function f is defined if : $4 - |x| \geq 0$

$\therefore |x| \leq 4$ $\qquad \therefore -4 \leq x \leq 4$

$\therefore$ The domain $= [-4 , 4]$

(4) The function f is defined if : $|x| - 2 \geq 0$

$\therefore |x| \geq 2$ $\qquad \therefore x \geq 2$ or $x \leq -2$

$\therefore$ The domain $= \mathbb{R} -]-2 , 2[$

(5) The function f is defined if : $|x - 3| - 2 > 0$

$\therefore |x - 3| > 2$ $\qquad \therefore x - 3 > 2$ or $x - 3 < -2$

$\therefore x > 5$ or $x < 1$

$\therefore$ The domain $= \mathbb{R} - [1 , 5]$

(6) The function f is defined if : $3 - |x + 2| > 0$

$\therefore |x + 2| < 3$ $\qquad \therefore -3 < x + 2 < 3$

$\therefore -5 < x < 1$

$\therefore$ The domain $=]-5 , 1[$

Example 11

Write the absolute value inequality which expresses :

(1) Student's mark in an exam ranges from 70 to 90 marks.

(2) The depth that some fish live in under the water level in an aquarium with interior height 40 cm.

Solution

(1) Let the mark of the student be x

$\therefore 70 \le x \le 90$ (By adding – 80 to the terms of the inequality)

$\therefore 70 - 80 \le x - 80 \le 90 - 80$

$\therefore -10 \le x - 80 \le 10$

$\therefore$ The absolute value inequality is $|x - 80| \le 10$

Notice that : 80 is the arithmetic mean of the two numbers 70 and 90

(2) Let the depth that these fish live in be x cm.

$\therefore 0 < x < 40$ (By adding – 20 to the terms of the inequality)

$\therefore 0 - 20 < x - 20 < 40 - 20$

$\therefore -20 < x - 20 < 20$

$\therefore$ The absolute value inequality is $|x - 20| < 20$

Notice that : 20 is the arithmetic mean of the two numbers 0 and 40

Example 12

Discuss the type of each of the two functions defined by the following two rules (even , odd or otherwise) :

(1) $f(x) = \dfrac{\cos x}{|x| - 3}$

(2) $f(x) = \dfrac{|2 + x| - |2 - x|}{|2 + x| + |2 - x|}$

Solution

(1) $\because f(-x) = \dfrac{\cos(-x)}{|-x| - 3} = \dfrac{\cos x}{|x| - 3} = f(x)$

$\therefore$ The function f is even.

Notice that : $|x| = |-x|$

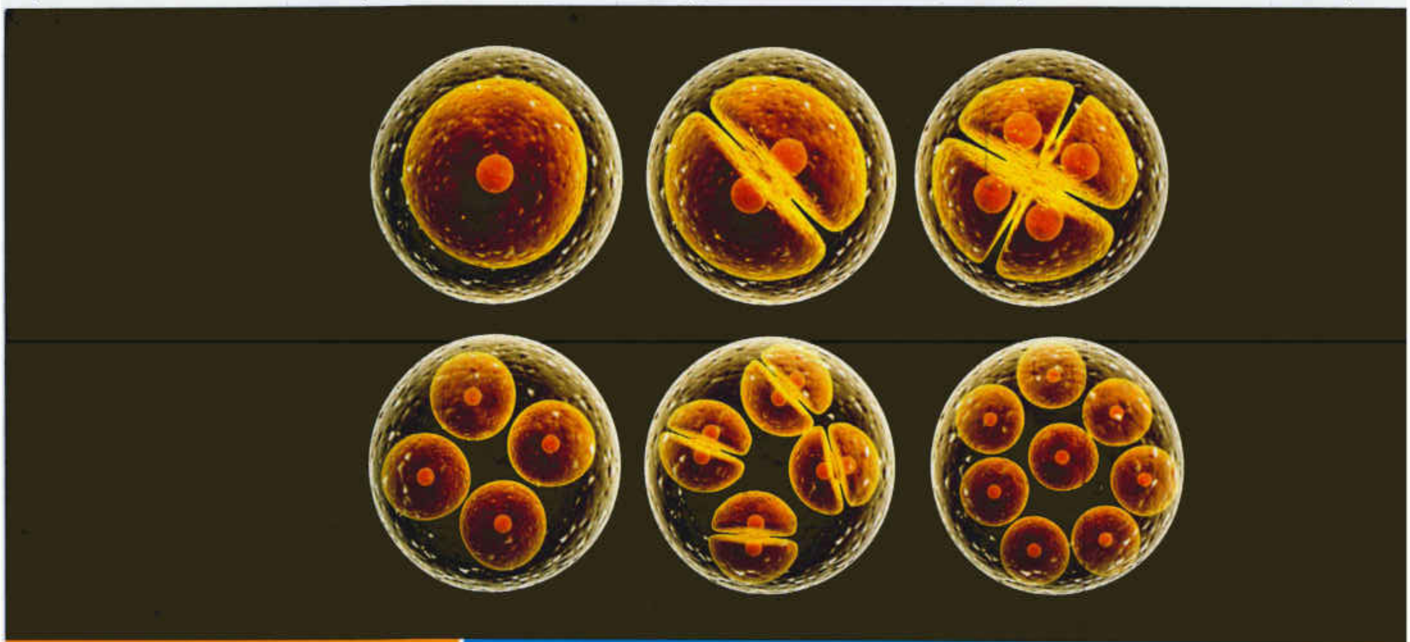
(2) $\because f(-x) = \dfrac{|2 - x| - |2 + x|}{|2 - x| + |2 + x|} = \dfrac{-\left(|2 + x| - |2 - x|\right)}{|2 + x| + |2 - x|} = -f(x)$

$\therefore$ The function f is odd.

UNIT 2

Exponents, logarithms and their applications

Unit Lessons



Lesson One — Rational exponents and exponential equations

The nth root

The equation $x^n = a$, $a \in \mathbb{R}$, $n \in \mathbb{Z}^+$ has n roots.

Let's study the following cases :

(1) If n is an even number , $a > 0$, then the equation $x^n = a$ has 2 real roots , one of them is positive and the other is negative and the other roots are complex not real numbers (when $n > 2$) and the two real roots denoted by $\sqrt[n]{a}$, $-\sqrt[n]{a}$ and the nth root which has the same sign of a is called the principle n^{th} root for the number a

For example : The equation $x^6 = 64$ has two real roots : $\sqrt[6]{64} = 2$, $-\sqrt[6]{64} = -2$ and there are four another complex not real roots (Try to get them by factorization).

(2) If n is an even number , $a < 0$, then the equation $x^n = a$ has no real roots. (Its roots are complex not real numbers).

For example : To solve the equation : $x^2 = -16$, then $x = \pm\sqrt{-16} = \pm 4\,i$ (Complex not real numbers)

(3) If n is an odd number , $a \in \mathbb{R} - \{0\}$, then the equation $x^n = a$ has only one real root which is $\sqrt[n]{a}$ and the other roots are complex not real numbers.

For example : The equation $x^3 = -27$ has only one real root which is $\sqrt[3]{-27} = -3$ and there are two complex not real roots (Try to get them by factorization)

(4) If $n \in \mathbb{Z}^+$, $a = 0$, then the equation $x^n = 0$ has only one real root which is $x = 0$ (The number of roots for the equation equals n and each of them $= 0$ when $n > 1$)

For example : The equation $x^3 = 0$ has three equal real roots and each of them $= 0$

Remark

$\sqrt[n]{a^n} = |a|$ if n is an even number , $\sqrt[n]{a^n} = a$ if n is an odd number

For example : $\sqrt[4]{(-4)^4} = |-4| = 4$, $\sqrt[3]{(-3)^3} = -3$

The properties of the nth root

If a and b are two real numbers , $\sqrt[n]{a}$, $\sqrt[n]{b} \in \mathbb{R}$, then :

(1) $\sqrt[n]{a\,b} = \sqrt[n]{a} \times \sqrt[n]{b}$

(2) $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$, $b \neq 0$

Notice that : $\sqrt[n]{a \pm b} \neq \sqrt[n]{a} \pm \sqrt[n]{b}$

For example : $\sqrt[3]{27\,b^{15}} = \sqrt[3]{27} \times \sqrt[3]{b^{15}} = 3\,b^5$, $\sqrt[4]{81\,x^4\,y^8} = \sqrt[4]{81} \times \sqrt[4]{x^4} \times \sqrt[4]{y^8} = 3\,|x|\,y^2$

Rational exponents

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Let a^m be the principle square root for the number a

$\therefore a^m = \sqrt{a}$ squaring both sides

$\therefore (a^m)^2 = a$ $\qquad$ $\therefore a^{2m} = a$

$\therefore 2\,m = 1$ $\qquad$ $\therefore m = \frac{1}{2}$

i.e. $a^{\frac{1}{2}} = \sqrt{a}$ $\quad$ similarly $a^{\frac{1}{3}} = \sqrt[3]{a}$, $a^{\frac{1}{4}} = \sqrt[4]{a}$

Definition

(1) If $n \in \mathbb{Z}^+ - \{1\}$, $a \in \mathbb{R}$, then $a^{\frac{1}{n}} = \sqrt[n]{a}$

taking in considerations that if : n is an even number , $a < 0$, then $a^{\frac{1}{n}} = \sqrt[n]{a} \notin \mathbb{R}$

For example :

$9^{\frac{1}{2}} = \sqrt{9} = 3 \in \mathbb{R}$, $\left(\frac{-1}{8}\right)^{\frac{1}{3}} = \sqrt[3]{\frac{-1}{8}} = -\frac{1}{2} \in \mathbb{R}$ but $(-16)^{\frac{1}{4}} = \sqrt[4]{-16} \notin \mathbb{R}$

(2) If m , n are two integers with no common factor , $n > 1$, $\sqrt[n]{a} \in \mathbb{R}$

, then $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$

For example :

$8^{\frac{2}{3}} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$, $(81)^{-\frac{3}{4}} = \left(\sqrt[4]{81}\right)^{-3} = 3^{-3} = \frac{1}{27}$

UNIT 2

Exponential Rules

If a , b are two real numbers , m , n are two rational numbers and by excluding the cases in which the denominator = zero , and cases in which both the base = zero and the index = zero and all expressions should be defined , then :

(1) $a^{zero} = 1$

(2) $a^{-n} = \frac{1}{a^n}$

(3) $a^m \times a^n = a^{m+n}$

(4) $\frac{a^m}{a^n} = a^{m-n}$

(5) $(a^m)^n = a^{mn}$

(6) $(ab)^n = a^n b^n$

(7) $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n = \frac{b^n}{a^n}$

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Remarks

1. If $a \in \mathbb{R}^-$, then : $a^n > 0$ when n is an even integer

, $a^n < 0$ when n is an odd integer

For example : $(-4)^2 = 16 > 0$ but $(-4)^3 = -64 < 0$

2. * If $x^{\frac{m}{n}} = a$, then $x = a^{\frac{n}{m}}$ where m is an odd number

* If $x^{\frac{m}{n}} = a$, then $x = \pm a^{\frac{n}{m}}$ where m is an even number

where m , n have no common factors $\left(\textit{i.e.}\ \frac{m}{n}\text{ is a rational in simplest form}\right)$

and if one of them is even , then a must be greater than or equal to zero.

3. *Common mistake* * $(-32)^{\frac{2}{10}} = \sqrt[10]{(-32)^2} = 2$ (wrong answer)

* $(-32)^{\frac{2}{10}} = \left(\sqrt[10]{-32}\right)^2 =$ undefined in $\mathbb{R}$ (wrong answer)

because the power $\frac{2}{10}$ is not in the simplest form and should be simplified first $\left(\frac{2}{10} = \frac{1}{5}\right)$

$\therefore (-32)^{\frac{2}{10}} = (-32)^{\frac{1}{5}} = \sqrt[5]{-32} = -2$ (the correct answer)

Example 1

Find the result of each of the following in the simplest form :

(1) $\sqrt[5]{a^3} \times \sqrt{a^3}$

(2) $\left(\sqrt[5]{x}\right)^2 \times \sqrt[3]{x^2}$

(3) $\left(\sqrt[3]{a^{-5}}\right)^2 \times \left(\sqrt[4]{a^3}\right)^3$

Solution

(1) $\sqrt[5]{a^3} \times \sqrt{a^3} = a^{\frac{3}{5}} \times a^{\frac{3}{2}} = a^{\frac{3}{5}+\frac{3}{2}} = a^{\frac{21}{10}} = a^2 \times a^{\frac{1}{10}} = a^2 \sqrt[10]{a}$

(2) $\left(\sqrt[5]{x}\right)^2 \times \sqrt[3]{x^2} = x^{\frac{2}{5}} \times x^{\frac{2}{3}} = x^{\frac{2}{5}+\frac{2}{3}} = x^{\frac{16}{15}} = x \times x^{\frac{1}{15}} = x \sqrt[15]{x}$

(3) $\left(\sqrt[3]{a^{-5}}\right)^2 \times \left(\sqrt[4]{a^3}\right)^3 = (a^{-\frac{5}{3}})^2 \times (a^{\frac{3}{4}})^3 = a^{-\frac{10}{3}} \times a^{\frac{9}{4}} = a^{-\frac{13}{12}} = \frac{1}{a^{\frac{13}{12}}} = \frac{1}{|a| \times a^{\frac{1}{12}}} = \frac{1}{|a| \sqrt[12]{a}}$

Example 2

Put in the simplest form : $\dfrac{\sqrt[4]{8} \times \sqrt[8]{0.01} \times 125}{\sqrt[4]{(15)^3} \times \sqrt[8]{4^5} \times (36)^{\frac{-3}{8}}}$

Solution

The expression $= \dfrac{\sqrt[4]{2^3} \times \sqrt[8]{(10)^{-2}} \times 5^3}{(15)^{\frac{3}{4}} \times 4^{\frac{5}{8}} \times (36)^{\frac{-3}{8}}} = \dfrac{2^{\frac{3}{4}} \times (2 \times 5)^{-\frac{1}{4}} \times 5^3}{(3 \times 5)^{\frac{3}{4}} \times (2^2)^{\frac{5}{8}} \times (2^2 \times 3^2)^{-\frac{3}{8}}}$

$= \dfrac{2^{\frac{3}{4}} \times 2^{-\frac{1}{4}} \times 5^{-\frac{1}{4}} \times 5^3}{3^{\frac{3}{4}} \times 5^{\frac{3}{4}} \times 2^{\frac{5}{4}} \times 2^{-\frac{3}{4}} \times 3^{-\frac{3}{4}}} = 2^{\frac{3}{4}-\frac{1}{4}-\frac{5}{4}+\frac{3}{4}} \times 3^{-\frac{3}{4}+\frac{3}{4}} \times 5^{-\frac{1}{4}+3-\frac{3}{4}}$

$= 2^0 \times 3^0 \times 5^2 = 1 \times 1 \times 25 = 25$

Example 3

Find the solution set in $\mathbb{R}$ for each of the following :

(1) $3x^5 = -96$

(2) $x^6 = -64$

(3) $(x-2)^4 = 81$

(4) $x^{\frac{3}{4}} = 27$

(5) $\sqrt[5]{x^2} = 1$

(6) $\sqrt[4]{(3x+2)^3} = 8$

(7) $x^{\frac{4}{3}} - 5x^{\frac{2}{3}} + 4 = 0$

(8) $(x^2 - 4x - 13)^{\frac{3}{5}} = 8$

Solution

Notice that : The required is the solution set in $\mathbb{R}$ ***i.e.*** The required is the real roots only.

(1) $\because 3x^5 = -96$ $\therefore x^5 = -32$

$\therefore x = \sqrt[5]{-32} = -2$ $\therefore$ S.S. $= \{-2\}$

(2) $x^6 = -64$ $\because -64 < 0$, 6 is an even number.

$\therefore$ S.S. $= \varnothing$

(3) $\because (x-2)^4 = 81$

$\therefore x - 2 = \sqrt[4]{81} = 3$ or $x - 2 = -\sqrt[4]{81} = -3$

$\therefore x = 3 + 2 = 5$ or $x = -3 + 2 = -1$

$\therefore$ S.S. $= \{5 , -1\}$

(4) $\because x^{\frac{3}{4}} = 27$ $\therefore x = 27^{\frac{4}{3}} = (3^3)^{\frac{4}{3}} = 3^4$

$\therefore x = 81$ $\therefore$ S.S. $= \{81\}$

(5) $\because \sqrt[5]{x^2} = 1$ $\therefore x^{\frac{2}{5}} = 1$

$\therefore x = \pm 1^{\frac{5}{2}}$ $\therefore x = \pm 1$

$\therefore$ S.S. $= \{1 , -1\}$

(6) $\because \sqrt[4]{(3x+2)^3} = 8$ $\therefore (3x+2)^{\frac{3}{4}} = 8$

$\therefore 3x+2 = 8^{\frac{4}{3}}$ $\therefore 3x+2 = (2^3)^{\frac{4}{3}}$

$\therefore 3x+2 = 16$ $\therefore x = \frac{14}{3}$

$\therefore$ S.S. $= \left\{\frac{14}{3}\right\}$

(7) $\because x^{\frac{4}{3}} - 5 \times x^{\frac{2}{3}} + 4 = 0$

$\therefore (x^{\frac{2}{3}} - 1)(x^{\frac{2}{3}} - 4) = 0$

$\therefore x^{\frac{2}{3}} = 1$ and hence $x = \pm 1^{\frac{3}{2}} = \pm 1$

or $x^{\frac{2}{3}} = 4$ and hence $x = \pm 4^{\frac{3}{2}} = \pm (2^2)^{\frac{3}{2}}$

$= \pm 2^3 = \pm 8$ $\therefore$ S.S. $= \{1 , -1 , 8 , -8\}$

Another solution :

Let $x^{\frac{2}{3}} = k$

$\therefore k^2 - 5k + 4 = 0$ $\therefore (k-4)(k-1) = 0$

$\therefore k = 4$ $\therefore x^{\frac{2}{3}} = 4$ $\therefore x = \pm\sqrt{4^3} = \pm 8$

or $k = 1$ $\therefore x^{\frac{2}{3}} = 1$ $\therefore x = \pm\sqrt{1^3} = \pm 1$

$\therefore$ S.S. $= \{1 , -1 , 8 , -8\}$

(8) $x^2 - 4x - 13 = 8^{\frac{5}{3}} = 32$ $\therefore x^2 - 4x - 45 = 0$

$\therefore (x-9)(x+5) = 0$ $\therefore x = 9$ or $x = -5$

$\therefore$ S.S. $= \{9 , -5\}$

Exponential equations

The exponential equation is an equation which contains a variable (unknown) in the power as $(2^{x+1} = 8)$

Laws of exponents

- **For every $m, n \in \mathbb{Z}$ and $a, b \in \mathbb{R} - \{-1, 0, 1\}$ we have :**

(1) If $a^n = 1$, then $n = \text{zero}$

(2) If $a^m = a^n$, then $m = n$

(3) If $a^n = b^n$,
- if n is an odd number, then $a = b$
- if n is an even number, then $a = \pm b$
- if $a \neq b$, then $n = \text{zero}$

Example 4

Find the value of x that satisfies each of the following equations :

(1) $2^{x+5} = 8$

(2) $3^{x^2-4} = 1$

(3) $4^{x+2} = x^{x+2}$

(4) $4^{x-3} = 3^{2x-6}$

(5) $\left(\frac{2}{3}\right)^{|x-5|} = \left(3\frac{3}{8}\right)^{-2}$

Solution

(1) $\because 2^{x+5} = 8$ $\quad \therefore 2^{x+5} = 2^3$

$\therefore x + 5 = 3$ $\quad \therefore x = -2$

(2) $\because 3^{x^2-4} = 1$

$\therefore x^2 - 4 = 0$ $\quad \therefore x = \pm 2$

(3) $\because 4^{x+2} = x^{x+2}$ $\quad \therefore x = \pm 4$ or $x + 2 = 0$, then $x = -2$

$\therefore x \in \{-2, 4, -4\}$

(4) $\because 4^{x-3} = 3^{2x-6}$ $\quad \therefore 4^{x-3} = 3^{2(x-3)}$

$\therefore 4^{x-3} = 9^{x-3}$ $\quad , \because 4 \neq 9$ $\quad \therefore x - 3 = 0$

$\therefore x = 3$

(5) $\because \left(\frac{2}{3}\right)^{|x-5|} = \left(\frac{27}{8}\right)^{-2}$ $\quad \therefore \left(\frac{2}{3}\right)^{|x-5|} = \left(\left(\frac{3}{2}\right)^3\right)^{-2}$

$\therefore \left(\frac{2}{3}\right)^{|x-5|} = \left(\frac{3}{2}\right)^{-6} = \left(\frac{2}{3}\right)^6$ $\quad \therefore |x - 5| = 6$ $\quad \therefore x - 5 = \pm 6$

$\therefore x - 5 = 6$ or $x - 5 = -6$

$\therefore x = 11$ $\quad$ | $\quad \therefore x = -1$

Example 5

Find in $\mathbb{R}$ the S.S. of each of the following equations :

(1) $2^{x} \times \sqrt[3]{4} = \left(\sqrt[3]{16}\right)^{-1}$

(2) $4^{x^2-1} = 8^{-x}$

(3) $\frac{3^{3x}+3^{2x}+3^{x}}{3^{2x}+3^{x}+1} = \frac{1}{9}$

Solution

(1) $\because 2^{x} \times 4^{\frac{1}{3}} = (16)^{-\frac{1}{3}}$ $\quad \therefore 2^{x} \times \left(2^2\right)^{\frac{1}{3}} = \left(2^4\right)^{-\frac{1}{3}}$

$\therefore 2^{x} \times 2^{\frac{2}{3}} = 2^{-\frac{4}{3}}$ $\quad \therefore 2^{x+\frac{2}{3}} = 2^{-\frac{4}{3}}$

$\therefore x + \frac{2}{3} = -\frac{4}{3}$ $\quad \therefore x = -2$

$\therefore$ S.S. $= \{-2\}$

(2) $\because 4^{x^2-1} = 8^{-x}$ $\quad \therefore (2^2)^{x^2-1} = (2^3)^{-x}$

$\therefore 2^{2x^2-2} = 2^{-3x}$ $\quad \therefore 2x^2 - 2 = -3x$

$\therefore 2x^2 + 3x - 2 = 0$ $\quad \therefore (2x-1)(x+2) = 0$

$\therefore 2x - 1 = 0$, then $2x = 1$ $\quad \therefore x = \frac{1}{2}$ or $x + 2 = 0$, then $x = -2$

$\therefore$ S.S. $= \left\{\frac{1}{2} , -2\right\}$

(3) $\because \frac{3^{3x}+3^{2x}+3^{x}}{3^{2x}+3^{x}+1} = \frac{1}{9}$ $\quad \therefore \frac{3^{x}(3^{2x}+3^{x}+1)}{3^{2x}+3^{x}+1} = 3^{-2}$

$\therefore 3^{x} = 3^{-2}$ $\quad \therefore x = -2$ $\quad \therefore$ S.S. $= \{-2\}$

Example 6

Find in $\mathbb{R}$ the S.S. of each of the following equations :

(1) $2^{x+1} + 2^{x-1} = 5$

(2) $5^{x} + \frac{125}{5^{x}} = 30$

Solution

(1) Taking 2^{x-1} as a common factor

$\therefore 2^{x-1}(2^2+1) = 5$ $\quad \therefore 2^{x-1}(4+1) = 5$

$\therefore 2^{x-1} = 1$ $\quad \therefore x - 1 = 0$

$\therefore x = 1$ $\quad \therefore$ S.S. $= \{1\}$

Another solution :

$\because 2^{x+1} + 2^{x-1} = 5$ $\therefore 2^{x} \times 2 + 2^{x} \times 2^{-1} = 5$

$\therefore 2^{x}\left[2 + \frac{1}{2}\right] = 5$ $\therefore 2^{x} \times \frac{5}{2} = 5$

$\therefore 2^{x} = 2$ $\therefore x = 1$

$\therefore$ S.S. $= \{1\}$

(2) Multiplying the two sides by 5^{x}

$\therefore 5^{2x} + 125 = 30 \times 5^{x}$ $\therefore 5^{2x} - 30 \times 5^{x} + 125 = 0$ and by factorizing

$\therefore (5^{x} - 5)(5^{x} - 25) = 0$

$\therefore 5^{x} - 5 = 0$ or $5^{x} - 25 = 0$

$\therefore 5^{x} = 5$ | $\therefore 5^{x} = 5^{2}$

$\therefore x = 1$ | $\therefore x = 2$

$\therefore$ S.S. $= \{1 , 2\}$

Another solution :

Putting $5^{x} = y$ $\therefore y + \frac{125}{y} = 30$

Multiplying the two sides by y

$\therefore y^{2} + 125 = 30\,y$

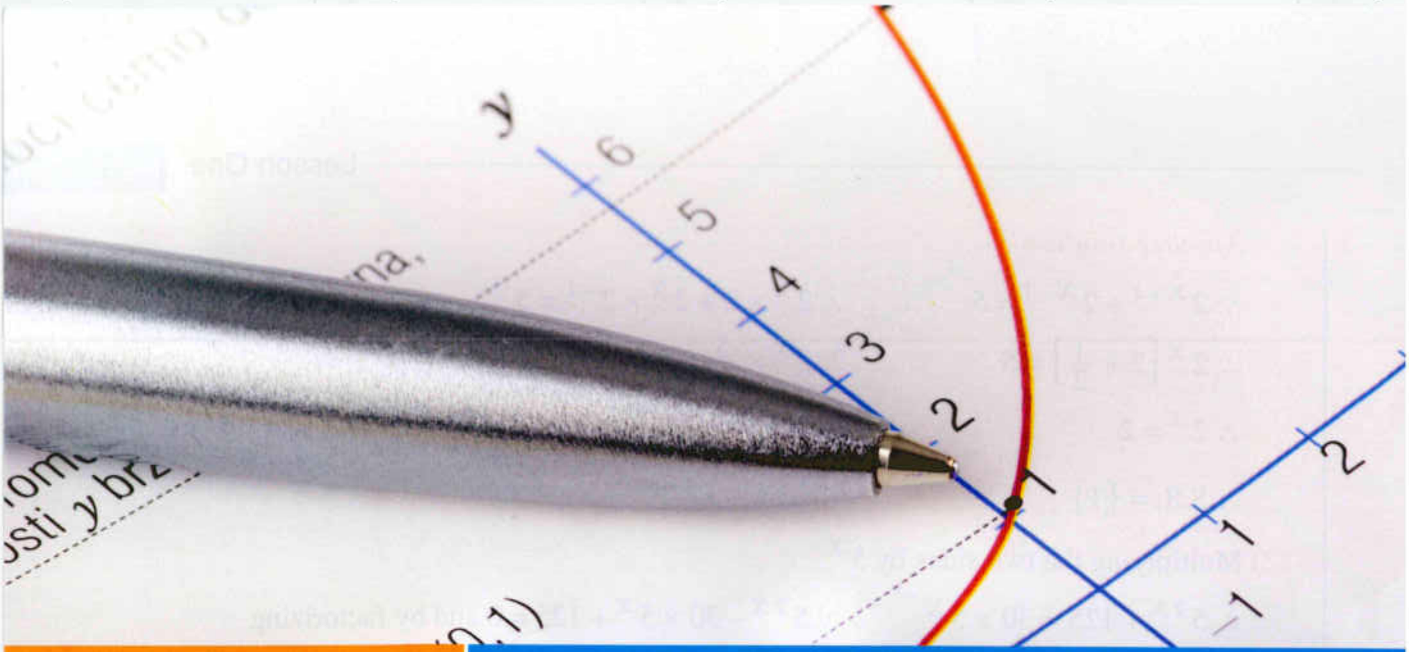
$\therefore y^{2} - 30\,y + 125 = 0$ $\therefore (y - 5)(y - 25) = 0$

$\therefore y = 5$ or $y = 25$

$\therefore 5^{x} = 5$ $\therefore x = 1$

or $5^{x} = 5^{2}$ $\therefore x = 2$

$\therefore$ S.S. $= \{1 , 2\}$



Lesson Two — Exponential function and its applications

Definition

If $a \in \mathbb{R}^+ - \{1\}$

, then the function $f : \mathbb{R} \longrightarrow \mathbb{R}^+$ where $f(X) = a^X$

is called an exponential function whose base is "a"

For example :

- $f : f(X) = 3^X$ is an exponential function whose base = 3 and its power = X
- $f : f(X) = \left(\frac{1}{2}\right)^{X+1}$ is an exponential function whose base = $\frac{1}{2}$ and its power = $X + 1$

Remark

Notice the difference between the algebraic function and the exponential function :

∗ In the algebraic function , the independent variable X is the base in the rule of the function while the power is a real number.

For example : $f : f(X) = X^2 - 3X + 1$ or $f : f(X) = (X - 3)^3$

∗ In the exponential function , the independent variable X is the power in the rule of the function while the base is a positive real number $\neq 1$

For example : $f : f(X) = 3^X$ or $f : f(X) = 3^{X-1} + 2$ are exponential functions

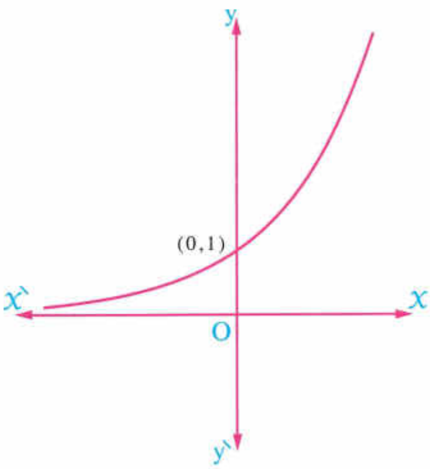
but : $f : f(X) = (-3)^X$ or $f(X) = (1)^X$ are not exponential functions.

The graphical representation of the exponential function

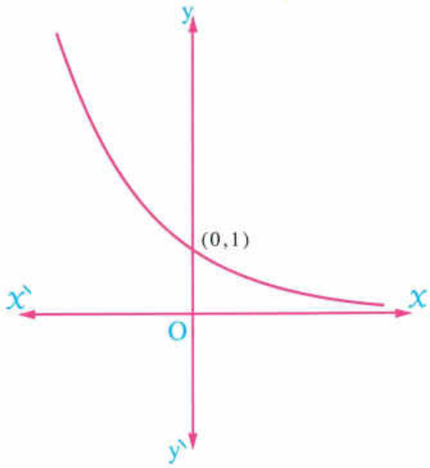
The general diagram of the graph of the function

$f : f(x) = a^x$ is as shown in the following two graphs :

a > 1

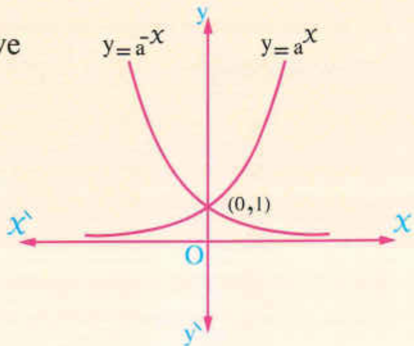


0 < a < 1



Some properties of the exponential function $f : f(x) = a^x$

- The domain = $\mathbb{R}$
- The range = $\mathbb{R}^+$ and its curve lies completely above the x-axis.
- The function is increasing on its domain $\mathbb{R}$ when $a > 1$ and is called an exponential growth function , its coefficient is a and the curve of the function approach to x-axis by the decreasing of the value of x
- The function is decreasing on its domain $\mathbb{R}$ when $0 < a < 1$ and is called an exponential decay function , its coefficient is a and the curve of the function approach to x-axis by the increasing of the value of x
- The curve of the exponential function passes through the point (0 , 1)
- $f : f(x) = a^x$ is one-to-one.
- If $f(x) = a^x$, then $f(-x) = a^{-x} = \left(\frac{1}{a}\right)^x$ and the curve $y = \left(\frac{1}{a}\right)^x$ is the image of the curve $y = a^x$ by reflection in y-axis.



- $a^x \longrightarrow \infty$ when $x \longrightarrow \infty$ where $a > 1$

  , $a^x \longrightarrow 0$ when $x \longrightarrow \infty$ where $0 < a < 1$

UNIT 2

Example 1

Graph the function $f : \mathbb{R} \longrightarrow \mathbb{R}^+$, $f(x) = 2^x$ taking $x \in [-3 , 4]$ and from the graph find an approximated value for each of the following :

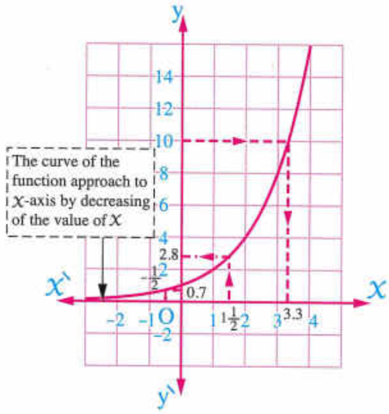
(1) $f(1.5)$, $f\left(-\frac{1}{2}\right)$

(2) The value of x when $f(x) = 10$

Solution

We form the following table :

x	-3	-2	-1	0	1	2	3	4
$y = 2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16



(1) Finding $f(1.5)$ and $f\left(-\frac{1}{2}\right)$:

At $x = 1.5$ we draw a straight line parallel to y-axis to cut the curve at a point, then read the corresponding value of y on y-axis we get it 2.8 approximately.

$\therefore f(1.5) \simeq 2.8$ similarly $f\left(-\frac{1}{2}\right) \simeq 0.7$

Remark

$f(x) = 2^x$ is an exponential growth function where $a > 1$

(2) Finding x when $f(x) = 10$ i.e. when $2^x = 10$:

At y = 10 we draw a straight line parallel to x-axis to cut the curve at a point , then read the corresponding value of x on x-axis to get it = 3.3 approximately.

$\therefore$ When $2^x = 10$, then $x \simeq 3.3$

Example 2

Graph the function $f : \mathbb{R} \longrightarrow \mathbb{R}^+$, $f(x) = \left(\frac{1}{2}\right)^x$ taking $x \in [-4 , 3]$, from the graph find an approximated value for each of the following :

(1) $f(-2.5)$

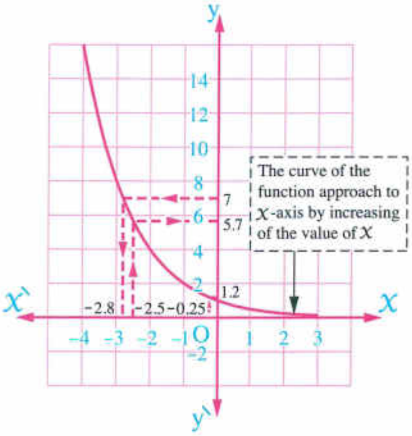
(2) $\sqrt[4]{2}$

(3) The value of x when $\left(\frac{1}{2}\right)^x = 7$

Solution

We form the following table :

x	-4	-3	-2	-1	0	1	2	3
$y = \left(\frac{1}{2}\right)^x$	16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$



From the graph we find that :

(1) $f(-2.5) \simeq 5.7$

(2) $\because \sqrt[4]{2} = 2^{\frac{1}{4}} = \left(2^{-1}\right)^{-\frac{1}{4}} = \left(\frac{1}{2}\right)^{-\frac{1}{4}} = f\left(-\frac{1}{4}\right)$

$\therefore f\left(-\frac{1}{4}\right) \simeq 1.2$

(3) When $\left(\frac{1}{2}\right)^x = 7$ i.e. $f(x) = 7$

$\therefore x \simeq -2.8$

Remark

$f(x) = \left(\frac{1}{2}\right)^x$ is an exponential decay function where $0 < a < 1$

Notice that :

In example (1) , example (2) : the curve $f : f(x) = 2^x$ is the image of the curve of the function $f : f(x) = \left(\frac{1}{2}\right)^x$ by reflection in y-axis.

Some geometric transformations of the function $f : f(x) = a^x$

1 The vertical displacement to the curve of the exponential

If $f(x) = a^x$, then $y = f(x) + c$

i.e. $y = a^x + c$ is represented graphically by the curve $y = a^x$ by vertical displacement of magnitude $|c|$

* In direction of $\overrightarrow{Oy}$ if $c > 0$ (displacement upwards)
* In direction of $\overrightarrow{Oy'}$ if $c < 0$ (displacement downwards)

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2 The horizontal displacement to the curve of the exponential function

If $f(x) = a^x$, then $y = f(x + b)$

i.e. $y = a^{x+b}$ is represented graphically by the curve $y = a^x$ by horizontal displacement of magnitude $|b|$

* In direction of $\overrightarrow{Ox}$ if $b < 0$
* In direction of $\overrightarrow{Ox'}$ if $b > 0$

3 The reflection of the curve of the exponential function in x-axis

If $f(x) = a^x$, then the curve of the function $y = -f(x)$

i.e. $y = -a^x$ is the image of the curve $y = a^x$ by reflection in x-axis.

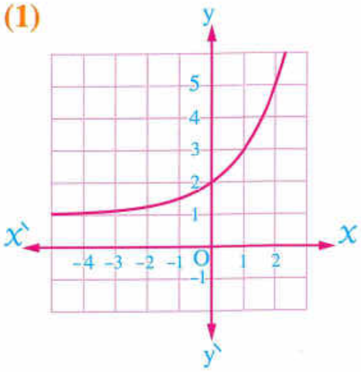
Example 3

Graph each of the following functions defined by the given rules , from the graph find the domain , the range and determine which of them is increasing and which is decreasing :

(1) $y = 2^x + 1$ (2) $y = 3^{x-1}$ (3) $y = -\left(\frac{1}{2}\right)^x$

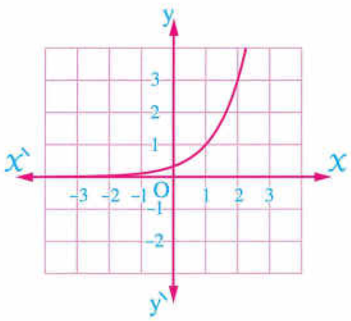
Solution

(1)



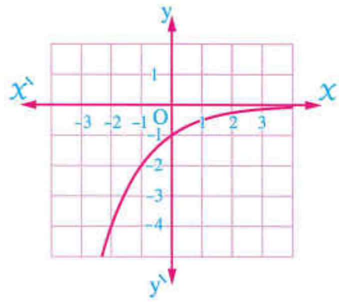
The domain = $\mathbb{R}$, the range = $]1 , \infty[$, the function is increasing on its domain.

(2)



The domain = $\mathbb{R}$, the range = $]0 , \infty[$, the function is increasing on its domain.

(3)



The domain = $\mathbb{R}$, the range = $]-\infty , 0[$, the function is increasing on its domain.

Example 4

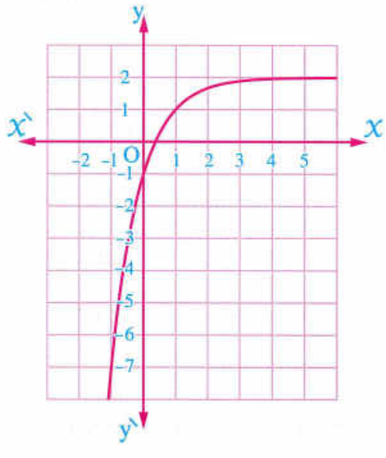
Graph the function defined by the given rule , from the graph find the domain , the range and show whether it is increasing or decreasing : $y = -3^{1-x} + 2$

Solution

$\because y = -3^{1-x} + 2 = -3^{-(x-1)} + 2$

$= -\left(\frac{1}{3}\right)^{x-1} + 2$

- The curve $y = -\left(\frac{1}{3}\right)^{x-1} + 2$ is the same as the curve $y = \left(\frac{1}{3}\right)^x$ by reflection in x-axis , then horizontal displacement one unit in the direction of $\overrightarrow{Ox}$, then vertical displacement 2 units in the direction of $\overrightarrow{Oy}$, the domain = $\mathbb{R}$, the range = $]-\infty , 2[$, the function is increasing on its domain.



Notice that :

The order of the operations is important on the curve $y = \left(\frac{1}{3}\right)^{x}$ to get the curve $y = -\left(\frac{1}{3}\right)^{x-1} + 2$ as the following

reflection in x-axis - horizontal displacement - vertical displacement.

If the order is reversed we get another curve differs than the required one.

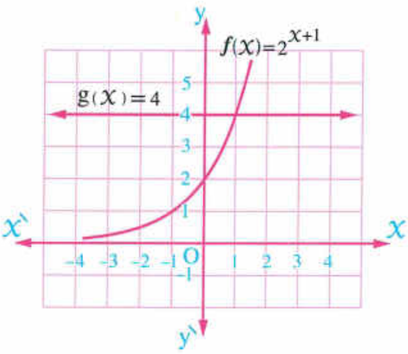
Solving the exponential equations graphically

The graphical solution for the exponential equation depends on supposing that the left hand side of the equation is an exponential function f , and by supposing the right hand side of the equation is another function g , then draw the two functions f , g in the same figure and then determine the x-coordinate of the point (points) of the intersection to get the solution set.

Example 5

Find graphically in $\mathbb{R}$ the S.S. of the equation : $2^{x+1} = 4$

Solution



Let the left hand side of the equation be the rule of the function $f : f(x) = 2^{x+1}$ and the right hand side be the rule of the function $g : g(x) = 4$ and by drawing the two curves in the same figure , from the graph :

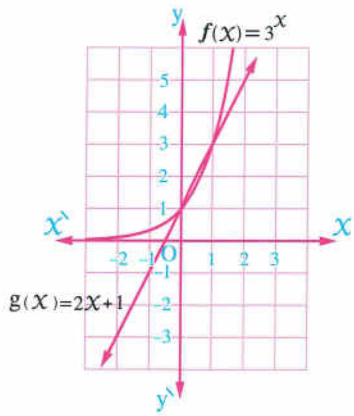
$\therefore$ The point of intersection is (1 , 4)

$\therefore x = 1$ $\therefore$ S.S. = $\{1\}$

Example 6

Find graphically in $\mathbb{R}$ the S.S. of the equation : $3^{x} = 2x + 1$

Solution



Let the left hand side of the equation be the rule of the function $f : f(x) = 3^{x}$ and the right hand side be the rule of the function $g : g(x) = 2x + 1$ and by drawing the two curves in the same figure , from the graph :

$\therefore$ The x-coordinates of the points of intersection are 0 , 1

$\therefore$ S.S. = $\{0 , 1\}$

Example 7

If $f : \mathbb{R} \longrightarrow \mathbb{R}^+$, $f(x) = 3^x$, prove that : $\frac{f(x+5)+f(x+3)}{f(x+3)+f(x+1)} = f(2)$

Solution

$$\text{L.H.S.} = \frac{3^{x+5}+3^{x+3}}{3^{x+3}+3^{x+1}} = \frac{3^{x+3}(3^2+1)}{3^{x+1}(3^2+1)} = \frac{3^{x+3}}{3^{x+1}} = 3^2$$

$\text{R.H.S.} = f(2) = 3^2$ $\therefore$ The two sides are equal.

Another solution :

$$\text{L.H.S.} = \frac{3^{x+5}+3^{x+3}}{3^{x+3}+3^{x+1}} = \frac{3^x(3^5+3^3)}{3^x(3^3+3)} = 9 = 3^2 = f(2)$$

Example 8

If $f(x) = 5^x$, find the value of x if : $f(2x-1) + f(2x+1) = \frac{26}{25}$

Solution

$\because f(2x-1)+f(2x+1) = \frac{26}{25}$ $\therefore 5^{2x-1}+5^{2x+1} = \frac{26}{25}$

$\therefore 5^{2x-1}(1+5^2) = \frac{26}{25}$ $\therefore 5^{2x-1} \times 26 = \frac{26}{25}$ $\therefore 5^{2x-1} = \frac{1}{25}$

$\therefore 5^{2x-1} = 5^{-2}$ $\therefore 2x-1 = -2$ $\therefore x = -\frac{1}{2}$

Another solution :

$\because f(2x-1)+f(2x+1) = \frac{26}{25}$ $\therefore 5^{2x-1}+5^{2x+1} = \frac{26}{25}$

$\therefore 5^{2x}(5^{-1}+5) = \frac{26}{25}$

$\therefore 5^{2x} = 5^{-1}$ $\therefore 2x = -1$ $\therefore x = -\frac{1}{2}$

Life applications on the exponential growth and decay

1 Exponential growth

- The function $f : f(t) = a(1+r)^t$ represents the exponential growth with a constant percentage during equal intervals of time , where a is the initial value , r is the percentage of the growth in a constant interval of time , t is the time interval.
- We can deduce this function by studying a phenomena such as the population :

  If the number of population in a city in one of the years is "a" and this number increases annually by constant percentage rate "r" , then the number of population after one year $= a + ra = a(1+r)$, after 2 years $= a(1+r) + ra(1+r) = a(1+r^2)$ and so on , then the number of population after n years $= a(1+r)^n$

Example 9

Wael bought a house by 1350000 L.E. and its price increases at the rate of 2.5% per year :

(1) Write the exponential function which represents the price of the house after n year.

(2) Estimate to the nearest pound the price of the house after 6 years.

Solution

$a = 1350000$, $r = \frac{2.5}{100} = 0.025$, $t = 6$

(1) The exponential growth $f : f(t) = a(1 + r)^t$

$\therefore f(t) = 1350000 (1 + 0.025)^t$

$\therefore f(t) = 1350000 (1.025)^t$

(2) By substituting at $t = 6$

$\therefore f(6) = 1350000 (1.025)^6 \approx 1565586$ L.E.

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The compound interest

If principal (P) is deposited in one of the banks at interest rate (r) as a percentage and compounded (n) times per year for a period of (t) years , then the accumulated value A is given by : $A = P\left(1 + \frac{r}{n}\right)^{nt}$

Example 10

A man deposited a capital of 15000 L.E. in one of the banks with annual compound interest 7% , find the sum of the capital after 10 years in each of the following :

(1) The interest compounded annually.

(2) The interest compounded quarter annually.

(3) The interest compounded monthly.

Solution

$\because A = P\left(1 + \frac{r}{n}\right)^{nt}$

(1) $\because$ The interest is annually *i.e.* The number of divided intervals = 1

$\therefore n = 1$

$\therefore A = 15000 (1 + 0.07)^{10} \approx 29507.27$ L.E.

(2) $\because$ The interest is quarter annually *i.e.* The number of divided intervals = 4

$\therefore n = 4$

$\therefore A = 15000 \left(1 + \frac{0.07}{4}\right)^{10 \times 4} \approx 30023.96$ L.E.

(3) $\because$ The interest is monthly *i.e.* The number of divided intervals = 12

$\therefore n = 12$

$\therefore A = 15000 \left(1 + \frac{0.07}{12}\right)^{10 \times 12} \approx 30144.92$ L.E.

2 Exponential decay

The function $f : f(t) = a(1-r)^t$ represents the exponential decay where a is the initial value , r is the percentage of the decay in a constant interval of time , t is the time interval.

Example 11

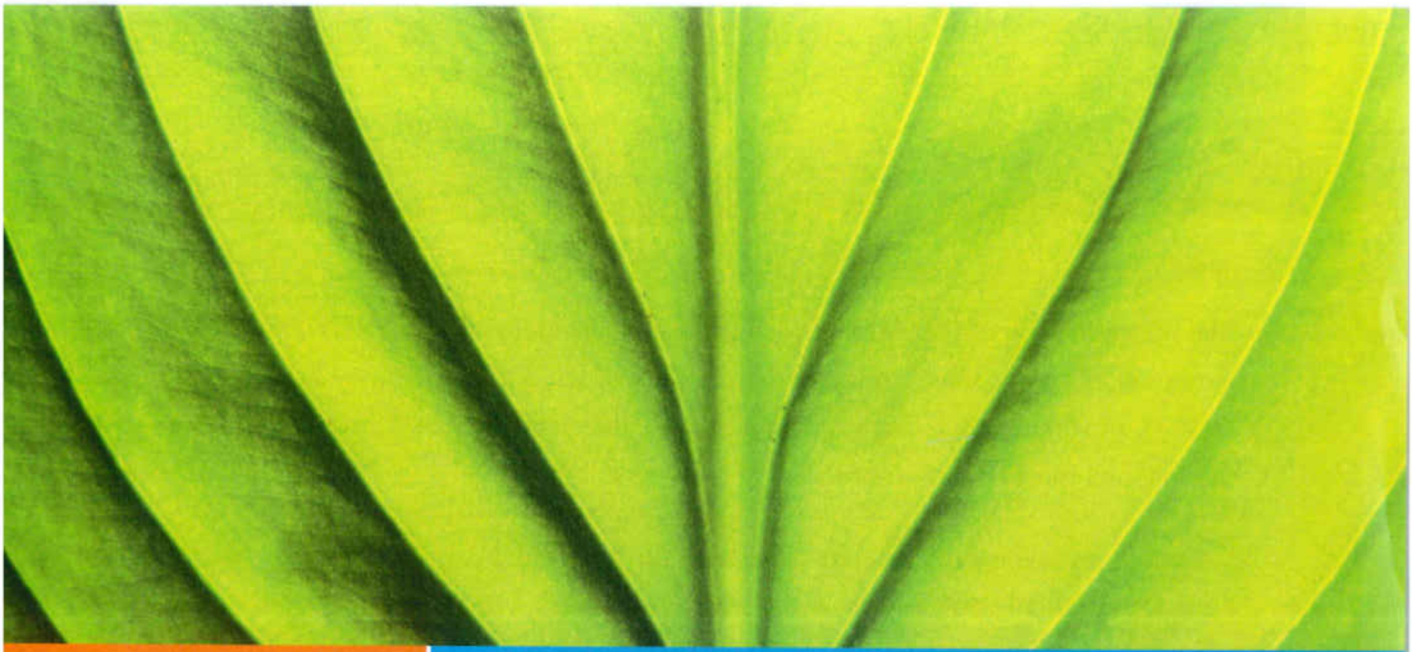
The number of infection persons by Hepatitis C is decreased at the rate 15% annually as a result of discovering the new treatment , if the number of infection persons in one of the countries 8000000 infections , write the exponential function which represents the number of infections persons after n years , then estimate the number of the infections persons after 8 years.

Solution

$a = 8000000$, $r = 0.15$, $t = 8$

The exponential function $f : f(t) = 8000000(1-0.15)^t = 8000000(0.85)^t$

when $t = 8$, then the number of infections persons $= 8000000(0.85)^8 \approx 2179924$ persons.



Lesson Three — The inverse function

The inverse function

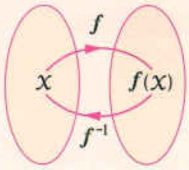
If f is one-to-one function its domain X and its range Y , then for each element y in the range a corresponding unique element x in the domain , therefore it is possible to determine an inverse function from Y to X which is denoted by f^{-1} where $f^{-1}(y) = x$

i.e. If for each $(x , y) \in f$, we find $(y , x) \in f^{-1}$

, Y is the range of f and the domain of f^{-1}

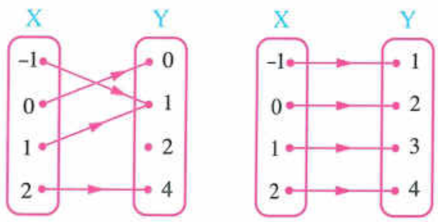
X is the domain of f and the range of f^{-1}

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Example 1

Which of the functions represented by the opposite arrow diagrams has an inverse function , then write the set of ordered pairs of the functions f , f^{-1} if exist :



(1) (2)

Solution

(1) The function f is not one-to-one , so it has no inverse function

set of ordered pairs of $f = \{(-1 , 1) , (0 , 0) , (1 , 1) , (2 , 4)\}$

Notice :

The image of each of the elements -1 , 1 in X is the element 1 in Y, so if we inverse the function , it becomes a relation the set of its ordered pairs $= \{(1, -1), (0, 0), (1, 1), (4, 2)\}$ and it is not a function because the element 1 has 2 images -1 , 1 , so the function must be one-to-one to find its inverse function.

(2) There is an inverse function for f because the function f is one-to-one and for each $(x, y) \in f$ there is $(y, x) \in f^{-1}$

Set of ordered pairs of $f = \{(-1, 1), (0, 2), (1, 3), (2, 4)\}$

Set of ordered pairs of $f^{-1} = \{(1, -1), (2, 0), (3, 1), (4, 2)\}$

Example 2

If the function f is defined from the set $X = \{2, 3, 4, 5\}$ to the set $Y = \{4, 5, 6, 7\}$, $f(x) = x + 2$

(1) Write the set of ordered pairs of f^{-1}

(2) Deduce the rule of the function f^{-1}

Solution

(1) $\because f(2) = 2 + 2 = 4$, $f(3) = 3 + 2 = 5$, $f(4) = 4 + 2 = 6$, $f(5) = 5 + 2 = 7$

$\therefore f = \{(2, 4), (3, 5), (4, 6), (5, 7)\}$

$\therefore f$ is one-to-one function and range of $f = Y$

$\therefore f^{-1} = \{(4, 2), (5, 3), (6, 4), (7, 5)\}$

(2) Take note of the ordered pairs of f^{-1} , we find the y-coordinate decreases by 2 than the x-coordinate

i.e. $y = x - 2$

$\therefore f^{-1}(x) = x - 2$

An important remark

We can find the rule of f^{-1} directly by replacing the two variables x , y , then finding y in terms of x in the previous example :

$\because f(x) = x + 2$ *i.e.* $y = x + 2$ by replacing the two variables x , y

$\therefore x = y + 2$, then $y = x - 2$

$\therefore f^{-1}(x) = x - 2$

Example 3

Find the inverse function of the function $f : f(x) = 3x - 2$, and represent each of them in the same figure.

Solution

* To find the inverse function , we replace the two variables then we find the value of y in terms of x

$\because y = 3x - 2$, by replacing the two variables.

$\therefore x = 3y - 2$, to find the value of y $\quad \therefore 3y = x + 2$

$\therefore y = \frac{1}{3}(x + 2)$ $\quad \therefore f^{-1}(x) = \frac{1}{3}(x + 2)$

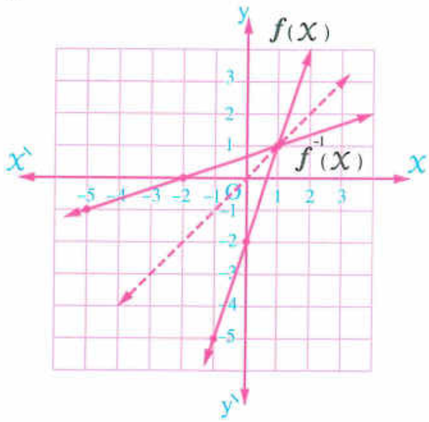
The graph :

* $f(x) = 3x - 2$

x	1	0	−1
$f(x)$	1	−2	−5

* $f^{-1}(x) = \frac{1}{3}(x + 2)$

x	1	−2	−5
$f^{-1}(x)$	1	0	−1



Remarks

1. Notice that : $f^{-1}(x) \neq \frac{1}{f(x)}$, in the previous example :

$f^{-1}(x) = \frac{1}{3}(x + 2)$ but $\frac{1}{f(x)} = \frac{1}{3x - 2}$

2. In the previous example , we note that : each of the function f and its inverse f^{-1} are symmetric about the straight line $y = x$, in general : for any function f , if it is possible to find its inverse function f^{-1} , then the two functions f , f^{-1} are symmetric about the straight line $y = x$

i.e. f , f^{-1} each of them is the image of the other by reflection in the straight line $y = x$

Properties of the inverse function

From the properties of the inverse function :

(1) The two functions f , g are said to be each of them is the inverse function of the other if $(f \circ g)(x) = x$, $(g \circ f)(x) = x$

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(2) The domain of f = the range of the inverse function f^{-1} , the range of f = the domain of the inverse function f^{-1}

Example 4

Verify that each of f , g where $f(x) = 4x + 9$, $g(x) = \frac{x-9}{4}$ is the inverse function of the other.

Solution

$\because (f \circ g)(x) = f\left(g(x)\right) = f\left(\frac{x-9}{4}\right) = 4\left(\frac{x-9}{4}\right) + 9 = (x-9) + 9 = x$

, $(g \circ f)(x) = g\left(f(x)\right) = g(4x+9) = \frac{4x+9-9}{4} = \frac{4x}{4} = x$

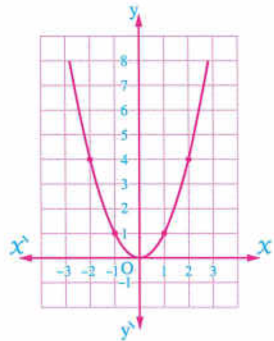
$\therefore$ Each of f , g is the inverse function of the other.

Example 5

Find the domain in which the function $f : f(x) = x^2$ has an inverse function , then find this inverse function.

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Solution



* If $x \in \mathbb{R}$, then the function $f : f(x) = x^2$ is not one-to-one (doesn't satisfy the condition of the horizontal line test) , so it has no inverse function in the domain $\mathbb{R}$

* If $x \in [0 , \infty[$, then the function $f : f(x) = x^2$ is one-to-one and in this case it has an inverse function.

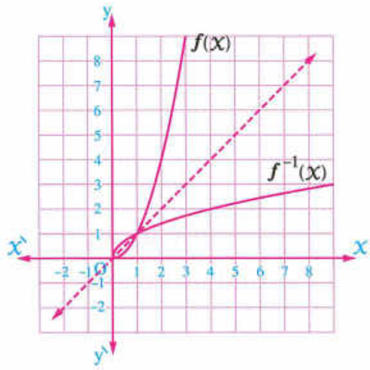
  $\because y = x^2$

  Where $x \geq 0$, $y \geq 0$, and by replacing the two variables

  $\therefore x = y^2$ $\qquad\qquad$ $\therefore y = \sqrt{x}$

  Where : $y \geq 0$, $x \geq 0$

  $\therefore f^{-1}(x) = \sqrt{x}$



* If $x \in]-\infty , 0]$, then $f : f(x) = x^2$ is one-to-one and in this case it has an inverse function.

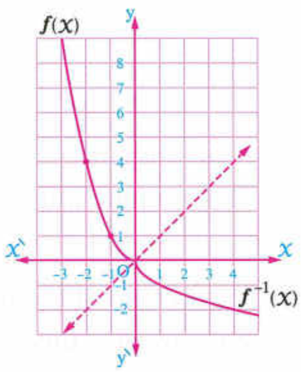
  $\because y = x^2$, where $x \leq 0$, $y \geq 0$ and by replacing the two variables

  $\therefore x = y^2$ where $x \geq 0$, $y \leq 0$

  $\therefore y = -\sqrt{x}$ where $x \geq 0$, $y \leq 0$

  $\therefore f^{-1}(x) = -\sqrt{x}$

  $\therefore$ The domain in which the function f has an inverse function $= [0 , \infty[$ or $]-\infty , 0]$



Example 6

If $f(x) = \frac{1}{x-2} + 3$, find :

(1) The domain and the range of the function f

(2) $f^{-1}(x)$ and determine the domain and the range of the function f^{-1}

Solution

(1) The domain of $f = \mathbb{R} - \{2\}$, the range of $f = \mathbb{R} - \{3\}$

(2) $\because y = \frac{1}{x-2} + 3$ by replacing the two variables

$\therefore x = \frac{1}{y-2} + 3$ $\qquad \therefore \frac{1}{y-2} = x - 3$

$\therefore y - 2 = \frac{1}{x-3}$ $\qquad \therefore y = \frac{1}{x-3} + 2$

$\therefore f^{-1}(x) = \frac{1}{x-3} + 2$

$\therefore$ The domain of $f^{-1} = \mathbb{R} - \{3\}$

, the range of $f^{-1} = \mathbb{R} - \{2\}$ Notice that : The domain of f^{-1} = the range of f

Example 7

If f is a function where $f(x) = 2 + \sqrt{x-3}$, find :

(1) The domain and the range of f

(2) $f^{-1}(x)$ and determine the domain and the range of f^{-1}

Solution

(1) $\because f(x)$ is defined for all the values $x - 3 \geq 0$ $\qquad \therefore x \geq 3$

$\therefore$ The domain of $f = [3, \infty[$

, $\because$ for each $x \geq 3$, then $2 + \sqrt{x-3} \geq 2$ $\quad \therefore y \geq 2$

$\therefore$ The range of the function $f = [2, \infty[$

(2) $\because y = 2 + \sqrt{x-3}$ where $x \geq 3$, $y \geq 2$ by replacing the two variables

$\therefore x = 2 + \sqrt{y-3}$ where $y \geq 3$, $x \geq 2$

$\therefore x - 2 = \sqrt{y-3}$ $\qquad \therefore (x-2)^2 = y - 3$

$\therefore y = (x-2)^2 + 3$

$\therefore f^{-1}(x) = (x-2)^2 + 3$ where $x \geq 2$, $y \geq 3$

, the domain of f^{-1} = the range of $f = [2, \infty[$, the range of f^{-1} = the domain of $f = [3, \infty[$

Example 8

Find the inverse function of the function f where $f(x) = (x-2)^2 + 3$, $x \leq 2$, determine the domain of f^{-1}

Solution

$\because y = (x-2)^2 + 3$, $x \leq 2$

, $\because$ for each $x \leq 2$, then $(x-2)^2 + 3 \geq 3$ $\therefore y \geq 3$

and by replacing the two variables

$\therefore x = (y-2)^2 + 3$, $y \leq 2$, $x \geq 3$

$\therefore (y-2)^2 = x - 3$, by taking the square root to both sides

$\therefore y - 2 = \pm\sqrt{x-3}$

, $\because y \leq 2$ $\therefore y - 2 = -\sqrt{x-3}$

$\therefore y = -\sqrt{x-3} + 2$

$\therefore f^{-1}(x) = -\sqrt{x-3} + 2$ where $x \geq 3$, $y \leq 2$

$\therefore$ The domain of $f^{-1} = [3 , \infty[$

Example 9

If $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ where $f(x) = \frac{1}{x^2+1}$, find $f^{-1}(x)$

, determine the domain and the range of f^{-1}

Solution

$\because y = \frac{1}{x^2+1}$, $x \in \mathbb{R}^+$, for each $x > 0$, then $0 < y < 1$ and by replacing the two variables

$\therefore x = \frac{1}{y^2+1}$, $y > 0$, $0 < x < 1$ $\therefore y^2 + 1 = \frac{1}{x}$

$\therefore y^2 = \frac{1}{x} - 1$

, $\because y > 0$ $\therefore y = \sqrt{\frac{1}{x} - 1}$

$\therefore f^{-1}(x) = \sqrt{\frac{1}{x} - 1}$ where $y > 0$, $0 < x < 1$

$\therefore$ The domain of $f^{-1} =]0 , 1[$

, the range of $f^{-1} = \mathbb{R}^+$

Remark

The function which is symmetric about the line $y = x$ its inverse function is itself.

For example :

(1) The linear function $f : f(x) = -x + b$ as $f : f(x) = -x + 1$

$, f : f(x) = \frac{1}{2} - x$, ...

(2) The rational function $f : f(x) = \frac{a}{x - k} + k$ where $k \in \mathbb{R}$

as $f : f(x) = \frac{1}{x - 5} + 5$

$, f : f(x) = \frac{-2}{x}$, ...

Example 10

Prove that the inverse function of each of the functions defined by the following rules is itself :

(1) $f(x) = 2 - x$

(2) $f(x) = \frac{1}{x - 3} + 3$

Solution

(1) $y = 2 - x$, by replacing the two variables.

$\therefore x = 2 - y$ $\qquad$ $\therefore y = 2 - x$

$\therefore f^{-1}(x) = 2 - x = f(x)$

$\therefore$ The inverse function of f is itself.

(2) $y = \frac{1}{x - 3} + 3$, by replacing the two variables.

$\therefore x = \frac{1}{y - 3} + 3$ $\qquad$ $\therefore x - 3 = \frac{1}{y - 3}$

$\therefore y - 3 = \frac{1}{x - 3}$ $\qquad$ $\therefore y = \frac{1}{x - 3} + 3$

$\therefore f^{-1}(x) = \frac{1}{x - 3} + 3 = f(x)$

$\therefore$ The inverse function of f is itself.



Lesson Four | Logarithmic function and its graph

You know that the number 8 can be written as : $8 = 2^3$, the number (3) which is written as a power to the number (2) to get (8) is called the logarithm (8) to the base (2) and denoted by : $\log_2 8$

Thus we find that every exponential form , whose base is a positive real number $\neq 1$ has an equivalent form called the logarithmic form

$$y = \log_a x \Leftrightarrow x = a^y \text{ where } a \in \mathbb{R}^+ - \{1\} , x \in \mathbb{R}^+ \text{ and } y \in \mathbb{R}$$

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For example :

$\log_3 81 = 4 \Leftrightarrow 3^4 = 81$, $\log_3 \frac{1}{9} = -2 \Leftrightarrow 3^{-2} = \frac{1}{9}$,

$4^2 = 16 \Leftrightarrow \log_4 16 = 2$, $2^{-3} = \frac{1}{8} \Leftrightarrow \log_2 \frac{1}{8} = -3$, and so on ....

Remarks

1. The logarithm of a non-positive number is meaningless

i.e. Each of $\log_2 -3$, $\log_5 -8$ and $\log_6 0$ is meaningless.

2. The base "a" must be a positive number differs "1"

i.e. Each of $\log_{-2} 8$, $\log_0 5$, $\log_1 4$ is meaningless.

3. If the base of the logarithm = 10 , then the logarithm is called the common logarithm and the base is called a common base. It is agreed to omit this base in writing.

For example :

$\log_{10} 3$ is written as $\log 3$

The logarithmic function

If $a \in \mathbb{R}^+ - \{1\}$, then the function $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ where $f(X) = \log_a X$ is called the logarithmic function

The relation between the exponential function and the logarithmic function

We had studied before the graph of the exponential function $f : f(X) = 3^X$ *i.e.* $y = 3^X$, then we form the following table :

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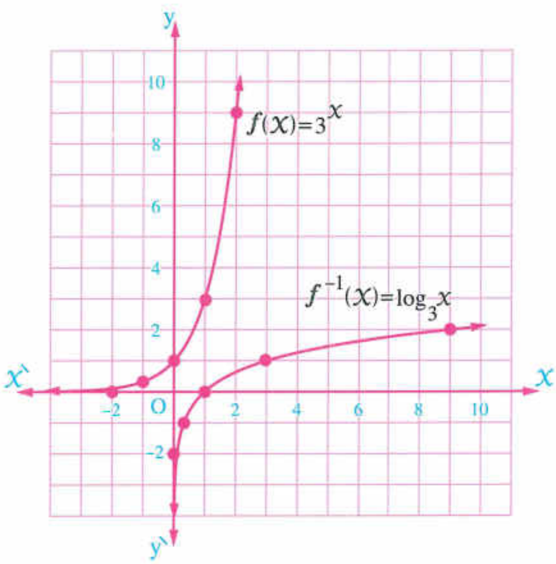
X	-2	-1	0	1	2
$f(X) = 3^X$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

By replacing the two variables we get the inverse function $X = 3^y$ which is equivalent to the logarithmic form $y = \log_3 X$

i.e. $f^{-1}(X) = \log_3 X$

, to draw this function , we replace the values of X by the values of y in the previous table :

X	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
$f^{-1}(X) = \log_3 X$	-2	-1	0	1	2



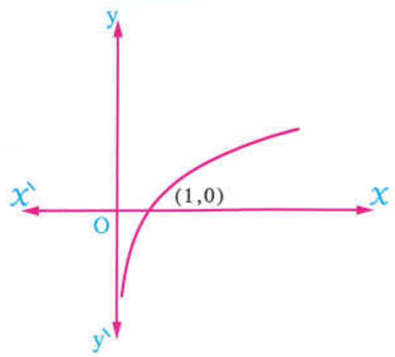
* From the properties of the inverse function and the opposite figure we notice that : The two curves of the two functions are symmetric about the straight line $y = X$, the domain of the exponential function is $\mathbb{R}$ and the range $=]0 , \infty[$, the domain of the logarithmic function is $]0 , \infty[$ and the range is $\mathbb{R}$

i.e. The logarithmic function is the inverse function for the exponential function.

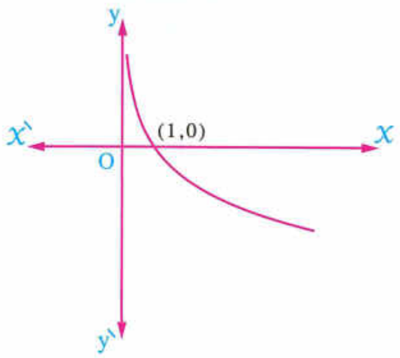
The graphical representation of the logarithmic function $f : f(x) = \log_a x$

- The graph of the logarithmic function will be in the shape of one of the following figures according to the value of the base a :

If a > 1



If 0 < a < 1



Some properties of the logarithmic function $f : f(x) = \log_a x$

(1) The domain of the logarithmic function = $\mathbb{R}^+$

(2) The range of the logarithmic function = $\mathbb{R}$

(3) The logarithmic function is increasing when a > 1 and it is decreasing when 0 < a < 1

(4) All curves of the logarithmic functions for any positive base ≠ 1 pass through the point (1 , 0)

(5) The logarithmic function is one-to-one

i.e. If $\log_a x = \log_a y$, then $x = y$

Example 1

Express each of the following by the equivalent exponential form :

(1) $\log_2 64 = 6$

(2) $\log_2 8\sqrt{2} = \frac{7}{2}$

(3) $\log_3 \frac{1}{27} = -3$

(4) $\log 0.01 = -2$

Solution

(1) $\log_2 64 = 6 \Leftrightarrow 64 = 2^6$

(2) $\log_2 8\sqrt{2} = \frac{7}{2} \Leftrightarrow 8\sqrt{2} = 2^{\frac{7}{2}}$

(3) $\log_3 \frac{1}{27} = -3 \Leftrightarrow \frac{1}{27} = 3^{-3}$

(4) $\log 0.01 = -2 \Leftrightarrow 0.01 = 10^{-2}$

Example 2

Write the logarithmic form that is equivalent to each of the following exponential forms :

(1) $243 = \left(\sqrt{3}\right)^{10}$

(2) $10^{-2} = 0.01$

(3) $3^{\frac{5}{2}} = 9\sqrt{3}$

(4) $c = a^{x}$

Solution

(1) $243 = \left(\sqrt{3}\right)^{10} \Leftrightarrow \log_{\sqrt{3}} 243 = 10$

(2) $10^{-2} = 0.01 \Leftrightarrow \log_{10} 0.01 = -2$

(3) $3^{\frac{5}{2}} = 9\sqrt{3} \Leftrightarrow \log_3 9\sqrt{3} = \frac{5}{2}$

(4) $c = a^{x} \Leftrightarrow \log_a c = x$

Example 3

Find the value of each of :

(1) $\log_2 64$

(2) $\log_6 1$

(3) $\log_4 4\sqrt{2}$

(4) $\log_{\sqrt{3}} \frac{1}{27}$

(5) $\log 0.0001$

Solution

(1) Putting : $\log_2 64 = x$ $\quad \therefore 2^{x} = 64$

$\therefore 2^{x} = 2^{6}$ $\quad \therefore x = 6$ $\quad \therefore \log_2 64 = 6$

(2) Putting : $\log_6 1 = x$ $\quad \therefore 6^{x} = 1$

$\therefore x = \text{zero}$ $\quad \therefore \log_6 1 = \text{zero}$

(3) Putting : $\log_4 4\sqrt{2} = x$ $\quad \therefore 4^{x} = 4\sqrt{2}$ $\quad \therefore 2^{2x} = 2^{2} \times 2^{\frac{1}{2}}$

$\therefore 2^{2x} = 2^{\frac{5}{2}}$ $\quad \therefore 2x = \frac{5}{2}$

$\therefore x = \frac{5}{4}$ $\quad \therefore \log_4 4\sqrt{2} = \frac{5}{4}$

(4) Putting : $\log_{\sqrt{3}} \frac{1}{27} = x$ $\quad \therefore \left(\sqrt{3}\right)^{x} = \frac{1}{27}$ $\quad \therefore 3^{\frac{1}{2}x} = 3^{-3}$

$\therefore \frac{1}{2}x = -3$ $\quad \therefore x = -6$ $\quad \therefore \log_{\sqrt{3}} \frac{1}{27} = -6$

(5) Putting : $\log 0.0001 = x$ $\quad \therefore 10^{x} = 0.0001 = 10^{-4}$

$\therefore x = -4$ $\quad \therefore \log 0.0001 = -4$

Example 4

Find the value of x if :

(1) $\log_2 x = -4$

(2) $\log_9 81\sqrt{3} = x$

(3) $\log_{\frac{1}{2}} x = -3$

(4) $\log_x 8 = 6$

Solution

(1) $\because \log_2 x = -4$ $\quad \therefore x = 2^{-4}$

$\therefore x = \frac{1}{2^4} = \frac{1}{16}$

(2) $\because \log_9 81\sqrt{3} = x$ $\quad \therefore 9^x = 81\sqrt{3}$

$\therefore 3^{2x} = 3^4 \times 3^{\frac{1}{2}}$ $\quad \therefore 3^{2x} = 3^{\frac{9}{2}}$

$\therefore 2x = \frac{9}{2}$ $\quad \therefore x = \frac{9}{4}$

(3) $\because \log_{\frac{1}{2}} x = -3$ $\quad \therefore x = \left(\frac{1}{2}\right)^{-3} = 2^3$

$\therefore x = 8$

(4) $\because \log_x 8 = 6$ $\quad \therefore x^6 = 8$

$\therefore x^6 = 2^3$ $\quad \therefore x^6 = \left(\sqrt{2}\right)^6$

$\therefore x = \sqrt{2}$

Example 5

Find in $\mathbb{R}$ the solution set of each of the following equations :

(1) $\log_x 7x = 2$

(2) $\log_2 \left(x^2 + \frac{3}{4}x\right) = -2$

(3) $\log_2 \log_3 (x^2 - 7x + 21) = 1$

(4) $(\log_2 x)^2 - 3\log_2 x = 4$

Solution

(1) $\because \log_x 7x = 2$ $\quad \therefore x^2 = 7x$

$\therefore x^2 - 7x = 0$ $\quad \therefore x(x - 7) = 0$

$\therefore x = 0$ (refused) or $x = 7$ (verify) $\quad \therefore$ S.S. $= \{7\}$

Notice that :

When you solve the equations you must verify the values that you obtained in the original equation and the solution is the value (s) which verify this equation , as we know the logarithm of non-positive number is meaningless or finding the set of the available values of the variable x for substituting by them before starting of solving the equations and this is for avoidance the substituting operation by the values of x that we obtained

(2) $\because \log_2 (x^2 + \frac{3}{4} x) = -2$ $\therefore x^2 + \frac{3}{4} x = 2^{-2}$

$\therefore x^2 + \frac{3}{4} x = \frac{1}{4}$ $\therefore 4x^2 + 3x - 1 = 0$

$\therefore (x+1)(4x-1) = 0$ $\therefore x = -1$ (verify) or $x = \frac{1}{4}$ (verify)

$\therefore$ S.S. $= \left\{-1 , \frac{1}{4}\right\}$

(3) $\because \log_2 \log_3 (x^2 - 7x + 21) = 1$ $\therefore \log_3 (x^2 - 7x + 21) = 2^1 = 2$

$\therefore x^2 - 7x + 21 = 3^2 = 9$ $\therefore x^2 - 7x + 12 = 0$

$\therefore (x-3)(x-4) = 0$ $\therefore x = 3$ (verify) or $x = 4$ (verify)

$\therefore$ S.S. $= \{3 , 4\}$

(4) $\because (\log_2 x)^2 - 3\log_2 x - 4 = 0$ $\therefore (\log_2 x - 4)(\log_2 x + 1) = 0$

$\therefore \log_2 x = 4$ $\therefore x = 2^4 = 16$ (verify)

or $\log_2 x = -1$ $\therefore x = 2^{-1} = \frac{1}{2}$ (verify)

$\therefore$ S.S. $= \left\{16 , \frac{1}{2}\right\}$

Example 6

If the curve of the function $f : f(x) = \log_a x$ passes through the point (27 , 3) , find the value of a , then draw the graph of the function taking $x \in \left[\frac{1}{9} , 9\right]$, then from the graph :

(1) Deduce the domain , the range , monotonicity and the point of intersection with x-axis.

(2) Find an approximated value to the number $\log_3 6$

Solution

$\because f(x) = \log_a x$ for each $x > 0$, $a \in \mathbb{R}^+ - \{1\}$

, $\because$ the point (27 , 3) $\in$ the curve of the function

$\therefore 3 = \log_a 27$ $\therefore a^3 = 27 = 3^3$

$\therefore a = 3$ $\therefore f(x) = \log_3 x$

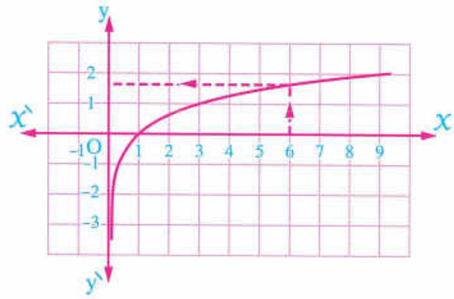
Form the following table : [Note the base = 3 > 1]

x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
$y = \log_3 x$	-2	-1	0	1	2

Notice that :

Choosing the values of x from the powers of the base 3 $\{3^{-2} , 3^{-1} , 3^0 , 3^1 , 3^2\}$

From the graph we find that :



* The domain = $\mathbb{R}^+$, the range = $\mathbb{R}$
* The function is increasing on its domain
* The curve intersects the X-axis at the point (1 , 0)
* $\log_3 6 \approx 1.6$

Example 7

If the curve of the function $f : f(X) = \log_a X$ passes through the point $\left(\frac{1}{16}, 4\right)$, find the value of a , then draw the graph of the function taking $X \in \left[\frac{1}{4}, 4\right]$, then from the graph deduce the range , monotonicity then find an approximated value to the number $\log_{\frac{1}{2}} 3.5$

Solution

$\because f(X) = \log_a X$ for each $X > 0$, $a \in \mathbb{R}^+ - \{1\}$

, $\because$ the point $\left(\frac{1}{16}, 4\right) \in$ the curve of the function

$\therefore 4 = \log_a \frac{1}{16}$ $\qquad \therefore a^4 = \frac{1}{16} = \left(\frac{1}{2}\right)^4$

$\therefore a = \frac{1}{2}$ (negative solution is refused)

$\therefore f(X) = \log_{\frac{1}{2}} X$

Form the following table :

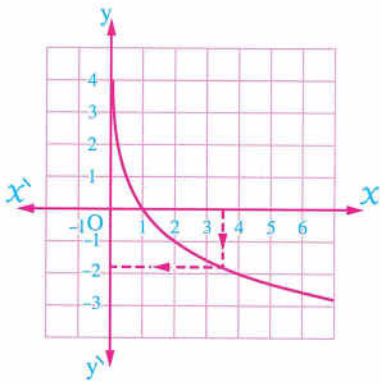
$\left[\text{Note the base} = \frac{1}{2} < 1\right]$

X	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$y = \log_{\frac{1}{2}} X$	2	1	0	− 1	− 2

Notice that :

Choosing the values of X from the powers of the base $\frac{1}{2}$

$\left\{\left(\frac{1}{2}\right)^{-2}, \left(\frac{1}{2}\right)^{-1}, \left(\frac{1}{2}\right)^{0}, \left(\frac{1}{2}\right)^{1}, \left(\frac{1}{2}\right)^{2}\right\}$



From the graph we find that :

* The range = $\mathbb{R}$
* The function is decreasing on its domain
* $\log_{\frac{1}{2}} 3.5 \simeq -1.8$

Example 8

Find the domain of each of the functions that are defined by the following rules :

(1) $f(X) = \log_4 (4 - X)$

(2) $f(X) = \log_{1-X} 5$

(3) $f(X) = \log_{X-3} X$

(4) $f(X) = \log_{3-X} X$

Solution

(1) The function is defined for all values of X which verify : $4 - X > 0$

i.e. $X < 4$

$\therefore$ The domain of $f =]-\infty , 4[$

(2) The function is defined for all values of X which verify : $\begin{cases} 1 - X > 0 \\ 1 - X \neq 1 \end{cases}$ *i.e.* $\begin{cases} X < 1 \\ X \neq 0 \end{cases}$

$\therefore$ The domain of $f =]-\infty , 1[- \{0\}$

(3) The function is defined for all values of X which verify : $\begin{cases} X > 0 \\ X - 3 > 0 \\ X - 3 \neq 1 \end{cases}$ *i.e.* $\begin{cases} X > 0 \\ X > 3 \\ X \neq 4 \end{cases}$

$\therefore$ The domain of $f =]3 , \infty[- \{4\}$

(4) The function is defined for all values of X which verify : $\begin{cases} X > 0 \\ 3 - X > 0 \\ 3 - X \neq 1 \end{cases}$ *i.e.* $\begin{cases} X > 0 \\ X < 3 \\ X \neq 2 \end{cases}$

$\therefore$ The domain of $f =]0 , 3[- \{2\}$

Remember that

The function f : $f(X) = \log_a X$ is defined for all the values of X , a which verify : $\begin{cases} X > 0 \\ a > 0 \\ a \neq 1 \end{cases}$

Example 9

Use the curve of the function $f : f(X) = \log_3 X$ to graph each of the functions that are defined by the following rules and from the graph find the domain , the range and the monotonicity :

(1) $g(X) = \log_3 X + 2$

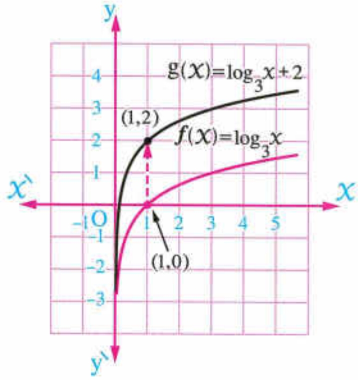
(2) $k(X) = \log_3 (X - 1)$

(3) $n(X) = -\log_3 X$

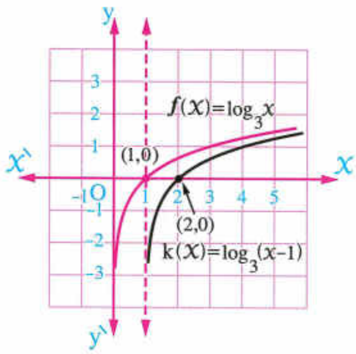
(4) $t(X) = \log_3 (-X)$

Solution

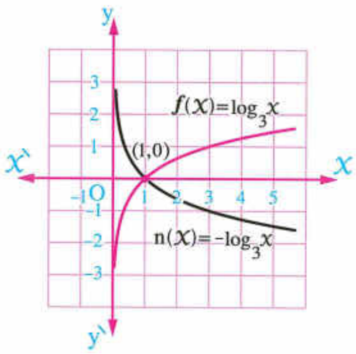
(1) The curve of the function g is the same curve of the function f by vertical displacement 2 units in the direction of $\overrightarrow{Oy}$, the domain = $]0, \infty[$, the range = $\mathbb{R}$, the function is increasing on its domain.



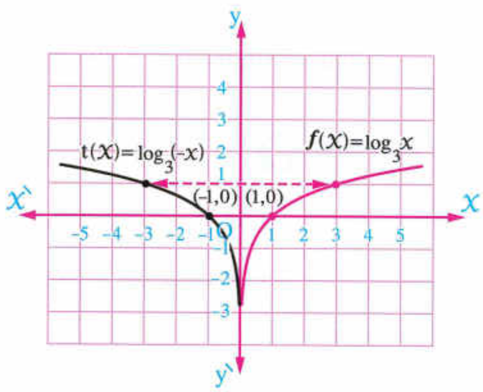
(2) The curve of the function k is the same curve of the function f by horizontal displacement 1 unit in the direction of $\overrightarrow{Ox}$, the domain = $]1, \infty[$, the range = $\mathbb{R}$, the function is increasing on its domain.



(3) The curve of the function n is the same curve of the function f by reflection in x-axis , the domain = $]0, \infty[$, the range = $\mathbb{R}$, the function is decreasing on its domain.



(4) The curve of the function t is the same curve of the function f by reflection in y-axis , the domain = $]-\infty, 0[$, the range = $\mathbb{R}$, the function is decreasing on its domain.



Using the calculator

* The key of the logarithm for any base is [log▪□] , the key of the common logarithm is [log]

For example :

(1) To find $\log_3 24$ we use the keys of the calculator successively as shown below

Start → [log▪□] [3] [REPLAY ►] [2] [4] [=] 2.892789261

i.e. $\log_3 24 \approx 2.8928$ to the nearest 4 decimals digits

(2) To find log 8.4 we use the keys of the calculator successively as shown below

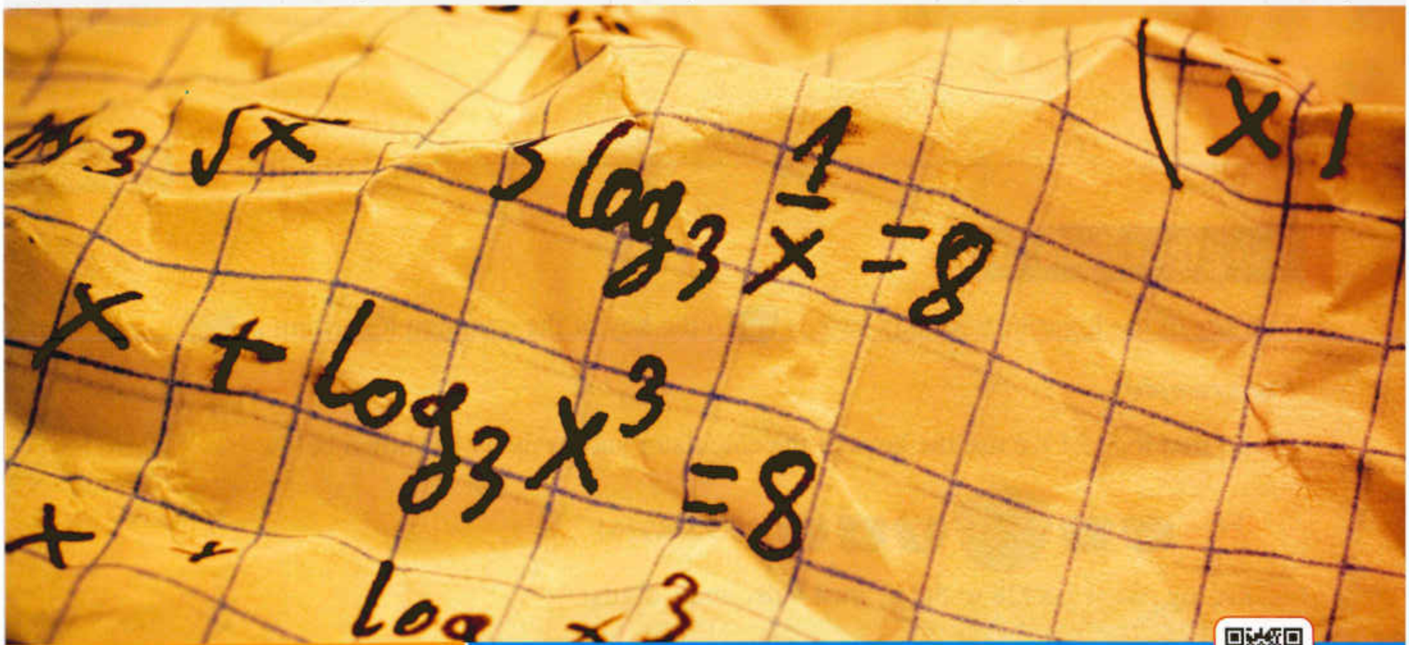
Start → [log] [8] [.] [4] [=] 0.9242792861

i.e. $\log 8.4 \approx 0.9243$ to the nearest 4 decimals digits.

(3) To evaluate the value of X which satisfies $\log X = 0.4572$, use the keys of the calculator as follow.

Start → [SHIFT] [$10^{\square}$ log] [0] [.] [4] [5] [7] [2] [=] 2.865497276

$\therefore X \approx 2.8655$ to the nearest 4 decimals digits.



Lesson Five Some properties of logarithms

WATCH VIDEO

1st Property

WATCH VIDEO

- If $a \in \mathbb{R}^+ - \{1\}$, then $\log_a a = 1$

For example :

$\log_7 7 = 1$, $\log_5 5 = 1$, $\log_{\sqrt{3}} \sqrt{3} = 1$

Proof :

$\because a^1 = a$ Converting into the logarithmic form

$\therefore \log_a a = 1$

2nd Property

- If $a \in \mathbb{R}^+ - \{1\}$, then $\log_a 1 = 0$

For example :

$\log_3 1 = 0$, $\log_5 1 = 0$, $\log_{\sqrt{7}} 1 = 0$

Proof :

$\because a^0 = 1$ Converting into the logarithmic form

$\therefore \log_a 1 = 0$

3rd Property Multiplication property

- If $x , y \in \mathbb{R}^+$, $a \in \mathbb{R}^+ - \{1\}$

, then $\log_a xy = \log_a x + \log_a y$

For example :

$\log_3 (2 \times 5) = \log_3 2 + \log_3 5$

and vice versa : $\log_5 2 + \log_5 11 = \log_5 (2 \times 11) = \log_5 22$

Proof : Put $\log_a x = b$, $\log_a y = c$

$\therefore x = a^b$, $y = a^c$ $\qquad \therefore x y = a^b \times a^c$ $\qquad \therefore x y = a^{b+c}$

Converting into the logarithmic form

$\therefore \log_a x y = b + c$ $\qquad \therefore \log_a x y = \log_a x + \log_a y$

Corollary

If $x_1, x_2, x_3, \cdots, x_n \in \mathbb{R}^+$, $a \in \mathbb{R}^+ - \{1\}$, then

$\log_a (x_1 \times x_2 \times x_3 \times \cdots \times x_n) = \log_a x_1 + \log_a x_2 + \log_a x_3 + \cdots + \log_a x_n$

For example :

$\log_2 (3 \times 5 \times 7) = \log_2 3 + \log_2 5 + \log_2 7$

and vice versa :

$\log_a 75 + \log_a \frac{4}{9} + \log_a 0.06 = \log_a \left(75 \times \frac{4}{9} \times \frac{6}{100}\right) = \log_a 2$

An important remark

Remember very well that :

$\log_a (x + y) \neq \log_a x + \log_a y$, and $\log_a (x \times y) \neq \log_a x \times \log_a y$

4th Property Division property

- If $x , y \in \mathbb{R}^+$, $a \in \mathbb{R}^+ - \{1\}$, then $\log_a \frac{x}{y} = \log_a x - \log_a y$

For example :

$\log_5 \frac{2}{3} = \log_5 2 - \log_5 3$

and vice versa : $\log_5 11 - \log_5 2 = \log_5 \frac{11}{2}$

Proof : Put $\log_a x = b$, $\log_a y = c$

$\therefore x = a^b$, $y = a^c$ $\qquad \therefore \frac{x}{y} = \frac{a^b}{a^c} = a^{b-c}$

Converting into the logarithmic form :

$\therefore \log_a \frac{x}{y} = b - c$ $\qquad \therefore \log_a \frac{x}{y} = \log_a x - \log_a y$

Corollary

$\log_a \frac{x y}{z \ell} = \log_a x + \log_a y - \log_a z - \log_a \ell$

UNIT 2

An important remark

Remember very well that : $\log_a (x - y) \neq \log_a x - \log_a y$, and $\log_a \left(\frac{x}{y}\right) \neq \log_a x \div \log_a y$

5th Property The power property

- If $x \in \mathbb{R}^+$, $a \in \mathbb{R}^+ - \{1\}$, $n \in \mathbb{R}$, then $\log_a x^n = n \log_a x$

For example :

$\log_2 125 = \log_2 5^3 = 3 \log_2 5$

and vice versa : $7 \log_5 2 = \log_5 2^7 = \log_5 128$

Proof :

$\log_a x^n = \log_a (x \times x \times x \times .... \text{ to n terms})$

$= \log_a x + \log_a x + ... \text{ to n terms}$

$= n \log_a x$

6th Property Base changing property

- If $x \in \mathbb{R}^+$, $y \in \mathbb{R}^+ - \{1\}$, $a \in \mathbb{R}^+ - \{1\}$, then $\log_y x = \frac{\log_a x}{\log_a y}$

For example :

$\log_5 7 = \frac{\log 7}{\log 5}$, $\log_3 2 = \frac{\log_{11} 2}{\log_{11} 3}$

Proof :

Put $\log_y x = z$

$\therefore y^z = x$ by taking logarithms to both sides for the base "a"

$\therefore z \log_a y = \log_a x$ $\therefore z = \frac{\log_a x}{\log_a y}$

$\therefore \log_y x = \frac{\log_a x}{\log_a y}$

7th Property The multiplicative inverse property

- If x , $y \in \mathbb{R}^+ - \{1\}$, then $\log_y x = \frac{1}{\log_x y}$

For example :

$\log_7 5 = \frac{1}{\log_5 7}$, then $\log_7 5 \times \log_5 7 = 1$

Proof :

$\because \log_y x = \frac{\log x}{\log y}$, $\log_x y = \frac{\log y}{\log x}$ $\therefore \log_y x \times \log_x y = 1$

$\therefore \log_y x = \frac{1}{\log_x y}$

Example 1

Without using the calculator , find the value of each of the following :

(1) $\log_3 15 + \log_3 6 - \log_3 10$

(2) $\log_5 100 - 3 \log_5 2 - \log_5 18 + \log_5 36$

(3) $\log_5 \frac{3}{5} + 2 \log_5 \frac{15}{2} - \log_5 \frac{5}{36} + \log_5 \frac{5}{243}$

(4) $\log_2 7 \times \log_7 11 \times \log_{11} 9 \times \log_3 2$

(5) $\frac{\log_2 243 - \log_3 32}{\log_2 27 - \log_3 8}$

Solution

(1) The value $= \log_3 \frac{15 \times 6}{10} = \log_3 9 = \log_3 3^2 = 2 \log_3 3 = 2 \times 1 = 2$

(2) The value $= \log_5 100 - \log_5 2^3 - \log_5 18 + \log_5 36$

$$= \log_5 \frac{100 \times 36}{8 \times 18} = \log_5 25 = \log_5 5^2 = 2 \log_5 5 = 2 \times 1 = 2$$

(3) The value $= \log_5 \frac{3}{5} + \log_5 \left(\frac{15}{2}\right)^2 - \log_5 \frac{5}{36} + \log_5 \frac{5}{243}$

$$= \log_5 \frac{\frac{3}{5} \times \frac{15}{2} \times \frac{15}{2} \times \frac{5}{243}}{\frac{5}{36}} = \log_5 \frac{3 \times 15 \times 15 \times 5 \times 36}{5 \times 2 \times 2 \times 243 \times 5} = \log_5 5 = 1$$

(4) The value $= \frac{\log 7}{\log 2} \times \frac{\log 11}{\log 7} \times \frac{\log 9}{\log 11} \times \frac{\log 2}{\log 3}$

$$= \frac{\log 9}{\log 3} = \frac{\log 3^2}{\log 3} = \frac{2 \log 3}{\log 3} = 2$$

(5) The value $= \frac{\log_2 3^5 - \log_3 2^5}{\log_2 3^3 - \log_3 2^3} = \frac{5 \log_2 3 - 5 \log_3 2}{3 \log_2 3 - 3 \log_3 2} = \frac{5 (\log_2 3 - \log_3 2)}{3 (\log_2 3 - \log_3 2)} = \frac{5}{3}$

Example 2

Without using the calculator , prove that :

(1) $3 \log 5 + 2 \log 6 - \log 9 + \log 0.2 = \log_5 25$

(2) $\frac{\log 30 - \log 6 + \log 5}{\log 12 - \log 3 + \log 25} = 1 - \log 2$

Solution

(1) L.H.S. $= \log 5^3 + \log 6^2 - \log 9 + \log \frac{2}{10}$

$$= \log \frac{5^3 \times 6^2 \times \frac{2}{10}}{9} = \log \frac{125 \times 36 \times 2}{9 \times 10}$$

$$= \log 100 = \log 10^2 = 2 \log 10 = 2$$

, R.H.S. $= \log_5 5^2 = 2 \log_5 5 = 2$

$\therefore$ The two sides are equal.

(2) L.H.S. $= \dfrac{\log \frac{30 \times 5}{6}}{\log \frac{12 \times 25}{3}} = \dfrac{\log 25}{\log 100} = \dfrac{\log 5^2}{\log 10^2} = \dfrac{2 \log 5}{2 \log 10} = \log 5$

, R.H.S. $= 1 - \log 2 = \log 10 - \log 2 = \log \frac{10}{2} = \log 5$

$\therefore$ The two sides are equal.

Example 3

If $\log_3 7 \simeq 1.771$

, find the value of each of the following in the simplest form , then verify your answer by using the calculator :

(1) $\log_3 21$　　**(2)** $\log_3 63$　　**(3)** $\log_3 \frac{7}{9}$

Solution

(1) $\log_3 21 = \log_3 (3 \times 7) = \log_3 3 + \log_3 7 = 1 + 1.771 = 2.771$

(Verifying by using the calculator : Start → [log▫□] [3] [REPLAY ▸] [2] [1] [=] ...)

(2) $\log_3 63 = \log_3 (9 \times 7) = \log_3 9 + \log_3 7$

$$= \log_3 3^2 + \log_3 7 = 2 \log_3 3 + \log_3 7$$

$$= 2 + 1.771 = 3.771$$

(Verifying by using the calculator : Start → [log▫□] [3] [REPLAY ▸] [6] [3] [=] ...)

(3) $\log_3 \frac{7}{9} = \log_3 7 - \log_3 9$

$$= \log_3 7 - \log_3 3^2$$

$$= \log_3 7 - 2 \log_3 3$$

$$= (\log_3 7) - 2 = 1.771 - 2 = -0.229$$

(Verifying by using the calculator : Start → [log▫□] [3] [REPLAY ▸] [÷] [7] [REPLAY ▸] [9] [=] ...)

Example 4

Find the value of each of the following in the simplest form :

(1) $\log_2 \sqrt[7]{32}$

(2) $\frac{1}{\log_x xyz} + \frac{1}{\log_y xyz} + \frac{1}{\log_z xyz}$

Solution

(1) $\log_2 \sqrt[7]{32} = \log_2 (2^5)^{\frac{1}{7}} = \log_2 2^{\frac{5}{7}} = \frac{5}{7} \log_2 2 = \frac{5}{7}$

(2) $\frac{1}{\log_x xyz} + \frac{1}{\log_y xyz} + \frac{1}{\log_z xyz} = \log_{xyz} x + \log_{xyz} y + \log_{xyz} z$

$= \log_{xyz} xyz = 1$

Example 5

Using the calculator , find the value of x to the nearest 2 decimal digits in each of the following :

(1) $5^x = 17$

(2) $2^{x-1} = 7$

(3) $5^{x+1} = 2^{4x-3}$

(4) $5^{x-2} = 3 \times 4^{x+1}$

(5) $3^{2x} - 14 \times 3^x + 45 = 0$

Solution

(1) $\because 5^x = 17$ "taking logarithms of the two sides"

$\therefore \log 5^x = \log 17$ $\therefore x \log 5 = \log 17$

$\therefore x = \frac{\log 17}{\log 5}$, then by using the calculator $x \simeq 1.76$

(2) $\because 2^{x-1} = 7$ "taking logarithms of the two sides"

$\therefore \log 2^{x-1} = \log 7$ $\therefore (x-1) \log 2 = \log 7$

$\therefore x \log 2 - \log 2 = \log 7$ $\therefore x = \frac{\log 7 + \log 2}{\log 2} \simeq 3.81$

(3) $\because 5^{x+1} = 2^{4x-3}$ "taking logarithms of the two sides"

$\therefore \log 5^{x+1} = \log 2^{4x-3}$ $\therefore (x+1) \log 5 = (4x-3) \log 2$

$\therefore x \log 5 + \log 5 = 4x \log 2 - 3 \log 2$ $\therefore 4x \log 2 - x \log 5 = 3 \log 2 + \log 5$

$\therefore x (4 \log 2 - \log 5) = 3 \log 2 + \log 5$ $\therefore x = \frac{3 \log 2 + \log 5}{4 \log 2 - \log 5} \simeq 3.17$

(4) $\because 5^{x-2} = 3 \times 4^{x+1}$ "taking logarithms of the two sides"

$\therefore (x-2)\log 5 = \log 3 + (x+1)\log 4$

$\therefore x\log 5 - 2\log 5 = \log 3 + x\log 4 + \log 4$

$\therefore x\log 5 - x\log 4 = \log 3 + \log 4 + 2\log 5$

$\therefore x(\log 5 - \log 4) = \log 3 + \log 4 + 2\log 5$

$\therefore x = \frac{\log 3 + \log 4 + 2\log 5}{\log 5 - \log 4} \simeq 25.56$

Another solution :

$\because 5^{x-2} = 3 \times 4^{x+1}$

$\therefore \frac{5^x}{5^2} = 3 \times 4^x \times 4$

$\therefore \frac{5^x}{4^x} = 3 \times 4 \times 5^2$

$\therefore \left(\frac{5}{4}\right)^x = 300$

$\therefore x = \frac{\log 300}{\log \frac{5}{4}} \simeq 25.56$

(5) $\because 3^{2x} - 14 \times 3^x + 45 = 0$

$\therefore (3^x - 9)(3^x - 5) = 0$

$\therefore 3^x - 9 = 0$ or $3^x - 5 = 0$

$\therefore 3^x = 9$	$\therefore 3^x = 5$ taking logarithms
$\therefore 3^x = 3^2$	$\therefore x\log 3 = \log 5$
$\therefore x = 2$	$\therefore x = \frac{\log 5}{\log 3} \simeq 1.46$

Important remark at solving logarithmic equation

If $x \in \mathbb{R}^*$ and m is an even number $\neq 0$ and $a \in \mathbb{R}^+ - \{1\}$

, then $\log_a x^m = m \log_a |x|$

For example : $\log_5 x^4 = 4\log_5 |x|$

Example 6

Find in $\mathbb{R}$ the S.S. of each of the following equations :

(1) $2\log x - \log(x+2) = 0$

(2) $\log x^2 = \log 4 + \log 9$

(3) $\log_2 x + \log_2 (x-2) = 3$

(4) $\log x + \log(x+2) = \log(x+6)$

(5) $\log \sqrt[3]{2x-1} + \log \sqrt[3]{x-2} = \log 30 - 1$

(6) $\frac{\log 49 - (\log 7)^2}{\log 0.07} = \log x$

Solution

Remember that

(1) The logarithmic function is one-to-one function.

i.e. If $\log_a x = \log_a y$, then $x = y$

(2) By substitution by the values which we obtained in the original equation , then the solution is the value that satisfies the equation , where the logarithm of non-positive number is meaningless.

(1) $\because 2\log x - \log(x+2) = 0$

$\therefore \log x^2 = \log(x+2)$

$\therefore x^2 = x + 2$

$\therefore x^2 - x - 2 = 0$

$\therefore (x-2)(x+1) = 0$

$\therefore x = 2$ or $x = -1$ (refused.)

$\therefore$ S.S. $= \{2\}$

(2) $\because \log x^2 = \log 4 + \log 9$

$\therefore \log x^2 = \log(4 \times 9)$

$\therefore \log x^2 = \log 36$

$\therefore x^2 = 36$

$\therefore x = \pm 6$

$\therefore$ S.S. $= \{6 , -6\}$

Another solution :

$\because \log x^2 = \log 4 + \log 9$

$\therefore \log x^2 = \log 36 = \log 6^2$

$\therefore 2\log|x| = 2\log 6$

$\therefore |x| = 6$

$\therefore x = \pm 6$

$\therefore$ S.S. $= \{6 , -6\}$

(3) $\because \log_2 x + \log_2(x-2) = 3$

$\therefore \log_2 x(x-2) = 3$

$\therefore x^2 - 2x = 2^3 = 8$

$\therefore x^2 - 2x - 8 = 0$

$\therefore (x-4)(x+2) = 0$

$\therefore x = 4$ or $x = -2$ (refused.)

$\therefore$ S.S. $= \{4\}$

(4) $\because \log x + \log(x+2) = \log(x+6)$

$\therefore \log x \times (x+2) = \log(x+6)$

$\therefore x(x+2) = x + 6$

$\therefore x^2 + 2x - x - 6 = 0$

$\therefore x^2 + x - 6 = 0$

$\therefore (x+3)(x-2) = 0$

$\therefore x = -3$ (refused) or $x = 2$

$\therefore$ S.S. $= \{2\}$

Remember that

$\log 10 = 1$

(5) $\because \log \sqrt[3]{2x-1} + \log \sqrt[3]{x-2} = \log 30 - 1$

$\therefore \log \sqrt[3]{(2x-1)(x-2)} = \log 30 - \log 10$

$\therefore \log \sqrt[3]{2x^2 - 5x + 2} = \log 3$

$\therefore \sqrt[3]{2x^2 - 5x + 2} = 3$ "raising the two sides to the power 3"

$\therefore 2x^2 - 5x + 2 = 27$

$\therefore 2x^2 - 5x - 25 = 0$

$\therefore (2x+5)(x-5) = 0$

$\therefore x = \frac{-5}{2}$ (refused) or $x = 5$

$\therefore$ S.S. $= \{5\}$

(6) $\because \frac{\log 49 - (\log 7)^2}{\log 0.07} = \log x$ $\therefore \frac{\log 7^2 - (\log 7)^2}{\log \frac{7}{100}} = \log x$

$\therefore \frac{2\log 7 - (\log 7)^2}{\log 7 - \log 100} = \log x$

Remember that

$\log 100 = 2$

$\therefore \frac{\log 7\,(2 - \log 7)}{\log 7 - 2} = \log x$

$\therefore -\log 7 = \log x$ $\therefore \log 7^{-1} = \log x$

$\therefore x = 7^{-1} = \frac{1}{7}$ $\therefore$ S.S. $= \left\{\frac{1}{7}\right\}$

Example 7

Find in $\mathbb{R}$ the S.S. of each of the following equations :

(1) $\log_3 (x^2 - 3x + 2) - \log_3 (x - 2) = \log_7 49$

(2) $x^{\log x} = 10$

(3) $\log_2 x = \log_4 25$

(4) $\log_3 x \times \log_9 x = 2$

(5) $\log_4 x + \log_x 4 = 2$

Solution

(1) $\because \log_3 (x^2 - 3x + 2) - \log_3 (x - 2) = \log_7 49$

$\therefore \log_3 \frac{x^2 - 3x + 2}{x - 2} = \log_7 7^2$ $\therefore \log_3 \frac{(x-2)(x-1)}{x-2} = 2\log_7 7$

$\therefore \log_3 (x - 1) = 2$ $\therefore x - 1 = 3^2$

$\therefore x = 1 + 9 = 10$

$\therefore$ S.S. $= \{10\}$

(2) $\because x^{\log x} = 10$ "taking logarithms of the two sides"

$\therefore \log x^{\log x} = \log 10$ $\therefore (\log x)(\log x) = 1$

$\therefore (\log x)^2 = 1$ $\therefore \log x = \pm 1$

$\therefore \log x = 1$ $\therefore x = 10^1 = 10$

or $\log x = -1$ $\therefore x = 10^{-1} = 0.1$

$\therefore$ S.S. $= \{10, 0.1\}$

(3) $\because \log_2 x = \log_4 25$ $\therefore \frac{\log x}{\log 2} = \frac{\log 25}{\log 4}$

$\therefore \log x = \frac{\log 5^2 \times \log 2}{\log 2^2} = \frac{2\log 5 \times \log 2}{2\log 2} = \log 5$

$\therefore x = 5$ (verify) $\therefore$ S.S. $= \{5\}$

(4) $\because \log_3 X \times \log_9 X = 2$ $\therefore \log_3 X \times \frac{\log_3 X}{\log_3 9} = 2$

$\therefore (\log_3 X)^2 = 2 \log_3 9 = 2 \log_3 3^2 = 4 \log_3 3 = 4$

$\therefore \log_3 X = \pm 2$ $\therefore X = 3^2 = 9$ (verify)

or $X = 3^{-2} = \frac{1}{9}$ (verify) $\therefore$ S.S. $= \left\{9 , \frac{1}{9}\right\}$

(5) $\because \log_4 X + \log_X 4 = 2$ $\therefore \log_4 X + \frac{1}{\log_4 X} = 2$ (multiply $\times \log_4 X$)

$\therefore (\log_4 X)^2 + 1 = 2 \log_4 X$ $\therefore (\log_4 X)^2 - 2 \log_4 X + 1 = 0$

$\therefore \left((\log_4 X) - 1\right)^2 = 0$ $\therefore \log_4 X = 1$

$\therefore X = 4$ (verify) $\therefore$ S.S. $= \{4\}$

Example 8

Find in $\mathbb{R}$ the S.S. of each of the following equations :

(1) $\log 49 \times \log \sqrt{8} = \log 343 \times \log X^3$

(2) $\log X^2 = (\log X)^2$

(3) $(\log X + 1)\left(\log \frac{X}{10}\right) = 3$

(4) $\log X - \log_X 100 = 1$

Solution

(1) $\because \log 49 \times \log \sqrt{8} = \log 343 \times \log X^3$

$\therefore \log X^3 = \frac{\log 49 \times \log \sqrt{8}}{\log 343} = \frac{\log 7^2 \times \log \sqrt{2^3}}{\log 7^3} = \frac{2 \log 7 \times \log 2^{\frac{3}{2}}}{3 \log 7}$

$\therefore \log X^3 = \frac{2}{3} \times \frac{3}{2} \log 2$ $\therefore \log X^3 = \log 2$

$\therefore X^3 = 2$ $\therefore X = \sqrt[3]{2}$

$\therefore$ S.S. $= \left\{\sqrt[3]{2}\right\}$

(2) $\because \log X^2 = (\log X)^2$ $\therefore 2 \log X = (\log X)^2$

$\therefore (\log X)^2 - 2 \log X = 0$ $\therefore \log X (\log X - 2) = 0$

$\therefore \log X = 0$, then $X = 10^0 = 1$ or $\log X = 2$, then $X = 10^2 = 100$

$\therefore$ S.S. $= \{1 , 100\}$

(3) $\because (\log x + 1)\left(\log \frac{x}{10}\right) = 3$ $\therefore (\log x + 1)(\log x - \log 10) = 3$

$\therefore (\log x + 1)(\log x - 1) = 3$ $\therefore (\log x)^2 - 1 = 3$

$\therefore (\log x)^2 = 4$ $\therefore \log x = \pm 2$

$\therefore \log x = 2$, then $x = 10^2 = 100$

or $\log x = -2$, then $x = 10^{-2} = 0.01$

$\therefore$ S.S. $= \{100 , 0.01\}$

(4) $\because \log x - \log_x 100 = 1$ $\therefore \log x - \log_2 10^2 = 1$

$\therefore \log x - 2\log_x 10 = 1$ $\therefore \log x - \frac{2}{\log x} = 1$ (multiply $\times \log x$)

$\therefore (\log x)^2 - 2 = \log x$ $\therefore (\log x)^2 - \log x - 2 = 0$

$\therefore (\log x + 1)(\log x - 2) = 0$ $\therefore \log x = -1$, then $x = 10^{-1} = 0.1$

or $\log x = 2$, then $x = 10^2 = 100$ $\therefore$ S.S. $= \{100 , 0.1\}$

Example 9

(1) If $x y = 16$, **prove that :** $3\log_2 x + 4\log_2 y - \log_2 x y^2 = 8$

(2) If $x^2 + y^2 = 6 x y$, **prove that :** $2\log(x + y) = \log x + \log y + 3\log 2$

Solution

(1) L.H.S. $= \log_2 x^3 + \log_2 y^4 - \log_2 x y^2$

$= \log_2 \frac{x^3 y^4}{x y^2} = \log_2 x^2 y^2 = \log_2 (x y)^2$

$= 2\log_2 x y = 2\log_2 16 = 2\log_2 2^4 = 2 \times 4 \log_2 2$

$= 2 \times 4 \times 1 = 8 =$ R.H.S.

(2) $\because x^2 + y^2 = 6 x y$ $\therefore x^2 + y^2 + 2 x y = 6 x y + 2 x y$

$\therefore x^2 + 2 x y + y^2 = 8 x y$ $\therefore (x + y)^2 = 8 x y$

$\therefore \log (x + y)^2 = \log 8 x y$

$\therefore 2\log(x + y) = \log 8 + \log x + \log y$

$= \log 2^3 + \log x + \log y = 3\log 2 + \log x + \log y$

Example 10

If the magnitude of the intensity M (I) of an earthquake on Richter scale is given by the relation $M(I) = \log \frac{I}{I_o}$ where I is the earthquake intensity , I_o is the smallest earth movement that can be recorded , called the reference intensity.

(1) Find on Richter scale the magnitude of the earthquake of intensity 1.6×10^8 times the reference intensity.

(2) If the magnitude of the earthquake = 7 on Richter scale find how many times the intensity of this earthquake equals the reference intensity.

Solution

(1) $M(I) = \log \frac{I}{I_0}$, $I = 1.6 \times 10^8 \ I_0$

$\therefore M = \log \frac{1.6 \times 10^8 \ I_0}{I_0} = \log (1.6 \times 10^8) \simeq 8.2$

i.e. The intensity of the earthquake on Richter = 8.2

(2) $\because$ The magnitude of the earthquake (M) = 7

$\therefore 7 = \log \frac{I}{I_0}$ $\qquad$ $\therefore \frac{I}{I_0} = 10^7$

$\therefore I = 10^7 \ I_0$

$\therefore$ The intensity of the earthquake = 10 000 000 times the reference intensity.

Enrich your knowledge

For any number $X \in \mathbb{Z}^+$, if $n \le \log X < n + 1$

, then the number of digits of the number X equals $(n + 1)$ where $n \in \mathbb{N}$

For example :

* To find the number of digits of the number 5^3

  $\because \log 5^3 \simeq 2.097$, then $2 \le \log 5^3 < 3$

  $\therefore$ The number of digits of 5^3 = 3 digits.

* To find the number of digits of the number 3^{17}

  $\because \log 3^{17} \simeq 8.111$, then $8 \le \log 3^{17} < 9$

  $\therefore$ The number of digits of 3^{17} = 9 digits.

Second

Calculus and Trigonometry



UNIT 3 Limits and continuity.

UNIT 4 Trigonometry.

UNIT 3

Limits and continuity.

Unit Lessons

Lesson 1 Introduction to limits of functions "Evaluation of the limit numerically and graphically".

Lesson 2 Finding the limit of a function algebraically.

Lesson 3 Theorem (4) "The law".

Lesson 4 Limit of the function at infinity.

Lesson 5 Limits of trigonometric functions.

Lesson 6 Existence of the limit of a piecewise function.

Lesson 7 Continuity.



Lesson One — Introduction to limits of functions "Evaluation of the limit numerically and graphically"

Specified , unspecified and undefined quantities

When we do an arithmetic operation on $\mathbb{R}$, we will get one of the following three types of quantities.

1 Specified quantity

It is the quantity which has determined result.

For example : $\frac{8}{5}$ is a specified quantity

i.e. It has a determined result which is 1.6

because : The real number which if multiplied by 5 , the result will be 8 is 1.6

Examples for the specified quantities : $\frac{0}{3}$, 5 ± 0 , 7×3 , ...

2 Unspecified quantity

It is the quantity which has no determined answer.

For example : $\frac{0}{0}$ is an unspecified quantity

i.e. It has an infinite number of answers in $\mathbb{R}$.

because : The product of any real number $\times$ zero = zero

Noticing that there are other unspecified quantities we shall study later.

3 Undefined quantity

It is the quantity which is meaningless.

For example : $\frac{5}{0}$ is undefined quantity

i.e. It has no meaning to divide by zero.

because : There is no real number if multiplied by zero , the result will be 5

Generally : $\frac{a}{0}$ where $a \in \mathbb{R} - \{0\}$ is undefined quantity.

The symbols ∞ and $-\infty$

* The symbol ∞ (infinity) is not a real number but it represents a quantity greater than any positive real number can be recognized.
* The symbol $-\infty$ (negative infinity) is not a real number but it represents a quantity smaller than any negative real number can be recognized.
* Let a be a real number , then :

(1) $\infty \pm a = \infty$, $-\infty \pm a = -\infty$

(2) $\infty \times a = \begin{cases} \infty & \text{at } a > 0 \\ -\infty & \text{at } a < 0 \\ \text{unspecified} & \text{at } a = 0 \end{cases}$

, $-\infty \times a = \begin{cases} -\infty & \text{at } a > 0 \\ \infty & \text{at } a < 0 \\ \text{unspecified} & \text{at } a = 0 \end{cases}$

Enrich your knowledge

The unspecified quantities are seven and they are :

$\frac{\text{zero}}{\text{zero}}$, $\frac{\infty}{\infty}$, $\infty - \infty$, $\infty \times \text{zero}$, $(\text{zero})^{\text{zero}}$, $(\infty)^{\text{zero}}$ and $(1)^{\infty}$

For example : $\infty \pm 7 = \infty$, $-\infty \pm 2 = -\infty$, $\infty \times 15 = \infty$, $-\infty \times 7 = -\infty$

, $-\infty \times -2 = \infty$, $\infty + \infty = \infty$

The concept of the limit of a function at a point

Illustrated Example

If we want to find the value of the function $f : f(X) = \frac{X^2 - 1}{X - 1}$ at $X = 1$

We find that : $f(1) = \frac{1^2 - 1}{1 - 1} = \frac{0}{0}$ "unspecified quantity"

that means we can not determine the value of the function at $X = 1$

So , we go to study the approaching of $f(X)$ to a specified quantity , when X approaches to the number 1 , that by one of the following two methods :

(1) Evaluation of the limit numerically

Give values for the variable X approaches to one through taking values more than 1 and less than 1 in which X does not take the value 1 , the following table shows the values X takes approaching to «1» and their corresponding values of $f(X)$:

X approaches to 1 from the left - - - - - - → ← - - - - X approaches to 1 from the right

X	0.5	0.6	0.7	0.8	0.9		1.1	1.2	1.3	1.4	1.5
$f(X)$	1.5	1.6	1.7	1.8	1.9		2.1	2.2	2.3	2.4	2.5

$f(X)$ approaches to 2 from the left - - - - → ← - - - - $f(X)$ approaches to 2 from the right

* **We find that :**

When X approaches to the number 1 from the right side

(*i.e.* from the values of the variable X greater than 1) which is written mathematically as «$X \longrightarrow 1^+$» and is read as «X tends to 1 from the right».

, then $f(X)$ approaches to the number 2 and the number 2 is called the right limit of the function and it is written mathematically as $\boxed{\lim\limits_{X \to 1^+} f(X) = 2}$ or $\boxed{f(1^+) = 2}$

and is read as the limit of the function when $(X \longrightarrow 1^+)$ equals 2

* And when X approaches to the number 1 from the left side (*i.e.* from the values of the variable X smaller than 1) which is written mathematically as «$X \longrightarrow 1^-$» and is read as «X tends to 1 from the left» , then $f(X)$ approaches to the number 2 and the number 2 is called the left limit of the function and it is written mathematically as $\boxed{\lim\limits_{X \to 1^-} f(X) = 2}$ or $\boxed{f(1^-) = 2}$ and is read as the limit of the function when $(X \longrightarrow 1^-)$ equals 2

Definition

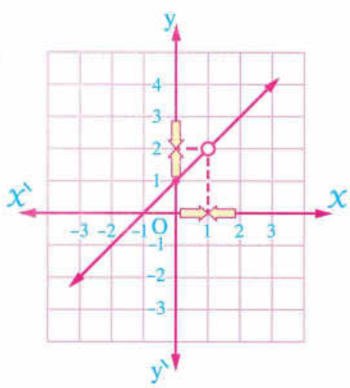
If the value of the function f approaches to a unique value ℓ when X approaches to a from the two sides right and left , then the limit of $f(X)$ equals ℓ and it is written symbolically $\lim\limits_{X \to a} f(X) = \ell$

i.e. If $\boxed{f(a^+) = f(a^-) = \ell}$, then $\boxed{\lim\limits_{X \to a} f(X) = \ell}$

In the previous example :

$\because f(1^+) = f(1^-) = 2 \qquad \therefore \lim\limits_{X \to 1} f(X) = 2$

2 Evaluation of the limit graphically



$\because f(X) = \dfrac{X^2 - 1}{X - 1}$ is undefined at $X = 1$

$\therefore f(X) = \dfrac{(X-1)(X+1)}{(X-1)} = X + 1$, where $X \neq 1$

i.e. It is represented by a straight line with an open dot at the point whose X-coordinate = 1 as in the opposite figure , and from the figure we notice that :

when $X \xrightarrow{\text{tends to}} 1$ (from the right and the left) , then $f(X) \xrightarrow{\text{tends to}} 2$

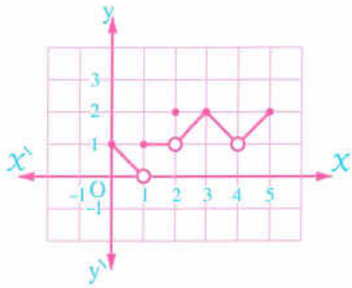
i.e. $\lim\limits_{X \to 1} f(X) = 2$

Remarks

1. At finding $\lim\limits_{x \to a} f(x)$, it is not necessary that the function be defined at $x = a$, it should be defined only in an interval on the left of a and another interval on the right of a
2. If $f(a^+) \neq f(a^-)$, then $\lim\limits_{x \to a} f(x)$ does not exist.

Important remarks at finding the limit of the function graphically

(1) In the opposite figure :



We find that :

First : At $x = 0$: $\lim\limits_{x \to 0^+} f(x) = 1$

, $\lim\limits_{x \to 0^-} f(x)$ and $\lim\limits_{x \to 0} f(x)$ don't exist

[Because the function is undefined at the left of $x = 0$]

Second : At $x = 1$: $\lim\limits_{x \to 1^-} f(x) = 0$, $\lim\limits_{x \to 1^+} f(x) = 1$

$\therefore \lim\limits_{x \to 1} f(x)$ does not exist.

[Because the right limit ≠ the left limit]

Notice that : Although f is defined at $x = 1$ «$f(1) = 1$» , the limit does not exist.

Third : At $x = 2$: $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2} f(x) = 1$

[Notice that , it is not necessary that the value of the function equals the value of the limit where $f(2) = 2$]

Fourth : At $x = 3$: $\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^+} f(x) = \lim\limits_{x \to 3} f(x) = f(3) = 2$

Fifth : At $x = 4$: $\lim\limits_{x \to 4^-} f(x) = \lim\limits_{x \to 4^+} f(x) = \lim\limits_{x \to 4} f(x) = 1$

[Notice that $f(4)$ is undefined *i.e.* Although the function is undefined , the limit exists]

Sixth : At $x = 5$: $\lim\limits_{x \to 5^-} f(x) = 2$, $\lim\limits_{x \to 5^+} f(x)$ and $\lim\limits_{x \to 5} f(x)$ do not exist.

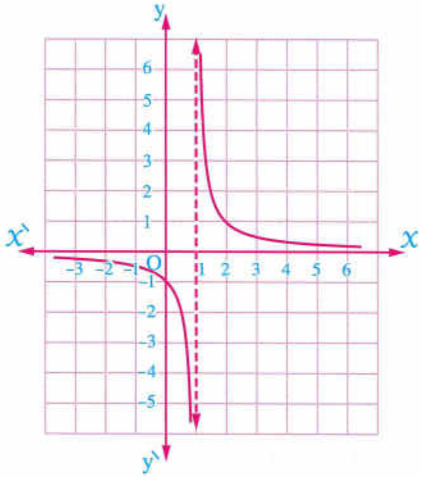
[Because the function is undefined on the right of $x = 5$]

Remark

From the graph of the function in the previous figure , then :

* The point which is represented by an open dot does not affect on the existing of a limit at it as in third and fifth.
* The point which has a brumpt break (jump) dues to non existence a limit as in second.

(2) The opposite figure represents the function :



$f : f(x) = \frac{1}{x-1}$

and we find that :

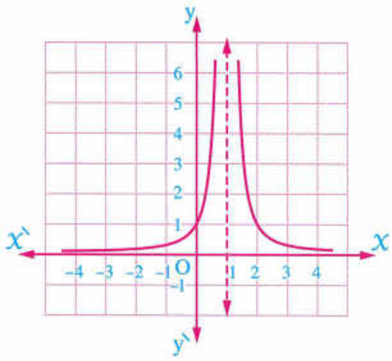
$f(1^+) = \infty$

$, f(1^-) = -\infty$

$\therefore f(1^+) \neq f(1^-)$

$\therefore \lim_{x \to 1} f(x)$ does not exist.

(3) The opposite figure represents the function :



$f : f(x) = \frac{1}{(x-1)^2}$

and we find that :

$f(1^+) = \infty$

$, f(1^-) = \infty$

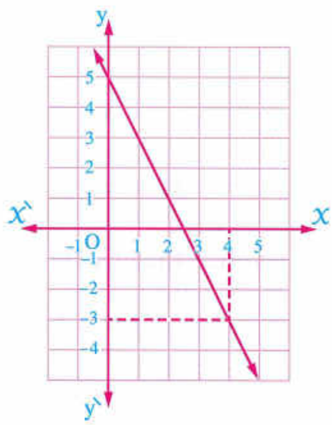
$, \lim_{x \to 1} f(x) = \infty$

Example 1

Evaluate : $\lim_{x \to 4} (5 - 2x)$ graphically and numerically.

Solution

*** Graphically :**



We represent the linear function $f : f(x) = 5 - 2x$

as in the opposite figure :

We notice that , when $x \longrightarrow 4$

, then $f(x) \longrightarrow -3$

i.e. $\lim_{x \to 4} (5 - 2x) = -3$

*** Numerically :**

We form a table for the values of $f(x)$ and this by choosing values of x approaches to the number 4 from the right and the left as follows :

x	3.9	3.99	3.999	..... 4	4.001	4.01	4.1
$f(x)$	– 2.8	– 2.98	– 2.998	..... – 3	– 3.002	– 3.02	– 3.2

From the table , we notice that , when x approaches to the number 4 from the right or the left , the values of $f(x)$ approaches to the number – 3 $\therefore \lim_{x \to 4} (5 - 2x) = -3$

Example 2

Study each of the following figures , then find the value of :

(1) $f(2)$ (2) $\lim\limits_{x \to 2^+} f(x)$ (3) $\lim\limits_{x \to 2^-} f(x)$ (4) $\lim\limits_{x \to 2} f(x)$

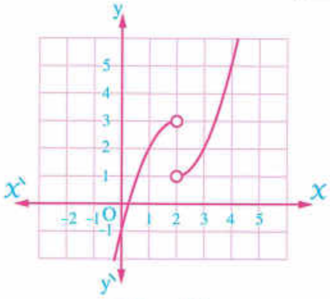


Fig. (1)

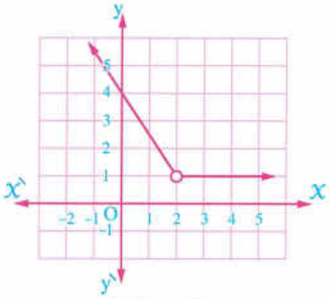


Fig. (2)

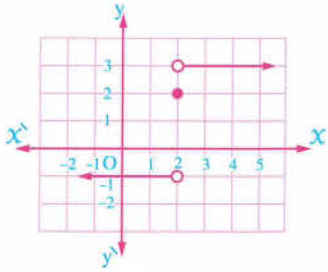


Fig. (3)

Solution

In Fig. (1) : $f(2)$ is undefined , $\lim\limits_{x \to 2^+} f(x) = 1$, $\lim\limits_{x \to 2^-} f(x) = 3$, $\lim\limits_{x \to 2} f(x)$ does not exist.

In Fig. (2) : $f(2)$ is undefined , $\lim\limits_{x \to 2^+} f(x) = 1$, $\lim\limits_{x \to 2^-} f(x) = 1$, $\lim\limits_{x \to 2} f(x) = 1$

In Fig. (3) : $f(2) = 2$, $\lim\limits_{x \to 2^+} f(x) = 3$, $\lim\limits_{x \to 2^-} f(x) = -1$, $\lim\limits_{x \to 2} f(x)$ does not exist.

Example 3

Study each of the following figures , then find the value of :

(1) $f(0)$ (2) $f(0^+)$ (3) $f(0^-)$ (4) $\lim\limits_{x \to 0} f(x)$

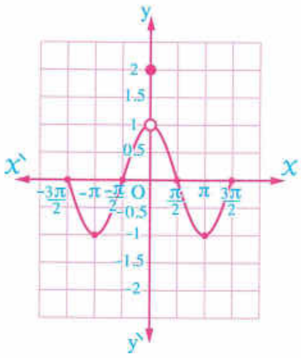


Fig. (1)

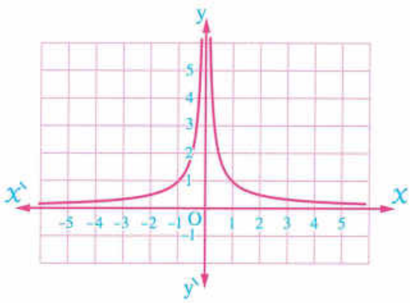


Fig. (2)

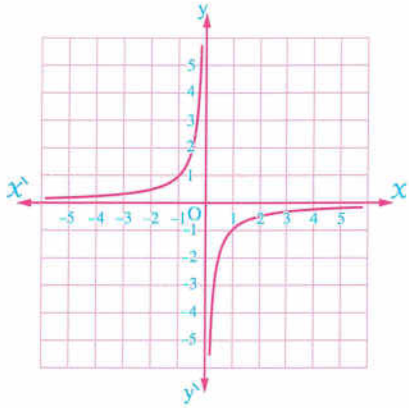


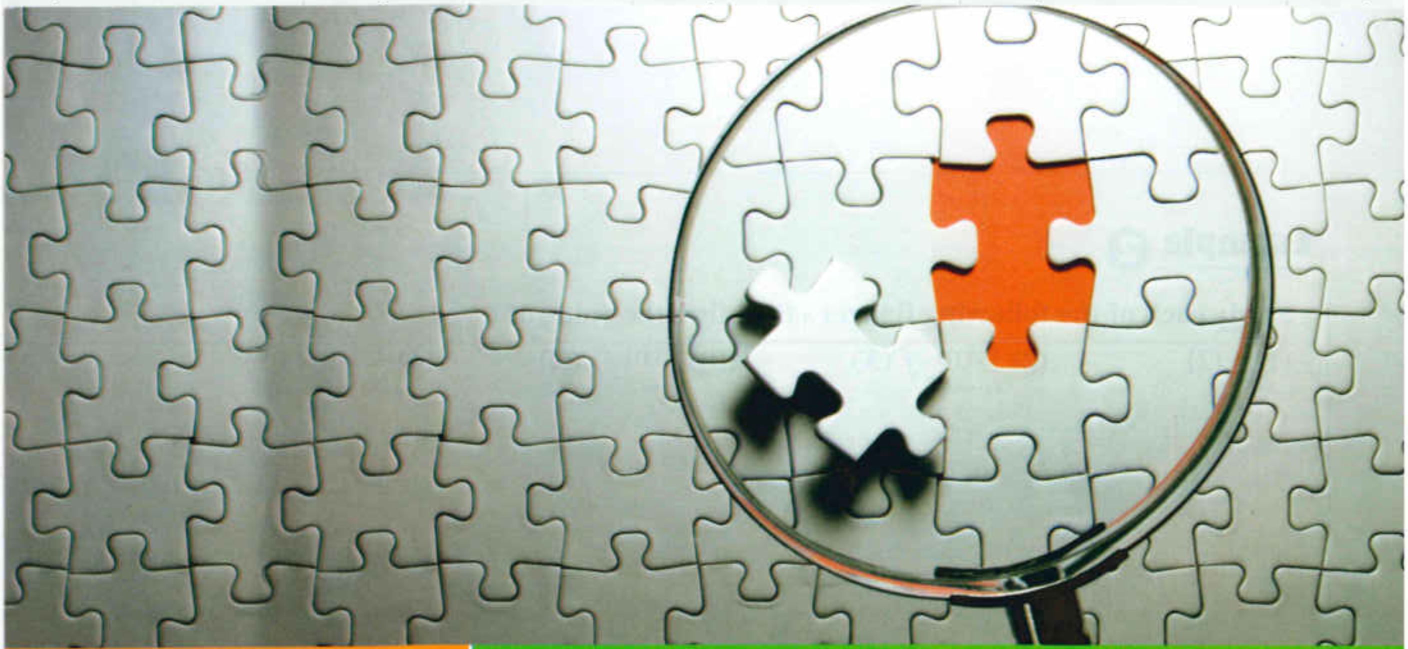
Fig. (3)

Solution

In Fig. (1) : $f(0) = 2$, $f(0^+) = f(0^-) = 1$ $\quad \therefore \lim\limits_{x \to 0} f(x) = 1$

In Fig. (2) : $f(0)$ is undefined , $f(0^+) = f(0^-) = \infty$ $\quad \therefore \lim\limits_{x \to 0} f(x) = \infty$

In Fig. (3) : $f(0)$ is undefined , $f(0^+) = -\infty$, $f(0^-) = \infty$ $\quad \therefore \lim\limits_{x \to 0} f(x)$ does not exist.



Lesson Two — Finding the limit of a function algebraically

The following are some fundamental theorems and corollaries which help for finding the limit of a function without restoring to the graphing or studying the values of the function.

Theorem (1) ***(Limit of a polynomial function) :***

If : $f(x)$ is a polynomial function in x, then $\lim\limits_{x \to a} f(x) = f(a)$

For example :

$\lim\limits_{x \to 2} (2x + 5) = f(2) = 2(2) + 5 = 9$

, $\lim\limits_{x \to 1} (x^2 - 3x + 2) = f(1) = 1 - 3 + 2 = \text{zero}$

Corollary

Limit of the constant function.

If $f(x) = k$ where k is constant , then $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} k = k$

For example : $\lim\limits_{x \to 3} 4 = 4$, $\lim\limits_{x \to -1} -5 = -5$

Theorem (2)

If f , g are two real functions in x , $\lim\limits_{x \to a} f(x) = \ell$, $\lim\limits_{x \to a} g(x) = m$ where ℓ and $m \in \mathbb{R}$, then :

(1) $\lim\limits_{x \to a} [f(x) \pm g(x)] = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x) = \ell \pm m$

i.e. Limit of the algebraic sum of two functions = the algebraic sum of their limits.

This rule can be generalized for the sum of a finite number of functions.

(2) $\lim\limits_{x \to a} [f(x) \times g(x)] = \lim\limits_{x \to a} f(x) \times \lim\limits_{x \to a} g(x) = \ell \times m$

i.e. Limit of the product of two functions = the product of their limits.

This rule can be generalized for the product of a finite number of functions.

(3) $\lim\limits_{x \to a} k\, f(x) = k \lim\limits_{x \to a} f(x) = k\,\ell$, where k is constant.

i.e. Limit of the product of a constant × function = the constant × limit of this function.

(4) $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \frac{\ell}{m}$ where $m \neq 0$

i.e. Limit of the quotient of two functions = the quotient of their limits regarding that the denominator $\neq 0$

This rule can be generalized for the product of a finite number of functions divided by the product of a finite number of functions under condition that the denominator $\neq 0$

(5) $\lim\limits_{x \to a} [f(x)]^n = [\lim\limits_{x \to a} f(x)]^n = \ell^n$, $n \in \mathbb{Z}^+$

Example 1

Find each of the following limits :

(1) $\lim\limits_{x \to 0} x^2 + 3x - 2 + \frac{2x}{x-1}$

(2) $\lim\limits_{x \to -2} (x^2 + 3)(x^3 - 2x^2 + 4x - 1)$

(3) $\lim\limits_{x \to 1} \frac{(2x^2 - 5x + 6)(x+1)}{(x^2+2)(3x-1)}$

Solution

(1) $\lim\limits_{x \to 0} x^2 + 3x - 2 + \frac{2x}{x-1} = \lim\limits_{x \to 0} (x^2 + 3x - 2) + \lim\limits_{x \to 0} \frac{2x}{x-1}$

$= \lim\limits_{x \to 0} (x^2 + 3x - 2) + \frac{\lim\limits_{x \to 0} 2x}{\lim\limits_{x \to 0} (x-1)} = -2 + \frac{0}{-1} = -2$

(2) $\lim\limits_{x \to -2} (x^2 + 3)(x^3 - 2x^2 + 4x - 1)$

$= \lim\limits_{x \to -2} (x^2 + 3) \times \lim\limits_{x \to -2} (x^3 - 2x^2 + 4x - 1)$

$= (4 + 3)(-8 - 8 - 8 - 1) = 7 \times (-25) = -175$

(3) $\lim\limits_{x \to 1} \frac{(2x^2 - 5x + 6)(x+1)}{(x^2+2)(3x-1)} = \frac{\lim\limits_{x \to 1} (2x^2 - 5x + 6) \times \lim\limits_{x \to 1} (x+1)}{\lim\limits_{x \to 1} (x^2+2) \times \lim\limits_{x \to 1} (3x-1)}$

$= \frac{(2-5+6)(1+1)}{(1+2)(3-1)} = \frac{3 \times 2}{3 \times 2} = 1$

Notice that :

We can solve the previous example by using direct substitution without separating limits.

Remark

We can use the direct substitution in $\lim\limits_{x \to a} f(x) = f(a)$ if the function is polynomial or rational with denominator $\neq 0$ at $x = a$

Theorem (3)

If f , g are two functions in the variable x , $f(x) = g(x)$ for all the values of $x \in \mathbb{R} - \{a\}$ and $\lim\limits_{x \to a} g(x) = \ell$, then : $\lim\limits_{x \to a} f(x) = \ell$

The use of the previous theorem :

This theorem is used to find the limit of a rational function (fraction each of its numerator and denominator is polynomial) say $f(x)$ at $x \longrightarrow a$ where $f(a) = \frac{\text{zero}}{\text{zero}}$

This means that $(x - a)$ is a common factor between the numerator and the denominator.

Notice that :

$x \longrightarrow a$ means $(x - a) \longrightarrow 0$

i.e. $(x - a) \neq 0$ and because that the simplifying is done.

In this case , to find $\lim\limits_{x \to a} f(x)$, we cancel the factor $(x - a)$ using factorization or long division to get a new function $g(x)$ equal to $f(x)$ when $x \neq a$,

then $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x)$ and the next example illustrates this process.

Example 2

Find each of the following :

(1) $\lim\limits_{x \to 4} \frac{x^2 - 16}{x - 4}$ (2) $\lim\limits_{x \to 2} \frac{x^3 - 8}{x^2 - 5x + 6}$ (3) $\lim\limits_{x \to -1} \frac{(2x + 3)^2 - 1}{x^2 + x}$

Solution

(1) Let : $f(x) = \frac{x^2 - 16}{x - 4}$ $\therefore f(4) = \frac{4^2 - 16}{4 - 4} = \frac{\text{zero}}{\text{zero}}$

$$\therefore \lim_{x \to 4} \frac{x^2 - 16}{x - 4} = \lim_{x \to 4} \frac{(x - 4)(x + 4)}{x - 4} = \lim_{x \to 4} x + 4 = 4 + 4 = 8$$

(2) Let : $f(x) = \frac{x^3 - 8}{x^2 - 5x + 6}$ $\therefore f(2) = \frac{2^3 - 8}{2^2 - 5(2) + 6} = \frac{\text{zero}}{\text{zero}}$

$$\therefore \lim_{x \to 2} \frac{x^3 - 8}{x^2 - 5x + 6} = \lim_{x \to 2} \frac{(x - 2)(x^2 + 2x + 4)}{(x - 2)(x - 3)}$$

$$= \lim_{x \to 2} \frac{x^2 + 2x + 4}{x - 3} = \frac{2^2 + 2(2) + 4}{2 - 3} = -12$$

(3) Let : $f(x) = \dfrac{(2x+3)^2 - 1}{x^2 + x}$ $\qquad \therefore f(-1) = \dfrac{(-2+3)^2 - 1}{(-1)^2 - 1} = \dfrac{\text{zero}}{\text{zero}}$

$$\therefore \lim_{x \to -1} \frac{(2x+3)^2 - 1}{x^2 + x} = \lim_{x \to -1} \frac{(2x+3-1)(2x+3+1)}{x(x+1)}$$

$$= \lim_{x \to -1} \frac{(2x+2)(2x+4)}{x(x+1)} = \lim_{x \to -1} \frac{2(x+1) \times 2(x+2)}{x(x+1)}$$

$$= \lim_{x \to -1} \frac{4(x+2)}{x} = \frac{4(-1+2)}{-1} = -4$$

Example 3

Find : $\lim_{x \to 2} \dfrac{x^3 - 7x + 6}{3x^2 - 8x + 4}$

Solution

Let : $f(x) = \dfrac{x^3 - 7x + 6}{3x^2 - 8x + 4}$

$\therefore f(2) = \dfrac{(2)^3 - 7(2) + 6}{3(2)^2 - 8(2) + 4} = \dfrac{\text{zero}}{\text{zero}}$

$\therefore (x - 2)$ is a common factor between the numerator and the denominator , then divide the numerator by the factor $(x - 2)$ because its factorization is difficult.

$$\therefore \begin{array}{l|lll}
x - 2 & x^3 & & -7x+6 \\
\hline
x^2 + 2x - 3 & x^3 - 2x^2 & & \\
\hline
 & & 2x^2 & -7x+6 \\
 & & 2x^2 & -4x \\
\hline
 & & & -3x+6 \\
 & & & -3x+6 \\
\hline
 & & & 00 \quad 00
\end{array}$$

$\therefore$ The numerator $= (x - 2)(x^2 + 2x - 3)$

$$\therefore \lim_{x \to 2} \frac{x^3 - 7x + 6}{3x^2 - 8x + 4} = \lim_{x \to 2} \frac{(x-2)(x^2 + 2x - 3)}{(x-2)(3x-2)} = \lim_{x \to 2} \frac{x^2 + 2x - 3}{3x - 2}$$

$$= \frac{4 + 4 - 3}{6 - 2} = \frac{5}{4}$$

Remember that

For the long division operation :

(1) Arrange the terms of each dividend and divisor according to the powers of x ascendingly or descendingly by the same way with leave an empty place for the powers which do not exist.

(2) Divide the first term of the dividend by the first term of the divisor , then write the quotient.

(3) Multiply the quotient by the divisor , and subtract the result from the dividend to get the left.

(4) Continue with the same way until the division operation finished.

Remark

In the case of dividing by an expression of the first degree and the coefficient of $x = 1$

i.e. In the form of $(x - a)$, you can use the synthetic division method to make the long division easier , and you can use it in the previous example as follows to divide $(x^3 - 7x + 6)$ by $(x - 2)$

(1) Arrange the coefficients according to the ascendingly or descendingly powers of x and put (0) as a coefficient for the powers which do not exist and put the number (2) (Zero of the divisor) in the place of divisor.

(2)	1	0	−7	6

(2) The coefficient of the greatest power is brought down to the third row , then multiply it by 2 and put the product in the second row place of the neighboring column directly.

(2)	1	0	−7	6
	×2 →	2		
	1			

(3) Add the coefficient of the next power to the product which you got immediately.

(2)	1	0	−7	6
		+ 2		
	1	2		

(4) Repeat multiplying and adding to get the factors of the quotient 1 , 2 and − 3

$\therefore$ The quotient is : $x^2 + 2x - 3$

(2)	1	0	−7	6
	×2	+ 2	+ 4	+ −6
	1	2	−3	0

Another solution of example (3) :

Factorizing by grouping can be used , known that $(x - 2)$ is a factor of each numerator and denominator as follows :

$$\lim_{x \to 2} \frac{x^3 - 7x + 6}{3x^2 - 8x + 4} = \lim_{x \to 2} \frac{(x^3 - 8) + (-7x + 14)}{(x - 2)(3x - 2)} = \lim_{x \to 2} \frac{(x - 2)(x^2 + 2x + 4) - 7(x - 2)}{(x - 2)(3x - 2)}$$

$$= \lim_{x \to 2} \frac{\cancel{(x - 2)}[x^2 + 2x + 4 - 7]}{\cancel{(x - 2)}(3x - 2)} = \frac{2^2 + 2 \times 2 - 3}{3 \times 2 - 2} = \frac{5}{4}$$

Important remark

In case of existence of a difference of two square roots of algebraic expressions (in numerator or denominator or both) , we multiply each of the numerator and denominator by the conjugate of (the numerator or the denominator or both) when the result of the direct substitution equals $\frac{\text{zero}}{\text{zero}}$ and the next example illustrates this.

Example 4

WATCH VIDEO

Find each of the following :

(1) $\lim\limits_{x \to 0} \dfrac{x^2 + 2x}{\sqrt{x+9} - 3}$ (2) $\lim\limits_{x \to 0} \dfrac{\sqrt{4+x} - 2}{\sqrt{9+x} - \sqrt{9-x}}$

Solution

By substitution in each of the two functions by $x = 0$, we find that the value of each $= \frac{\text{zero}}{\text{zero}}$

(1) $\lim\limits_{x \to 0} \dfrac{x^2 + 2x}{\sqrt{x+9} - 3} = \lim\limits_{x \to 0} \dfrac{x(x+2)}{\sqrt{x+9} - 3} \times \dfrac{\sqrt{x+9} + 3}{\sqrt{x+9} + 3}$

(multiplying by the conjugate of the denominator)

$$= \lim_{x \to 0} \frac{x(x+2)\left[\sqrt{x+9} + 3\right]}{x + 9 - 9}$$

$$= \lim_{x \to 0} (x+2)\left[\sqrt{x+9} + 3\right] = 2 \times 6 = 12$$

(2) $\lim\limits_{x \to 0} \dfrac{\sqrt{4+x} - 2}{\sqrt{9+x} - \sqrt{9-x}} = \lim\limits_{x \to 0} \dfrac{\sqrt{4+x} - 2}{\sqrt{9+x} - \sqrt{9-x}} \times \dfrac{\sqrt{4+x} + 2}{\sqrt{4+x} + 2} \times \dfrac{\sqrt{9+x} + \sqrt{9-x}}{\sqrt{9+x} + \sqrt{9-x}}$

(multiplying by the conjugates of numerator and denominator)

$$= \lim_{x \to 0} \frac{4 + x - 4}{(9+x) - (9-x)} \times \frac{\sqrt{9+x} + \sqrt{9-x}}{\sqrt{4+x} + 2}$$

$$= \lim_{x \to 0} \frac{x\left(\sqrt{9+x} + \sqrt{9-x}\right)}{2x\left(\sqrt{4+x} + 2\right)} = \lim_{x \to 0} \frac{\sqrt{9+x} + \sqrt{9-x}}{2\left(\sqrt{4+x} + 2\right)}$$

$$= \frac{\sqrt{9+0} + \sqrt{9-0}}{2\left(\sqrt{4+0} + 2\right)} = \frac{3}{4}$$

From the previous we can deduce that

To find $\lim_{x \to a} f(x)$ find $f(a)$ by direct substitution $x = a$ in the function , if the result :

(1) Is a real number b , then $\lim_{x \to a} f(x) = b$

(2) $\frac{\text{zero}}{\text{zero}}$ "unspecified quantity" , then divide both numerator and denominator by $(x - a)$

Example 5

Find each of the following :

(1) $\lim_{x \to -2} \frac{x-3}{x^2+1}$

(2) $\lim_{x \to 2} \left(\frac{x^2 - x}{x-2} - \frac{2}{x-2} \right)$

Solution

(1) $\because f(-2) = \frac{-2-3}{(-2)^2+1} = -1$ (by direct substitution it gives a real number)

$\therefore \lim_{x \to -2} \frac{x-3}{x^2+1} = -1$

(2) Let $f(x) = \frac{x^2 - x}{x-2} - \frac{2}{x-2}$ $\therefore f(x) = \frac{x^2 - x - 2}{x-2}$

$\therefore f(2) = \frac{4-2-2}{2-2} = \frac{\text{zero}}{\text{zero}}$ (by direct substitution it gives $\frac{\text{zero}}{\text{zero}}$ "unspecified quantity")

$\therefore \lim_{x \to 2} \left(\frac{x^2 - x}{x-2} - \frac{2}{x-2} \right) = \lim_{x \to 2} \frac{x^2 - x - 2}{x-2}$

$= \lim_{x \to 2} \frac{(x-2)(x+1)}{x-2} = \lim_{x \to 2} (x+1) = 2 + 1 = 3$

Example 6

If : $\lim_{x \to 3} \frac{f(x) - 7}{x-3} = 4$ where $f(x)$ is a polynomial function , find $\lim_{x \to 3} f(x)$

Solution

$\because \lim_{x \to 3} \frac{f(x) - 7}{x-3}$ exists and equals 4

, $\because$ the denominator = zero at $x = 3$

$\therefore$ The numerator = zero at $x = 3$

$\therefore f(3) - 7 = \text{zero}$ $\therefore f(3) = 7$

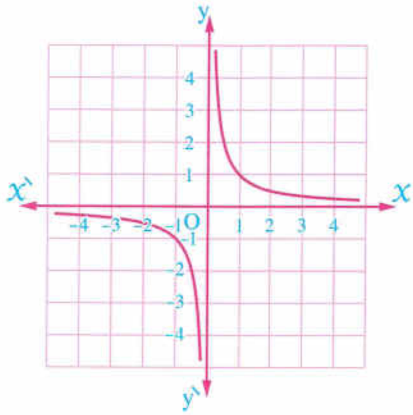
, $\because f(x)$ is a polynomial function

$\therefore \lim_{x \to 3} f(x) = f(3) = 7$

Enrich your knowledge

The limit gives $\frac{\text{real number} \neq 0}{\text{zero}}$ by direct substitution is "undefined quantity" :

The opposite figure represents the function $f : f(X) = \frac{1}{X}$ and we notice that :



(1) $\lim\limits_{x \to 0^+} \frac{1}{X} = \infty$

(2) $\lim\limits_{x \to 0^-} \frac{1}{X} = -\infty$

(3) $\lim\limits_{x \to 0} \frac{1}{X}$ in not exist because $f(0^+) \neq f(0^-)$

From the previous , other limits gives by direct substitution $\frac{\text{real number} \neq \text{zero}}{\text{zero}}$ can be found without using the graph as follows :

(1) $\lim\limits_{x \to 2} \frac{1}{X-2} = \lim\limits_{(X-2) \to 0} \frac{1}{(X-2)}$

$\lim\limits_{(X-2) \to 0^+} \frac{1}{X-2} = \infty$

$\lim\limits_{(X-2) \to 0^-} \frac{1}{X-2} = -\infty$

$\therefore$ The right limit $\neq$ the left limit

$\therefore \lim\limits_{x \to 2} \frac{1}{X-2}$ is not exist.

(2) $\lim\limits_{x \to 0} \frac{1}{X^2}$

$\lim\limits_{x \to 0^+} \left(\frac{1}{X}\right)^2 = \left(\lim\limits_{x \to 0^+} \frac{1}{X}\right)^2 = (\infty)^2 = \infty$

$\lim\limits_{x \to 0^-} \left(\frac{1}{X}\right)^2 = \left(\lim\limits_{x \to 0^-} \frac{1}{X}\right)^2 = (-\infty)^2 = \infty$

$\therefore$ The right limit = the left limit

$\therefore \lim\limits_{x \to 0} \frac{1}{X^2} = \infty$

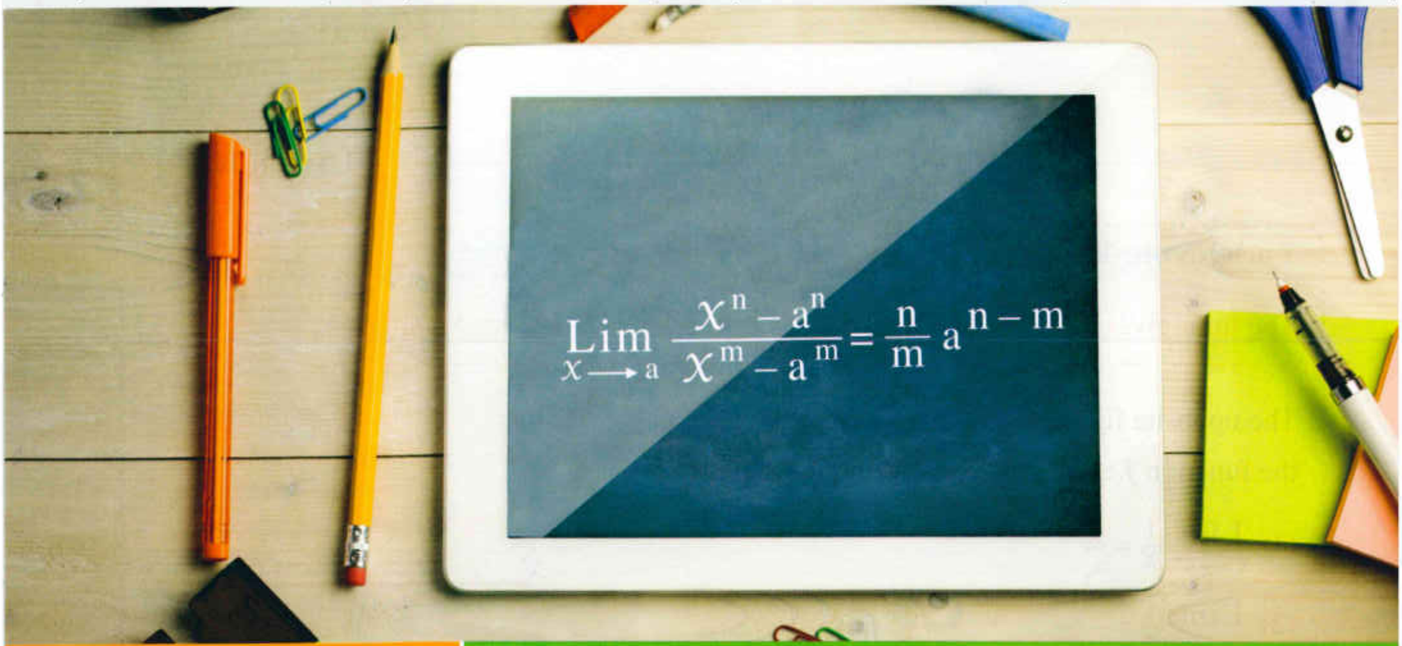
(3) $\lim\limits_{x \to -1} \frac{3X+4}{X+1} = \lim\limits_{x+1 \to 0} (3X+4) \times \frac{1}{(X+1)}$

$\lim\limits_{(X+1) \to 0^+} (3X+4) \times \frac{1}{(X+1)} = 1 \times \infty = \infty$

$\lim\limits_{(X+1) \to 0^-} (3X+4) \times \frac{1}{(X+1)} = 1 \times -\infty = -\infty$

$\therefore$ The right limit $\neq$ the left limit

$\therefore \lim\limits_{x \to -1} \frac{3X+4}{X+1}$ is not exist



Lesson Three — Theorem (4) "The law"

Theorem (4)

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = n\, a^{n-1} \quad \text{for every } n \in \mathbb{R} - \{0\}$$

To use this theorem , we must note that :

(1) The function must be in the form (or we can put it in the form) $\frac{x^n - a^n}{x - a}$

(2) The required is finding the limit when $x \longrightarrow a$

Example 1

Find each of the following :

(1) $\lim_{x \to 3} \frac{x^4 - 81}{x - 3}$

(2) $\lim_{x \to \sqrt{5}} \frac{x^7 - 125\sqrt{5}}{x - \sqrt{5}}$

(3) $\lim_{x \to -2} \frac{x^5 + 32}{x^2 + 5x + 6}$

(4) $\lim_{x \to \frac{1}{2}} \frac{32x^5 - 1}{2x - 1}$

Solution

(1) $\lim_{x \to 3} \frac{x^4 - 81}{x - 3} = \lim_{x \to 3} \frac{x^4 - 3^4}{x - 3} = 4 \times (3)^3 = 108$

Notice that : by direct substitution , we get $f(3) = \frac{\text{zero}}{\text{zero}}$

(2) $\lim_{x \to \sqrt{5}} \frac{x^7 - 125\sqrt{5}}{x - \sqrt{5}} = \lim_{x \to \sqrt{5}} \frac{x^7 - (\sqrt{5})^7}{x - \sqrt{5}} = 7 \times (\sqrt{5})^6 = 875$

(3) $\lim\limits_{x \to -2} \frac{x^5 + 32}{(x+2)(x+3)} = \lim\limits_{x \to -2} \frac{x^5 - (-32)}{x - (-2)} \times \lim\limits_{x \to -2} \frac{1}{x+3} = \lim\limits_{x \to -2} \frac{x^5 - (-2)^5}{x - (-2)} \times 1$

$= 5 \times (-2)^4 = 80$

(4) $\lim\limits_{x \to \frac{1}{2}} \frac{32x^5 - 1}{2x - 1} = \lim\limits_{x \to \frac{1}{2}} \frac{32\left[x^5 - \frac{1}{32}\right]}{2\left[x - \frac{1}{2}\right]}$

$= \lim\limits_{x \to \frac{1}{2}} 16 \times \frac{x^5 - \left(\frac{1}{2}\right)^5}{x - \frac{1}{2}} = 16 \lim\limits_{x \to \frac{1}{2}} \frac{x^5 - \left(\frac{1}{2}\right)^5}{x - \frac{1}{2}} = 16 \times 5 \times \left(\frac{1}{2}\right)^4 = 5$

Another solution :

As $x \longrightarrow \frac{1}{2}$, then $2x \longrightarrow 1$

$\therefore \lim\limits_{x \to \frac{1}{2}} \frac{32x^5 - 1}{2x - 1} = \lim\limits_{2x \to 1} \frac{(2x)^5 - 1^5}{(2x) - 1} = 5 \times (1)^4 = 5$

Third solution :

Let $2x = y$ as $x \longrightarrow \frac{1}{2}$, then $y \longrightarrow 1$

$\therefore \lim\limits_{x \to \frac{1}{2}} \frac{32x^5 - 1}{2x - 1} = \lim\limits_{x \to \frac{1}{2}} \frac{(2x)^5 - 1^5}{2x - 1} = \lim\limits_{y \to 1} \frac{y^5 - 1^5}{y - 1} = 5 \times (1)^4 = 5$

Corollaries

(1) $\lim\limits_{x \to 0} \frac{(x+a)^n - a^n}{x} = n\, a^{n-1}$

(2) $\lim\limits_{x \to a} \frac{x^n - a^n}{x^m - a^m} = \frac{n}{m} (a)^{n-m}$, where $n \in \mathbb{R} - \{0\}$, $m \in \mathbb{R} - \{0\}$

Example 2

Find each of the following :

(1) $\lim\limits_{x \to 1} \frac{x^5 - 1}{x^3 - 1}$

(2) $\lim\limits_{x \to -3} \frac{x^5 + 243}{x^4 - 81}$

(3) $\lim\limits_{x \to 3} \frac{x^4 - 27x}{3x^4 - 243}$

(4) $\lim\limits_{x \to 1} \frac{(x+1)^6 - 64}{(x+1)^3 - 8}$

Solution

(1) $\lim\limits_{x \to 1} \frac{x^5 - 1}{x^3 - 1} = \lim\limits_{x \to 1} \frac{x^5 - 1^5}{x^3 - 1^3} = \frac{5}{3} \times (1)^{5-3} = \frac{5}{3}$

(2) $\lim\limits_{x \to -3} \frac{x^5 + 243}{x^4 - 81} = \lim\limits_{x \to -3} \frac{x^5 - (-243)}{x^4 - 81} = \lim\limits_{x \to -3} \frac{x^5 - (-3)^5}{x^4 - (-3)^4} = \frac{5}{4} \times (-3)^{5-4} = \frac{-15}{4}$

(3) $\lim\limits_{x \to 3} \frac{x^4 - 27x}{3x^4 - 243} = \lim\limits_{x \to 3} \frac{x(x^3 - 27)}{3(x^4 - 81)}$

$$= \lim\limits_{x \to 3} \frac{x}{3} \times \lim\limits_{x \to 3} \frac{x^3 - 3^3}{x^4 - 3^4} = \frac{3}{3} \times \frac{3}{4} \times 3^{3-4} = \frac{1}{4}$$

(4) $\because x \longrightarrow 1 \qquad \therefore x + 1 \longrightarrow 2$

$\therefore \lim\limits_{x \to 1} \frac{(x+1)^6 - 64}{(x+1)^3 - 8} = \lim\limits_{x+1 \to 2} \frac{(x+1)^6 - 2^6}{(x+1)^3 - 2^3} = \frac{6}{3}(2)^{6-3} = 16$

Another solution by using factorization :

$$\lim\limits_{x \to 1} \frac{(x+1)^6 - 64}{(x+1)^3 - 8} = \lim\limits_{x \to 1} \frac{[(x+1)^3 - 8][(x+1)^3 + 8]}{(x+1)^3 - 8}$$

$$= \lim\limits_{x \to 1} \left((x+1)^3 + 8\right) = 8 + 8 = 16$$

Example 3

Find each of the following :

(1) $\lim\limits_{x \to 0} \frac{(x+5)^4 - 625}{x}$ (2) $\lim\limits_{x \to 6} \frac{(x-5)^7 - 1}{x - 6}$ (3) $\lim\limits_{h \to 0} \frac{(a+2h)^6 - a^6}{5h}$

Solution

(1) $\lim\limits_{x \to 0} \frac{(x+5)^4 - 625}{x} = \lim\limits_{x \to 0} \frac{(x+5)^4 - (5)^4}{x} = 4 \times 5^3 = 500$

(2) $\lim\limits_{x \to 6} \frac{(x-5)^7 - 1}{x - 6} = \lim\limits_{x-5 \to 1} \frac{(x-5)^7 - 1^7}{(x-5) - 1} = 7 \times (1)^6 = 7$

(3) $\lim\limits_{h \to 0} \frac{(a+2h)^6 - a^6}{5h} = \lim\limits_{h \to 0} \frac{[(a+2h)^6 - a^6] \times \frac{2}{5}}{5h \times \frac{2}{5}} = \lim\limits_{2h \to 0} \frac{\frac{2}{5}[(a+2h)^6 - a^6]}{2h}$

$$= \frac{2}{5} \times 6 \times a^5 = \frac{12}{5} a^5$$

Remarks

- $\sqrt[n]{a} = a^{\frac{1}{n}}$ where $n \in \mathbb{Z}^+ - \{1\}$, $a \in \mathbb{R}^+$

  For example :

  $\sqrt{x} = x^{\frac{1}{2}}$, $\sqrt[3]{8} = 8^{\frac{1}{3}} = 2$

- $\sqrt[n]{a^m} = a^{\frac{m}{n}}$ where $n \in \mathbb{Z}^+ - \{1\}$, $a \in \mathbb{R}^+$, $m \in \mathbb{Z}$

  For example :

  $\sqrt[4]{x^5} = x^{\frac{5}{4}}$, $\sqrt[5]{64} = \sqrt[5]{2^6} = 2^{\frac{6}{5}}$

Example 4

Find each of the following :

(1) $\lim\limits_{x \to 9} \frac{\sqrt{x}-3}{x-9}$ (2) $\lim\limits_{x \to 1} \frac{\sqrt[5]{x}-1}{\sqrt[3]{x}-1}$ (3) $\lim\limits_{x \to 16} \frac{\sqrt[4]{x^5}-32}{\sqrt{x^3}-64}$

Solution

(1) $\lim\limits_{x \to 9} \frac{\sqrt{x}-3}{x-9} = \lim\limits_{x \to 9} \frac{x^{\frac{1}{2}}-9^{\frac{1}{2}}}{x-9} = \frac{1}{2} \times 9^{\frac{1}{2}-1} = \frac{1}{2} \times 9^{-\frac{1}{2}} = \frac{1}{2\sqrt{9}} = \frac{1}{6}$

Another solution by using factorization :

$$\lim\limits_{x \to 9} \frac{\sqrt{x}-3}{x-9} = \lim\limits_{x \to 9} \frac{\sqrt{x}-3}{(\sqrt{x}-3)(\sqrt{x}+3)} = \lim\limits_{x \to 9} \frac{1}{\sqrt{x}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}$$

(2) $\lim\limits_{x \to 1} \frac{\sqrt[5]{x}-1}{\sqrt[3]{x}-1} = \lim\limits_{x \to 1} \frac{x^{\frac{1}{5}}-1^{\frac{1}{5}}}{x^{\frac{1}{3}}-1^{\frac{1}{3}}} = \frac{\frac{1}{5}}{\frac{1}{3}} \times 1^{\frac{1}{5}-\frac{1}{3}} = \frac{3}{5}$

(3) $\lim\limits_{x \to 16} \frac{\sqrt[4]{x^5}-32}{\sqrt{x^3}-64} = \lim\limits_{x \to 16} \frac{x^{\frac{5}{4}}-(16)^{\frac{5}{4}}}{x^{\frac{3}{2}}-(16)^{\frac{3}{2}}}$

$= \frac{\frac{5}{4}}{\frac{3}{2}} \times (16)^{\frac{5}{4}-\frac{3}{2}}$

$= \frac{5}{6} \times 16^{-\frac{1}{4}} = \frac{5}{12}$

Notice that :

$16^{\frac{5}{4}} = (2^4)^{\frac{5}{4}} = 2^{4 \times \frac{5}{4}} = 2^5 = 32$

i.e. $32 = 16^{\frac{5}{4}}$

also , $64 = 16^{\frac{3}{2}}$

Example 5

Find each of the following :

(1) $\lim\limits_{x \to 3} \frac{x^3+x^2-36}{x^4-81}$ (2) $\lim\limits_{x \to 1} \frac{\sqrt[4]{2x+14}-2}{x-1}$ (3) $\lim\limits_{x \to 2} \frac{\sqrt[3]{3x+2}-2}{\sqrt[5]{x-3}+1}$

Solution

(1) $\lim\limits_{x \to 3} \frac{x^3+x^2-36}{x^4-81} = \lim\limits_{x \to 3} \frac{x^3-27+x^2-9}{x^4-81} = \lim\limits_{x \to 3} \frac{x^3-3^3}{x^4-3^4} + \lim\limits_{x \to 3} \frac{x^2-3^2}{x^4-3^4}$

$= \frac{3}{4}(3)^{3-4} + \frac{2}{4}(3)^{2-4} = \frac{1}{4} + \frac{1}{18} = \frac{11}{36}$

(2) $\lim\limits_{x \to 1} \dfrac{\sqrt[4]{2x+14}-2}{x-1} = \lim\limits_{x \to 1} \dfrac{(2x+14)^{\frac{1}{4}}-(16)^{\frac{1}{4}}}{x-1} \times \dfrac{2}{2}$

$$= 2 \lim\limits_{(2x+14) \to 16} \frac{(2x+14)^{\frac{1}{4}}-16^{\frac{1}{4}}}{(2x+14)-16} = 2 \times \frac{1}{4} \times (16)^{\frac{1}{4}-1}$$

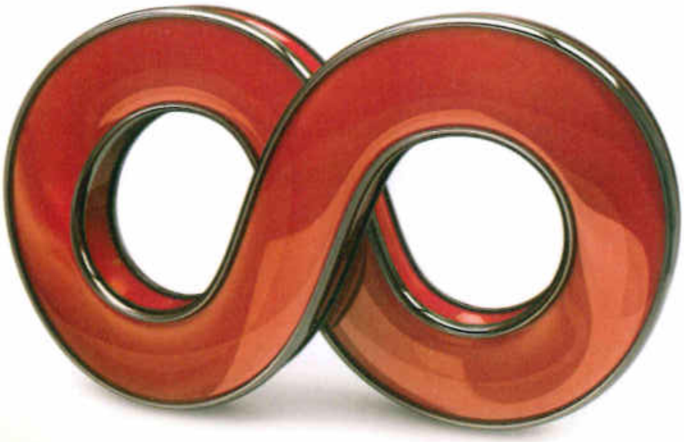
$$= 2 \times \frac{1}{4} \times \frac{1}{8} = \frac{1}{16}$$

(3) $\lim\limits_{x \to 2} \dfrac{\sqrt[3]{3x+2}-2}{\sqrt[5]{x-3}+1} = \lim\limits_{x \to 2} \dfrac{(3x+2)^{\frac{1}{3}}-8^{\frac{1}{3}}}{3x+2-8} \times \dfrac{3x+2-8}{(x-3)^{\frac{1}{5}}+1}$

$$= \lim\limits_{3x+2 \to 8} \frac{(3x+2)^{\frac{1}{3}}-8^{\frac{1}{3}}}{(3x+2)-8} \times \lim\limits_{x \to 2} \frac{3(x-2)}{(x-3)^{\frac{1}{5}}+1}$$

$$= \frac{1}{3} \times (8)^{\frac{1}{3}-1} \times 3 \lim\limits_{x \to 2} \frac{x-2}{(x-3)^{\frac{1}{5}}-(-1)^{\frac{1}{5}}}$$

$$= \frac{1}{4} \lim\limits_{x-3 \to -1} \frac{(x-3)-(-1)}{(x-3)^{\frac{1}{5}}-(-1)^{\frac{1}{5}}} = \frac{1}{4} \times \frac{1}{\frac{1}{5}} \times (-1)^{1-\frac{1}{5}} = \frac{5}{4}$$



Lesson Four — Limit of the function at infinity

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The meaning of finding the limit of a function at infinity , is studying the behavior of this function when X (the independent variable) takes very large values.

If $f(X)$ approaches a real number ℓ as X tends to infinity , then we say that $f(X)$ has a limit «ℓ» at infinity , and we write it as : $\lim\limits_{X \to \infty} f(X) = \ell$

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Illustrated Example 1

If $f : f(X) = \dfrac{2X+1}{X}$ and we want to study the behavior of the function f when X takes a very large values tend to ∞ let X takes the values : 1 , 10 , 100 , 1000 , 10000 , ...

We get the following table :

X	1	10	100	1000	10000	.....
$f(X) = \dfrac{2X+1}{X}$	3	2.1	2.01	2.001	2.0001	.....

From this table :

It's clear that when X takes values gradually increasing , we note that $f(X)$ approaches to 2 , then $f(X) \longrightarrow 2$ as $X \longrightarrow \infty$

and we write $\lim\limits_{X \to \infty} f(X) = 2$

Notice that :

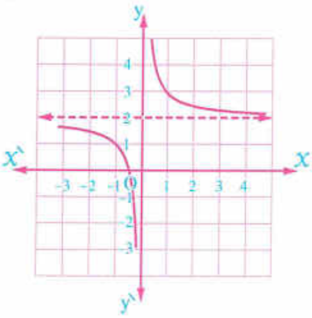
We can't get this result by the direct substitution which gives $f(X) = \frac{\infty}{\infty}$ (unspecified)

The graphical solution

At drawing the function

$f : f(X) = \dfrac{2X+1}{X}$

$= 2 + \dfrac{1}{X}$



From the graph , we notice that : when $X \longrightarrow \infty$, then $f(X) \longrightarrow 2$

Illustrated Example 2

If $f : f(x) = \frac{1}{x}$ and we want to study the behavior of this function as $x \longrightarrow \infty$

Form the table :

x	1	10	100	1000	10000	.....
$f(x) = \frac{1}{x}$	1	0.1	0.01	0.001	0.0001	.....

From this table we notice that :

as $x \longrightarrow \infty$, then $f(x) \longrightarrow 0$

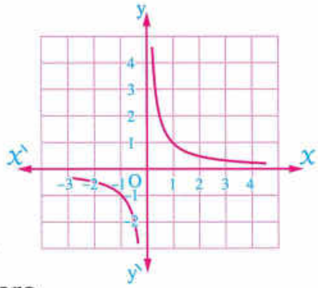
, then we can write $\lim\limits_{x \to \infty} \frac{1}{x} = 0$

This example leads us to the following theorem.

The graphical solution

At drawing the function $f : f(x) = \frac{1}{x}$

From the graph , we notice that : when $x \longrightarrow \infty$, then $f(x) \longrightarrow$ zero



Theorem (5)

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

Corollaries

If $a \in \mathbb{R}$, then :

(1) $\lim\limits_{x \to \infty} \frac{a}{x} = \text{zero}$

(2) $\lim\limits_{x \to \infty} \frac{a}{x^n} = \text{zero}$, $n \in \mathbb{R}^+$

Basic rules

* $\lim\limits_{x \to \infty} c = c$ where c is a constant
* $\lim\limits_{x \to \infty} x^n = \infty$ where n is a positive number
* Theorem (2) which is related by the limit of sum , difference , multiplying or dividing two functions at $x = a$ that we studied before is satisfied also when we put $x \longrightarrow \infty$ instead of $x \longrightarrow a$

Example 1

Find each of the following :

(1) $\lim\limits_{x \to \infty} \left(\frac{1}{x} + 2\right)$ (2) $\lim\limits_{x \to \infty} \left(3 - \frac{1}{x^2}\right)$

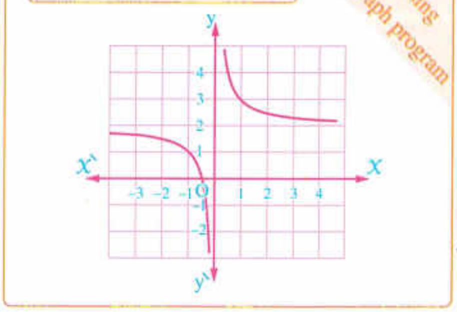
Solution

(1) $\lim\limits_{x \to \infty} \left(\frac{1}{x} + 2\right)$

$= \lim\limits_{x \to \infty} \frac{1}{x} + \lim\limits_{x \to \infty} 2$

$= 0 + 2 = 2$

The graph is for one's guidance only

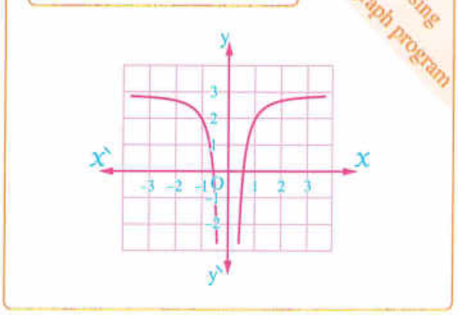


(2) $\lim\limits_{x \to \infty} \left(3 - \frac{1}{x^2}\right)$

$= \lim\limits_{x \to \infty} 3 - \lim\limits_{x \to \infty} \frac{1}{x^2}$

$= 3 - 0 = 3$

The graph is for one's guidance only



Getting the limit of a rational function at infinity

If the direct substitution by $x = \infty$ gives $\frac{\infty}{\infty}$ we divide each of numerator and denominator by x raised to the higher power in the denominator (degree of denominator) , then we use the theorem and its corollaries to get the limit (if it exists).

Example 2

Find each of the following :

(1) $\lim\limits_{x \to \infty} \frac{2x - 5}{3x - 7}$

(2) $\lim\limits_{x \to \infty} \frac{5x^2 - 3x + 6}{2x - 7x^2}$

(3) $\lim\limits_{x \to \infty} \frac{3x^2 - 5x}{2x^3 - 6x^2 + 4x - 1}$

(4) $\lim\limits_{x \to \infty} \frac{x^5 - 2x^2}{x^4 + 3x^3 - 1}$

Solution

(1) Dividing both numerator and denominator by x

$$\therefore \lim_{x \to \infty} \frac{2x-5}{3x-7} = \lim_{x \to \infty} \frac{2-\frac{5}{x}}{3-\frac{7}{x}} = \frac{2-0}{3-0} = \frac{2}{3}$$

(2) Dividing both numerator and denominator by x^2

$$\therefore \lim_{x \to \infty} \frac{5x^2-3x+6}{2x-7x^2} = \lim_{x \to \infty} \frac{5-\frac{3}{x}+\frac{6}{x^2}}{\frac{2}{x}-7} = -\frac{5}{7}$$

(3) Dividing both numerator and denominator by x^3

$$\therefore \lim_{x \to \infty} \frac{3x^2-5x}{2x^3-6x^2+4x-1} = \lim_{x \to \infty} \frac{\frac{3}{x}-\frac{5}{x^2}}{2-\frac{6}{x}+\frac{4}{x^2}-\frac{1}{x^3}} = \frac{\text{zero}}{2} = \text{zero}$$

(4) Dividing both numerator and denominator by x^4

$$\therefore \lim_{x \to \infty} \frac{x^5-2x^2}{x^4+3x^3-1} = \lim_{x \to \infty} \frac{x-\frac{2}{x^2}}{1+\frac{3}{x}-\frac{1}{x^4}} = \frac{\infty-0}{1+0-0} = \infty$$

Important remark

At finding $\lim_{x \to \infty} \frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are polynomial functions , then :

(1) The limit = a real number not equal to zero "if the degree of the numerator is equal to the degree of the denominator"

(2) The limit = zero "if the degree of the numerator is less than the degree of the denominator"

(3) The limit = $\pm\infty$ "if the degree of the numerator is greater than the degree of the denominator"

Example 3

Find each of the following :

(1) $\lim_{x \to \infty} \frac{(x-1)(x^2+1)}{x^2(5x-1)}$

(2) $\lim_{x \to \infty} \frac{(3x^3+2)^2(x^2-1)^3}{x^5(x+1)^7}$

Solution

(1) Dividing both numerator and denominator by x^3

$$\therefore \lim_{x \to \infty} \frac{(x-1)(x^2+1)}{x^2(5x-1)} = \lim_{x \to \infty} \frac{\left(1-\frac{1}{x}\right)\left(1+\frac{1}{x^2}\right)}{1\left(5-\frac{1}{x}\right)} = \frac{1 \times 1}{1 \times 5} = \frac{1}{5}$$

(2) Dividing both numerator and denominator by x^{12}

$$\therefore \lim_{x \to \infty} \frac{(3x^3+2)^2(x^2-1)^3}{x^5(x+1)^7} = \lim_{x \to \infty} \frac{\left(3+\frac{2}{x^3}\right)^2\left(1-\frac{1}{x^2}\right)^3}{1\left(1+\frac{1}{x}\right)^7} = \frac{9 \times 1}{1} = 9$$

Example 4

Find each of the following :

(1) $\lim_{x \to \infty} \frac{2x^3-9}{|3x|^3+7}$

(2) $\lim_{x \to \infty} \frac{5x-6}{\sqrt{9x^2+7}}$

(3) $\lim_{x \to \infty} \frac{\sqrt[3]{8x^3-5x+1}}{3x-2}$

(4) $\lim_{x \to \infty} \frac{\sqrt[5]{x^5+x^3}}{\sqrt[7]{x^7-x^2}}$

(5) $\lim_{x \to \infty} \left(\sqrt{x^2-x+1}-\sqrt{x^2+x+1}\right)$

Solution

(1) $\because x \to \infty \qquad \therefore |x| = x$

$\therefore$ The limit $= \lim_{x \to \infty} \frac{2x^3-9}{27x^3+7}$

Dividing both numerator and denominator by x^3

$$\therefore \lim_{x \to \infty} \frac{2x^3-9}{27x^3+7} = \lim_{x \to \infty} \frac{2-\frac{9}{x^3}}{27+\frac{7}{x^3}} = \frac{2}{27}$$

(2) Dividing both numerator and denominator by $x = \sqrt{x^2}$

$$\therefore \lim_{x \to \infty} \frac{5x-6}{\sqrt{9x^2+7}} = \lim_{x \to \infty} \frac{5-\frac{6}{x}}{\sqrt{9+\frac{7}{x^2}}} = \frac{5}{\sqrt{9}} = \frac{5}{3}$$

Notice that :

When $x \to \infty$

, then : $x = |x| = \sqrt{x^2}$

$= \sqrt[3]{x^3} = \sqrt[4]{x^4} = \ldots\ldots$

(3) Dividing both numerator and denominator by $x = \sqrt[3]{x^3}$

$$\therefore \lim_{x \to \infty} \frac{\sqrt[3]{8x^3-5x+1}}{3x-2} = \lim_{x \to \infty} \frac{\sqrt[3]{8-\frac{5}{x^2}+\frac{1}{x^3}}}{3-\frac{2}{x}} = \frac{\sqrt[3]{8}}{3} = \frac{2}{3}$$

(4) Dividing both numerator and denominator by $x = \sqrt[5]{x^5} = \sqrt[7]{x^7}$

$$\therefore \lim_{x \to \infty} \frac{\sqrt[5]{x^5+x^3}}{\sqrt[7]{x^7-x^2}} = \lim_{x \to \infty} \frac{\sqrt[5]{1+\frac{1}{x^2}}}{\sqrt[7]{1-\frac{1}{x^5}}} = \frac{1}{1} = 1$$

(5) $\lim\limits_{x \to \infty} \left(\sqrt{x^2 - x + 1} - \sqrt{x^2 + x + 1}\right)$

$= \lim\limits_{x \to \infty} \frac{\sqrt{x^2 - x + 1} - \sqrt{x^2 + x + 1}}{1} \times \frac{\sqrt{x^2 - x + 1} + \sqrt{x^2 + x + 1}}{\sqrt{x^2 - x + 1} + \sqrt{x^2 + x + 1}}$

$= \lim\limits_{x \to \infty} \frac{(x^2 - x + 1) - (x^2 + x + 1)}{\sqrt{x^2 - x + 1} + \sqrt{x^2 + x + 1}} = \lim\limits_{x \to \infty} \frac{-2x}{\sqrt{x^2 - x + 1} + \sqrt{x^2 + x + 1}}$

"dividing both numerator and denominator by $x = \sqrt{x^2}$"

$= \lim\limits_{x \to \infty} \frac{-2}{\sqrt{1 - \frac{1}{x} + \frac{1}{x^2}} + \sqrt{1 + \frac{1}{x} + \frac{1}{x^2}}} = \frac{-2}{1+1} = \frac{-2}{2} = -1$

Example 5

Find each of the following :

(1) $\lim\limits_{x \to \infty} \left(\frac{5}{x^2} - \frac{7x^3}{x^4} + 2\right)$

(2) $\lim\limits_{x \to \infty} (4x^2 + 3x + 7)$

(3) $\lim\limits_{x \to \infty} (5x^4 - x^3 - 3)$

(4) $\lim\limits_{x \to \infty} (4x^2 - 5x^3 + 13)$

Solution

(1) $\lim\limits_{x \to \infty} \left(\frac{5}{x^2} - \frac{7x^3}{x^4} + 2\right) = \text{zero} - \text{zero} + 2 = 2$

(2) $\lim\limits_{x \to \infty} (4x^2 + 3x + 7)$ [Notice that the direct substitution gives $\infty + \infty + 7$]

$\therefore \lim\limits_{x \to \infty} (4x^2 + 3x + 7) = \infty$

(3) $\lim\limits_{x \to \infty} (5x^4 - x^3 - 3)$ [Notice that the direct substitution gives $\infty - \infty - 3$ *i.e.* unspecified quantity]

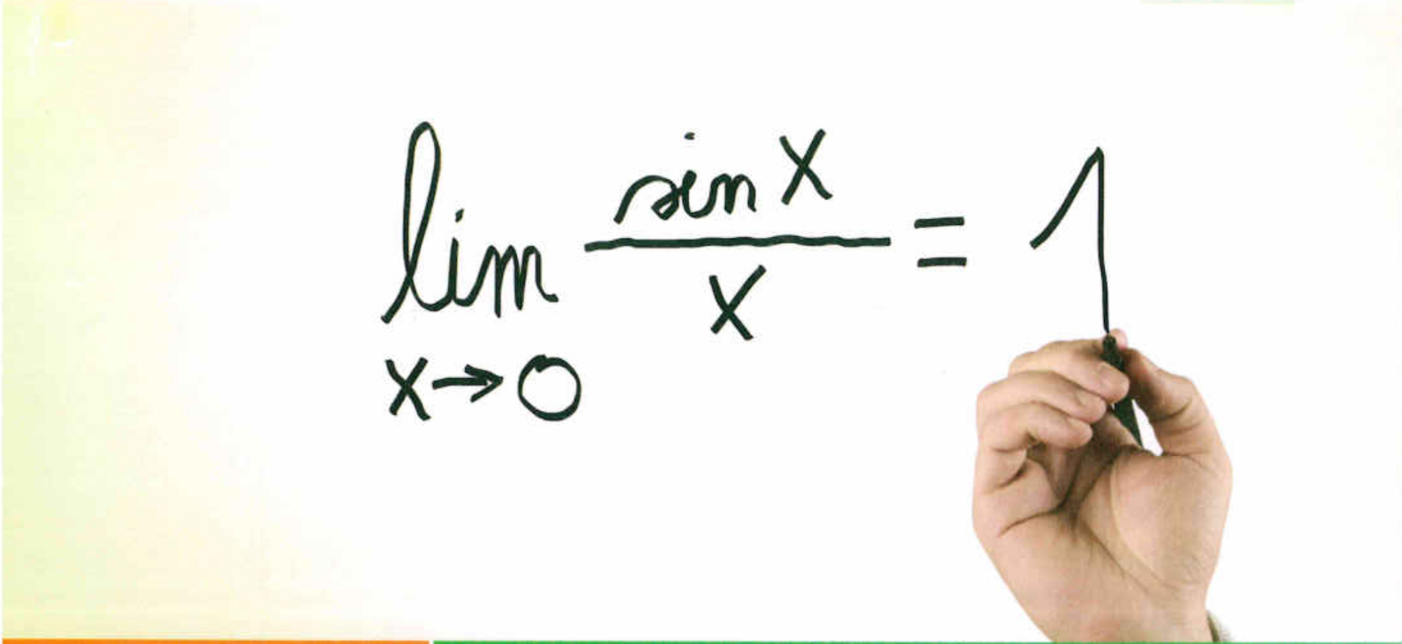
$= \lim\limits_{x \to \infty} x^4 \left(5 - \frac{1}{x} - \frac{3}{x^4}\right)$ [Taking x raised to the greatest power as a common factor]

$= \lim\limits_{x \to \infty} x^4 \times \lim\limits_{x \to \infty} \left(5 - \frac{1}{x} - \frac{3}{x^4}\right) = \infty \times 5 = \infty$

(4) $\lim\limits_{x \to \infty} (4x^2 - 5x^3 + 13)$ [Notice that the direct substitution gives $\infty - \infty + 13$ *i.e.* unspecified quantity]

$= \lim\limits_{x \to \infty} x^3 \left(\frac{4}{x} - 5 + \frac{13}{x^3}\right)$ [Taking x raised to the greatest power as a common factor]

$= \lim\limits_{x \to \infty} x^3 \times \lim\limits_{x \to \infty} \left(\frac{4}{x} - 5 + \frac{13}{x^3}\right) = \infty \times -5 = -\infty$



Lesson Five — Limits of trigonometric functions

Theorem

If x is the measure of an angle in radians , then :

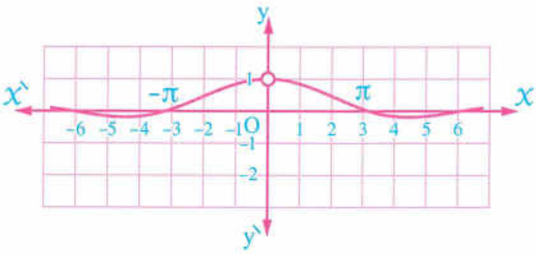
(1) $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$

(2) $\lim\limits_{x \to 0} \frac{\tan x}{x} = 1$

* At studying the values of the function $f : f(x) = \frac{\sin x}{x}$ when $x \longrightarrow 0$ where x is the measure of the angle in radians , we form the following table :

x	± 1	± 0.1	± 0.01	zero
$\frac{\sin x}{x}$	0.8415	0.9983	.... ⟶	1

And we notice that , when x approaches to zero , the ratio $\frac{\sin x}{x}$ approaches to the whole one and the opposite figure shows that graphically.

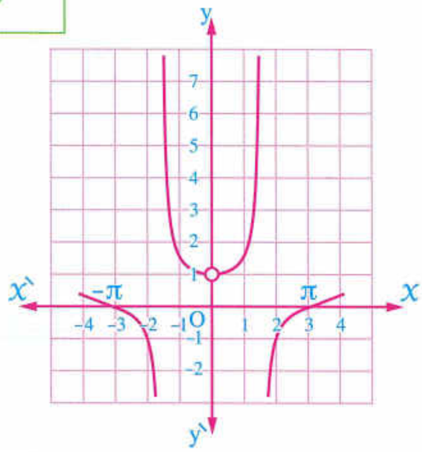


i.e. $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$

* Similarly , at studying the values of the function $f : f(x) = \frac{\tan x}{x}$ when $x \longrightarrow 0$ where x is the measure of the angle in radians , we form the following table :

x	± 1	± 0.1	± 0.01	zero
$\frac{\tan x}{x}$	1.5574	1.0033	.... →	1

And we notice that , when x approaches to zero , the ratio $\frac{\tan x}{x}$ approaches to the whole one and the opposite figure shows that graphically.



i.e. $\lim_{x \to 0} \frac{\tan x}{x} = 1$

Remarks

1. $\lim_{x \to 0} \frac{x}{\sin x} = 1$ **2.** $\lim_{x \to 0} \frac{x}{\tan x} = 1$ where x is in radian

Corollary (1)

If x is in radian, then :

(1) $\lim_{x \to 0} \frac{\sin a x}{x} = a$ and hence $\lim_{x \to 0} \frac{\sin a x}{b x} = \frac{a}{b}$, $\lim_{x \to 0} \frac{b x}{\sin a x} = \frac{b}{a}$

(2) $\lim_{x \to 0} \frac{\tan a x}{x} = a$ and hence $\lim_{x \to 0} \frac{\tan a x}{b x} = \frac{a}{b}$, $\lim_{x \to 0} \frac{b x}{\tan a x} = \frac{b}{a}$

For example :

* $\lim_{x \to 0} \frac{\sin 3 x}{x} = 3$
* $\lim_{x \to 0} \frac{\tan 2 x}{x} = 2$
* $\lim_{x \to 0} \frac{-5 x}{\sin 3 x} = \frac{-5}{3}$
* $\lim_{x \to 0} \frac{\sin 2 x}{5 x} = \frac{2}{5}$
* $\lim_{x \to 0} \frac{\tan 4 x}{3 x} = \frac{4}{3}$
* $\lim_{x \to 0} \frac{x}{\tan 7 x} = \frac{1}{7}$

Remarks

1. $\sin x$ and $\cos x$ are defined for all values of $x \in \mathbb{R}$, but $\tan x$ is defined for all values of x except when $x = \frac{2n+1}{2}\pi$, $n \in \mathbb{Z}$, then :

* $\lim_{x \to a} \sin x = \sin a$, $a \in \mathbb{R}$
* $\lim_{x \to a} \cos x = \cos a$, $a \in \mathbb{R}$
* $\lim_{x \to a} \tan x = \tan a$, $a \neq \frac{2n+1}{2}\pi$, $n \in \mathbb{Z}$

2. Notice that : $\lim_{x \to 0} \frac{\sin a x^2}{x^2}$ is different than $\lim_{x \to 0} \frac{\sin^2 a x}{x^2}$

Where $\lim_{x \to 0} \frac{\sin a x^2}{x^2} = a$ but $\lim_{x \to 0} \frac{\sin^2 a x}{x^2} = \lim_{x \to 0} \left(\frac{\sin a x}{x}\right)^2 = a^2$

For example : $\lim_{x \to 0} \frac{\sin 3 x^2}{x^2} = 3$, but $\lim_{x \to 0} \frac{\sin^2 3 x}{x^2} = \lim_{x \to 0} \left(\frac{\sin 3 x}{x}\right)^2 = 3^2 = 9$

Example 1

Find each of the following :

(1) $\lim\limits_{x\to 0} \frac{\sin 2x}{3x}$

(2) $\lim\limits_{x\to 0} \frac{4x}{\sin 5x}$

(3) $\lim\limits_{x\to 0} \frac{\sin 5x}{2\sin 3x}$

(4) $\lim\limits_{x\to 0} \frac{\tan x}{\sin 2x}$

Solution

(1) $\lim\limits_{x\to 0} \frac{\sin 2x}{3x} = \frac{2}{3}$

(2) $\lim\limits_{x\to 0} \frac{4x}{\sin 5x} = \frac{4}{5}$

(3) $\lim\limits_{x\to 0} \frac{\sin 5x}{2\sin 3x} = \frac{1}{2} \lim\limits_{x\to 0} \frac{\frac{\sin 5x}{x}}{\frac{\sin 3x}{x}} = \frac{1}{2} \times \frac{5}{3} = \frac{5}{6}$

(4) $\lim\limits_{x\to 0} \frac{\tan x}{\sin 2x} = \lim\limits_{x\to 0} \frac{\frac{\tan x}{x}}{\frac{\sin 2x}{x}} = \frac{1}{2}$

Example 2

Find each of the following :

(1) $\lim\limits_{x\to 0} \frac{\tan^2 2x}{3x^2}$

(2) $\lim\limits_{x\to 0} \frac{\sin 3x^2 \cos x^2}{5x^2}$

Solution

(1) $\lim\limits_{x\to 0} \frac{\tan^2 2x}{3x^2} = \frac{1}{3} \lim\limits_{x\to 0} \left(\frac{\tan 2x}{x}\right)^2 = \frac{1}{3} \times 2^2 = \frac{4}{3}$

(2) $\lim\limits_{x\to 0} \frac{\sin 3x^2 \cos x^2}{5x^2} = \lim\limits_{x\to 0} \frac{\sin 3x^2}{5x^2} \times \lim\limits_{x\to 0} \cos x^2 = \frac{3}{5} \times 1 = \frac{3}{5}$

Example 3

Find each of the following :

(1) $\lim\limits_{x\to 0} \frac{2x - 3\sin x}{3x + \tan x}$

(2) $\lim\limits_{x\to 0} \frac{x^2 + 5\tan^2 3x}{4x^2 - \sin 2x^2}$

Solution

(1) Dividing both numerator and denominator by x

$\therefore \lim\limits_{x\to 0} \frac{2x - 3\sin x}{3x + \tan x} = \lim\limits_{x\to 0} \frac{2 - \frac{3\sin x}{x}}{3 + \frac{\tan x}{x}} = \frac{2-3}{3+1} = \frac{-1}{4}$

(2) Dividing both numerator and denominator by x^2

$$\therefore \lim_{x\to 0} \frac{x^2 + 5\tan^2 3x}{4x^2 - \sin 2x^2} = \lim_{x\to 0} \frac{1 + 5\left(\frac{\tan 3x}{x}\right)^2}{4 - \frac{\sin 2x^2}{x^2}} = \frac{1 + 5 \times 9}{4 - 2} = \frac{46}{2} = 23$$

Corollary (2)

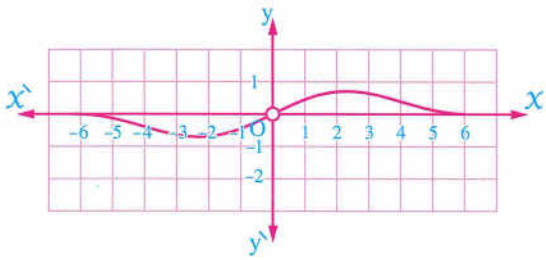
$$\lim_{x\to 0} \frac{1 - \cos x}{x} = 0$$

Proof of corollary (2)

$$\lim_{x\to 0} \frac{1 - \cos x}{x} = \lim_{x\to 0} \frac{1 - \cos x}{x} \times \frac{1 + \cos x}{1 + \cos x}$$

$$= \lim_{x\to 0} \frac{1 - \cos^2 x}{x(1 + \cos x)} = \lim_{x\to 0} \frac{\sin^2 x}{x(1 + \cos x)} = \lim_{x\to 0} \frac{\sin x}{x} \times \lim_{x\to 0} \frac{\sin x}{1 + \cos x} = 1 \times \text{zero} = \text{zero}$$

* The opposite figure represents the function $f : f(x) = \frac{1 - \cos x}{x}$ and we notice in the figure that when x approaches to zero , the ratio $\frac{1 - \cos x}{x}$ approaches to zero also where x is in radian measure.



i.e. $\lim_{x\to 0} \frac{1 - \cos x}{x} = 0$

Example 4

Find each of the following :

(1) $\lim_{x\to 0} \frac{1 - \cos x}{\sin x}$

(2) $\lim_{x\to 0} \frac{1 - \cos x}{x^2}$

Solution

(1) $\lim_{x\to 0} \frac{1 - \cos x}{\sin x} = \lim_{x\to 0} \left(\frac{1 - \cos x}{x} \times \frac{x}{\sin x}\right) = \lim_{x\to 0} \frac{1 - \cos x}{x} \times \lim_{x\to 0} \frac{x}{\sin x} = 0 \times 1 = 0$

(2) $\lim_{x\to 0} \frac{1 - \cos x}{x^2} = \lim_{x\to 0} \left(\frac{1 - \cos x}{x^2} \times \frac{1 + \cos x}{1 + \cos x}\right)$

$= \lim_{x\to 0} \left(\frac{1 - \cos^2 x}{x^2} \times \frac{1}{1 + \cos x}\right)$

$= \lim_{x\to 0} \left(\frac{\sin^2 x}{x^2} \times \frac{1}{1 + \cos x}\right)$

$= \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^2 \times \lim_{x\to 0} \left(\frac{1}{1 + \cos x}\right) = (1)^2 \times \frac{1}{1 + 1} = \frac{1}{2}$

Remember that

- $\sin^2 x + \cos^2 x = 1$
- $1 + \tan^2 x = \sec^2 x$
- $1 + \cot^2 x = \csc^2 x$

Example 5

Find each of the following :

(1) $\lim\limits_{x \to 4} \frac{\sin(3x-12)}{x-4}$

(2) $\lim\limits_{2x \to \pi} \frac{\cos x}{2x-\pi}$

(3) $\lim\limits_{x \to \pi} \frac{\tan x}{\pi - x}$

(4) $\lim\limits_{x \to \infty} x \sin x^{-1}$

Solution

(1) $\lim\limits_{x \to 4} \frac{\sin(3x-12)}{x-4} = \lim\limits_{(x-4) \to 0} \frac{\sin[3(x-4)]}{(x-4)} = 3$

(2) $\lim\limits_{2x \to \pi} \frac{\cos x}{2x-\pi} = \lim\limits_{\left(\frac{\pi}{2}-x\right) \to 0} \frac{\sin\left(\frac{\pi}{2}-x\right)}{-2\left(\frac{\pi}{2}-x\right)} = -\frac{1}{2}$

(3) $\lim\limits_{x \to \pi} \frac{\tan x}{\pi - x} = \lim\limits_{(\pi - x) \to 0} \frac{-\tan(\pi - x)}{(\pi - x)} = -1$

(4) $\lim\limits_{x \to \infty} x \sin x^{-1} = \lim\limits_{\frac{1}{x} \to 0} \frac{\sin \frac{1}{x}}{\frac{1}{x}} = 1$

Example 6

Find each of the following :

(1) $\lim\limits_{x \to 0} 4x \csc 5x$

(2) $\lim\limits_{x \to 0} \frac{\cos\left(\frac{\pi}{2}-x\right)}{x}$

(3) $\lim\limits_{x \to \frac{\pi}{2}} \frac{\cos x}{x - \frac{\pi}{2}}$

Solution

(1) $\lim\limits_{x \to 0} 4x \csc 5x = \lim\limits_{x \to 0} \frac{4x}{\sin 5x} = \frac{4}{5}$

(2) $\lim\limits_{x \to 0} \frac{\cos\left(\frac{\pi}{2}-x\right)}{x} = \lim\limits_{x \to 0} \frac{\sin x}{x} = 1$

(3) $\lim\limits_{x \to \frac{\pi}{2}} \frac{\cos x}{x - \frac{\pi}{2}} = \lim\limits_{\left(\frac{\pi}{2}-x\right) \to 0} \frac{\sin\left(\frac{\pi}{2}-x\right)}{-\left(\frac{\pi}{2}-x\right)} = -1$

Knowledge

If x is the measure of an angle in the degree measure , then :

(1) $\lim\limits_{x \to 0} \frac{\sin x}{x} = \frac{\pi}{180}$

(2) $\lim\limits_{x \to 0} \frac{\tan x}{x} = \frac{\pi}{180}$



Lesson Six — Existence of the limit of a piecewise function

We notice that the function $f : f(X) = \begin{cases} 2X + 1 , & X < 2 \\ 3 - X^2 , & X \geq 2 \end{cases}$

is defined by two rules and each one of them is specified to a part from the domain of the function.

i.e.

$f(X) = 2X + 1$ for the values of $X \in]-\infty , 2[$

$f(X) = 3 - X^2$ for the values of $X \in [2 , \infty[$

For example :

$f(-3) = 2(-3) + 1 = -5$, $f(4) = 3 - (4)^2 = -13$

and we explained before how to find the right and the left limit for these functions graphically , but now we explain how to verify that algebraically (if possible).

Definition

The function $f : f(X)$ tends to the limit ℓ when $X \longrightarrow$ a if and only if each of the right and the left limits at a exists and each of them equals ℓ

i.e. $\lim\limits_{X \to a} f(X) = \ell \iff f(a^+) = f(a^-) = \ell$

Remarks

1. When finding $\lim\limits_{x \to a} f(x)$ it is not necessary that the function is defined at $x = a$, it should be only defined on an interval on the left of a and on another interval on the right of a
2. When finding $\lim\limits_{x \to a} f(x)$, we should discuss the right and the left limits of the function and compare them (if they exist) , as follows :

   * If $f(a^+) = f(a^-) = \ell$, then $\lim\limits_{x \to a} f(x) = \ell$

   * If $f(a^+) \neq f(a^-)$, then $\lim\limits_{x \to a}$ does not exist
3. If the rule of the function is the same on both the right and left of a directly , then the limit of the function can be discussed without discussing the right and left limit.

Example 1

If $f(x) = \begin{cases} 3x + 1 & , x < 1 \\ 5 - x & , x > 1 \end{cases}$, find each of the following :

(1) $\lim\limits_{x \to -2} f(x)$ (2) $\lim\limits_{x \to 3} f(x)$ (3) $\lim\limits_{x \to 1} f(x)$

Solution

(1) $\because$ The function has the same rule on the right and on the left of $x = -2$ directly and it is $f(x) = 3x + 1$

$\therefore \lim\limits_{x \to -2} f(x) = \lim\limits_{x \to -2} (3x + 1) = 3(-2) + 1 = -5$

(2) $\because$ The function has the same rule on the right and on the left of $x = 3$ directly and it is $f(x) = 5 - x$

$\therefore \lim\limits_{x \to 3} f(x) = \lim\limits_{x \to 3} (5 - x) = 5 - 3 = 2$

(3) $\because$ The rule of the function on the left of $x = 1$ differs from the right of $x = 1$, then we should discuss each of the two limits of the function from the right and the left at $x = 1$

Notice that : The rule of the function f is defined on the left of $x = 1$ by $f(x) = 3x + 1$ and on the right of $x = 1$ by $f(x) = 5 - x$

$\because f(1^-) = \lim\limits_{x \to 1^-} (3x + 1) = 3 \times 1 + 1 = 4$, $f(1^+) = \lim\limits_{x \to 1^+} (5 - x) = 5 - 1 = 4$

$\therefore f(1^-) = f(1^+) = 4$ $\therefore \lim\limits_{x \to 1} f(x) = 4$

Example 2

If $f(x)=\begin{cases} x^2+2 & , x\le -2 \\ 2x+3 & , x>-2 \end{cases}$, then discuss the existence of : $\lim\limits_{x\to -2} f(x)$

Solution

$\because f(-2^-)=\lim\limits_{x\to -2^-}(x^2+2)=(-2)^2+2=6$

$, f(-2^+)=\lim\limits_{x\to -2^+}(2x+3)=2(-2)+3=-1$

$\therefore f(-2^-)\ne f(-2^+)$

$\therefore \lim\limits_{x\to -2} f(x)$ does not exist.

Notice that :

Although the function is defined at $x=-2$ where $f(-2)=6$, this does not affect on the existence of the limit of the function when $x=-2$

Example 3

If $f(x)=\frac{|x-3|}{x-3}$, then discuss the existence of : $\lim\limits_{x\to 3} f(x)$

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Solution

$\because |x-3|=\begin{cases} x-3 & , x\ge 3 \\ -(x-3) & , x<3 \end{cases}$

$\therefore f(x)=\begin{cases} \frac{x-3}{x-3} & , x>3 \\ \frac{-(x-3)}{(x-3)} & , x<3 \end{cases} = \begin{cases} 1 & , x>3 \\ -1 & , x<3 \end{cases}$

$\therefore f(3^+)=1 , f(3^-)=-1$

$\therefore f(3^+)\ne f(3^-)$

$\therefore \lim\limits_{x\to 3} f(x)$ does not exist.

Example 4

If $f(x)=\begin{cases} x|x|-1 & , x<0 \\ \frac{|x|}{x}-2 & , x>0 \end{cases}$, find : $\lim\limits_{x\to 0} f(x)$

Solution

$\because |x|=\begin{cases} x & , x\ge 0 \\ -x & , x<0 \end{cases}$

$$\therefore f(x) = \begin{cases} x(-x) - 1 & , x < 0 \\ \frac{x}{x} - 2 & , x > 0 \end{cases} = \begin{cases} -x^2 - 1 & , x < 0 \\ -1 & , x > 0 \end{cases}$$

$\therefore f(0^-) = \lim\limits_{x \to 0^-} (-x^2 - 1) = -1 \text{ , } f(0^+) = \lim\limits_{x \to 0^+} (-1) = -1$

$\therefore f(0^-) = f(0^+) = -1$

$\therefore \lim\limits_{x \to 0} f(x) = -1$

Remark

If $y = |f(x)|$ where f is a polynomial function , then $\lim\limits_{x \to a} y = |f(a)|$ by using the direct substitution (and it is not necessary to redefine the absolute value).

For example :

If $f(x) = |x^2 - 4x + 3|$, then $\lim\limits_{x \to 3} f(x) = 0$

, if $f(x) = |x + 1| - |x - 3|$, then $\lim\limits_{x \to 3} f(x) = 4$

Example 5

If $f(x) = \begin{cases} 2x + 4 & , x < -3 \\ 3x + 7 & , -3 < x < 5 \\ 5 - x & , x > 5 \end{cases}$

, then discuss the existence of : (1) $\lim\limits_{x \to -3} f(x)$ (2) $\lim\limits_{x \to 5} f(x)$

Solution

(1) $f(-3^-) = \lim\limits_{x \to -3^-} (2x + 4) = -6 + 4 = -2$

$, f(-3^+) = \lim\limits_{x \to -3^+} (3x + 7) = 3(-3) + 7 = -2$

$\therefore f(-3^-) = f(-3^+) = -2$

$\therefore \lim\limits_{x \to -3} f(x) = -2$

(2) $f(5^-) = \lim\limits_{x \to 5^-} (3x + 7) = 15 + 7 = 22 \text{ , } f(5^+) = \lim\limits_{x \to 5^+} (5 - x) = 5 - 5 = 0$

$\therefore f(5^-) \neq f(5^+)$

$\therefore \lim\limits_{x \to 5} f(x)$ does not exist.

Example 6

If $f(x) = \begin{cases} \dfrac{\sin(x-4)}{(x-4)} & , x < 4 \\ \tan \dfrac{\pi}{16} x & , 4 < x < 8 \end{cases}$, find : $\lim\limits_{x \to 4} f(x)$

Solution

$$f(4^-) = \lim_{x \to 4^-} \frac{\sin(x-4)}{(x-4)} = \lim_{(x-4) \to 0} \frac{\sin(x-4)}{(x-4)} = 1$$

$$, f(4^+) = \lim_{x \to 4^+} \tan \frac{\pi}{16} x = \tan \frac{\pi}{4} = 1$$

$\therefore f(4^-) = f(4^+) = 1$ $\qquad \therefore \lim\limits_{x \to 4} f(x) = 1$

The limit of the function defined on an interval at one of its terminals

If the function f is defined on the open interval $]a , b[$ or on the closed interval $[a , b]$, then we notice that :

(1) The function is not defined on the left of the point a , then we discuss the right limit $[f(a^+)]$ only , and in this case : $\lim\limits_{x \to a^-} f(x)$ and $\lim\limits_{x \to a} f(x)$ do not exist.

(2) The function is not defined on the right of the point b , then we discuss the left limit $[f(b^-)]$ only , and in this case : $\lim\limits_{x \to b^+} f(x)$ and $\lim\limits_{x \to b} f(x)$ do not exist.

i.e. The limit of the function at the terminal point does not exist , and the function has a limit at this point from one side only [right or left].

Example 7

If $f(x) = \begin{cases} \dfrac{2x}{\sin x} & , \dfrac{-\pi}{2} < x < 0 \\ 2 \cos \dfrac{4}{3} x & , 0 < x < \dfrac{\pi}{2} \end{cases}$

, discuss the existence of : (1) $f\left(-\dfrac{\pi^+}{2}\right)$, $f\left(-\dfrac{\pi^-}{2}\right)$, $\lim\limits_{x \to -\frac{\pi}{2}} f(x)$

(2) $\lim\limits_{x \to 0} f(x)$ $\qquad$ (3) $\lim\limits_{x \to \frac{\pi}{2}} f(x)$

Solution

The domain of the function f

is $\left]-\dfrac{\pi}{2} , \dfrac{\pi}{2}\right[- \{0\}$



(1) $f\left(-\frac{\pi^+}{2}\right) = \underset{x \to -\frac{\pi}{2}^+}{\text{Lim}} f(x) = \underset{x \to -\frac{\pi}{2}^+}{\text{Lim}} \frac{2x}{\sin x} = \frac{-\pi}{\sin -\frac{\pi}{2}} = \frac{-\pi}{-1} = \pi$

, $f\left(-\frac{\pi^-}{2}\right)$ does not exist.

$\therefore \underset{x \to -\frac{\pi}{2}}{\text{Lim}} f(x)$ does not exist.

[Because the function is undefined on the left of $x = \frac{-\pi}{2}$]

(2) $\because f$ is defined on the left and on the right of $x = 0$ by two rules.

$\therefore f(0^-) = \underset{x \to 0^-}{\text{Lim}} \frac{2x}{\sin x} = \underset{x \to 0^-}{\text{Lim}} \frac{2 \times 1}{\frac{\sin x}{x}} = 2 \times 1 = 2$

, $f(0^+) = \underset{x \to 0^+}{\text{Lim}} 2 \cos \frac{4x}{3} = 2 \cos 0 = 2 \times 1 = 2$

$\therefore f(0^-) = f(0^+) = 2$ $\therefore \underset{x \to 0}{\text{Lim}} f(x) = 2$

(3) $\because f$ is defined on the left of $x = \frac{\pi}{2}$ only.

$\therefore \underset{x \to \frac{\pi}{2}}{\text{Lim}} f(x)$ does not exist.

Example 8

Discuss the existence of the limit of the function $f : f(x) = \sqrt{x-3}$, when $x \to 3$

Solution

The domain of the function is $[3, \infty[$



$\therefore$ The function f is defined on the right of $x = 3$ only.

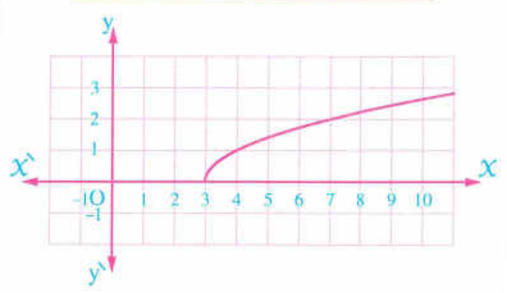
$\therefore \underset{x \to 3^+}{\text{Lim}} f(x) = \underset{x \to 3^+}{\text{Lim}} \sqrt{x-3} = 0$

, $\underset{x \to 3^-}{\text{Lim}} f(x)$ does not exist.

[Because the function is undefined on the left of 3]

$\therefore \underset{x \to 3}{\text{Lim}} f(x)$ does not exist.

The graphical drawing



Example 9

If $f(x) = \begin{cases} \dfrac{(x+2)^2-4}{x^2+x} & , \quad x > 0 \\ \dfrac{1-\cos^2 2x}{x^2} & , \quad x < 0 \end{cases}$, find : $\lim\limits_{x \to 0} f(x)$

Solution

$$\because f(0^+) = \lim_{x \to 0^+} \frac{(x+2)^2-4}{x^2+x} = \lim_{x \to 0^+} \frac{x^2+4x+\cancel{4}-\cancel{4}}{x(x+1)}$$

$$= \lim_{x \to 0^+} \frac{x(x+4)}{x(x+1)} = \lim_{x \to 0^+} \frac{x+4}{x+1} = 4$$

$$, f(0^-) = \lim_{x \to 0^-} \frac{1-\cos^2 2x}{x^2} = \lim_{x \to 0^-} \frac{\sin^2 2x}{x^2} = 2^2 = 4$$

$\therefore f(0^+) = f(0^-) = 4$

$\therefore \lim\limits_{x \to 0} f(x) = 4$

Example 10

If the function $f : f(x) = \begin{cases} \tan 4x \csc 9x & , \quad -\dfrac{\pi}{9} < x < 0 \\ \dfrac{1}{9}(a^2+2) & , \quad x > 0 \end{cases}$ has a limit when $x = 0$

, find the value of : a

Solution

$$f(0^-) = \lim_{x \to 0^-} \tan 4x \csc 9x = \lim_{x \to 0^-} \frac{\tan 4x}{\sin 9x} = \lim_{x \to 0^-} \frac{\left(\frac{\tan 4x}{x}\right)}{\left(\frac{\sin 9x}{x}\right)} = \frac{4}{9}$$

$, f(0^+) = \frac{1}{9}(a^2+2)$

, $\because$ The function has a limit when $x = 0$

$\therefore f(0^-) = f(0^+)$ $\qquad \therefore \frac{1}{9}(a^2+2) = \frac{4}{9}$

$\therefore a^2+2 = 4$ $\qquad \therefore a^2 = 2$ $\qquad \therefore a = \pm\sqrt{2}$

Example 11

If $f(x) = \begin{cases} x & , \quad x \geq 0 \\ \dfrac{1}{x} & , \quad x < 0 \end{cases}$

, find : (1) $\lim\limits_{x \to 0^+} f(x)$ $\qquad$ (2) $\lim\limits_{x \to 0^-} f(x)$ $\qquad$ (3) $\lim\limits_{x \to 0} f(x)$

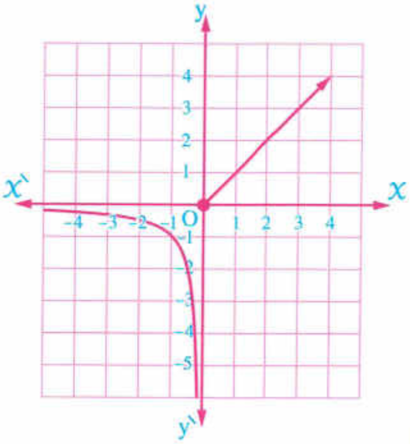
Solution

(1) $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} x = \text{zero}$

(2) $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} \frac{1}{x} = -\infty$

(3) $\lim\limits_{x \to 0} f(x)$ does not exist.

The graph is for one's guidance only



Example 12

Discuss the existence of : $\lim\limits_{x \to 0} \frac{\tan |x|}{x}$

Solution

$$\because |x| = \begin{cases} x & , \quad x \ge 0 \\ -x & , \quad x < 0 \end{cases}$$

$$\therefore \lim\limits_{x \to 0^+} \frac{\tan (x)}{x} = 1$$

$$\therefore \lim\limits_{x \to 0^-} \frac{\tan (-x)}{x} = -1$$

$\therefore$ The right limit $\ne$ the left limit.

$\therefore \lim\limits_{x \to 0} \frac{\tan |x|}{x}$ is not exist.



Lesson Seven — Continuity

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First — Continuity of a function at a point

Definition

A function f is said to be continuous at $X = a$, if the following three conditions are satisfied together :

(1) The function is defined at $X = a$ *i.e.* $f(a)$ exists.

(2) $\lim\limits_{X \to a} f(X)$ exists

(3) $\lim\limits_{X \to a} f(X) = f(a)$

* If the function is defined by more than one rule , then f is continuous at $X = a$ if $f(a^+) = f(a^-) = f(a)$

* If one condition from the three conditions does not satisfy , that is enough to decide that the function is not continuous at the point $X = a$

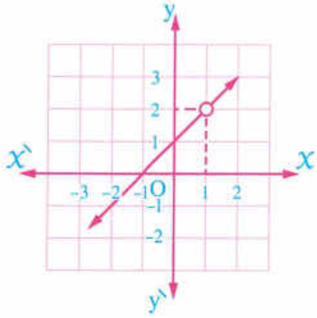
For example :

If the function is undefined at $X = a$, then it is not continuous at it , and we don't need to discuss the limit at it , and so on.

* Geometrically , we say that a function is continuous on an interval if we can draw the curve of the function on this interval without raising the tip of the pen from the paper which we draw on it , or the curve of the function , in this case has no open dots or brumpt break (jump) , but the curves that have open dots or breaks (jumps) are for discontinuous functions «separated».

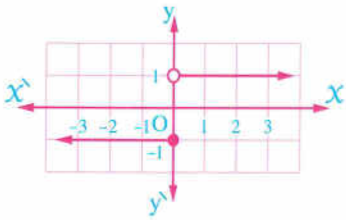
For example :

(1) $f(X) = \dfrac{X^2 - 1}{X - 1}$, $X \neq 1$



* $f(1)$ does not exist because the function is undefined at $X = 1$
* The curve of the function has an open dot at $X = 1$ because the function is undefined at $X = 1$, therefore the function f is discontinuous at $X = 1$
  Although the limit of the function exists $\left(f(1^+) = f(1^-) = 2\right)$

(2) $f(X) = \begin{cases} -1 & , \quad X \leq 0 \\ 1 & , \quad X > 0 \end{cases}$



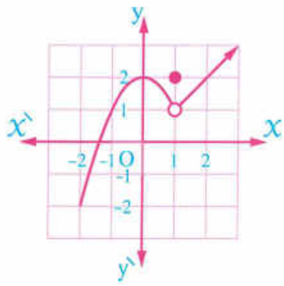
* $f(0) = -1$ (exists)
* $f(0^+) = 1$, $f(0^-) = -1$

  $\therefore f(0^+) \neq f(0^-)$

  $\therefore$ The curve of the function has a jump at $X = 0$

  , therefore the function f is discontinuous at $X = 0$

(3) $f(X) = \begin{cases} 2 - X^2 & , \quad X < 1 \\ 2 & , \quad X = 1 \\ X & , \quad X > 1 \end{cases}$



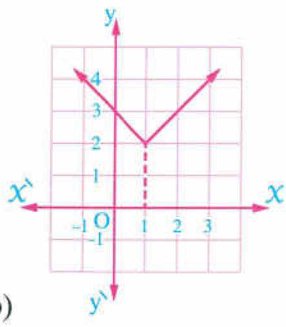
* $f(1) = 2$ (exists)
* $f(1^+) = f(1^-) = 1$ (exists)

  $\therefore \lim\limits_{X \to 1} f(X) = 1$

* The curve of the function has an open dot at $X = 1$ because $\lim\limits_{X \to 1} f(X) \neq f(1)$

  , therefore the function f is discontinuous at $X = 1$

(4) $f(X) = \begin{cases} X + 1 & , \quad X \geq 1 \\ 3 - X & , \quad X < 1 \end{cases}$



* $f(1) = 2$ (exists)
* $f(1^+) = f(1^-) = 2$ (exists)

  $\therefore \lim\limits_{X \to 1} f(X) = f(1)$

* The curve of the function has no open dot or brumpt break (jump)

  , $\lim\limits_{X \to 1} f(X) = f(1)$

  , therefore the function f is continuous at $X = 1$

UNIT 3

Example 1

Discuss the continuity of the function $f : f(x) = \begin{cases} 2x+3 & , \quad x > 2 \\ 5x-3 & , \quad x < 2 \end{cases}$, at $x = 2$

Solution

$\because$ The function is undefined at $x = 2$

i.e. $f(2)$ does not exist

$\therefore f$ is discontinuous at $x = 2$

Example 2

Discuss the continuity of the function $f : f(x) = \begin{cases} 5x-6 & , \quad x \geq 3 \\ 3x+4 & , \quad x < 3 \end{cases}$, at $x = 3$

Solution

$\because f(3) = 5 \times 3 - 6 = 9$

$, f(3^+) = \underset{x \longrightarrow 3^+}{\text{Lim}} f(x) = \underset{x \longrightarrow 3^+}{\text{Lim}} (5x - 6) = 9$

$, f(3^-) = \underset{x \longrightarrow 3^-}{\text{Lim}} f(x) = \underset{x \longrightarrow 3^-}{\text{Lim}} (3x + 4) = 13$

$\therefore f(3^+) \neq f(3^-)$

$\therefore$ The function has no limit at $x = 3$

$\therefore f$ is discontinuous at $x = 3$

Example 3

Discuss the continuity of the function $f : f(x) = \begin{cases} \frac{x^7 - 128}{x^4 - 16} & , \quad x \neq 2 \\ 14 & , \quad x = 2 \end{cases}$, at $x = 2$

Solution

$\because f(2) = 14$

$, \underset{x \longrightarrow 2}{\text{Lim}} f(x) = \underset{x \longrightarrow 2}{\text{Lim}} \frac{x^7 - 128}{x^4 - 16} = \underset{x \longrightarrow 2}{\text{Lim}} \frac{x^7 - 2^7}{x^4 - 2^4}$

$= \frac{7}{4} (2)^3 = 14$

$\therefore \underset{x \longrightarrow 2}{\text{Lim}} f(x) = f(2)$

$\therefore f$ is continuous at $x = 2$

Example 4

Discuss the continuity of the function $f : f(X) = |X - 1| + 2$, at $X = 1$

Solution

$\lim_{X \to 1} f(X) = \lim_{X \to 1} (|X - 1| + 2) = 2$

, $f(1) = |1 - 1| + 2 = 2$

$\therefore \lim_{X \to 1} f(X) = f(1)$

$\therefore f$ is continuous at $X = 1$

Remark

If $\lim_{X \to a} f(X)$ exists but the function f is discontinuous at $X = a$ because $f(a)$ is undefined or $\lim_{X \to a} f(X) \neq f(a)$, then we can redefine the function f to become continuous at a but if $\lim_{X \to a} f(X)$ does not exist , then we cannot redefine the function to become continuous at a

For example :

Redefine each of the functions defined by the following rules to be continuous at $X = 2$ if possible :

The function rule	$f(X) = \begin{cases} X + 2 & , X \neq 2 \\ 1 & , X = 2 \end{cases}$	$f(X) = \frac{X^2 + 4X - 12}{X - 2}$	$f(X) = \begin{cases} 1 & , X > 2 \\ 0 & , X = 2 \\ -1 & , X < 2 \end{cases}$
$f(2)$	1	undefined	0
$\lim_{X \to 2} f(X)$	4	8	does not exist.
The reason of discontinuity at $X = 2$	$f(2) \neq \lim_{X \to 2} f(X)$	The function is undefined at $X = 2$	$\lim_{X \to 2} f(X)$ does not exist.
Redefinition to become continuous at $X = 2$	$f(X) = \begin{cases} X + 2 & , X \neq 2 \\ 4 & , X = 2 \end{cases}$	$f(X) = \begin{cases} X + 6 & , X \neq 2 \\ 8 & , X = 2 \end{cases}$	It is not possible to redefine it to be continuous at $X = 2$

Example 5

Discuss the continuity of the function $f : f(x) = \dfrac{2x^2 - 7x - 4}{x - 4}$ at $x = 4$

, if f is discontinuous , can we redefine the function f to become continuous at $x = 4$?

Solution

$\because$ f is undefined at $x = 4$

$\therefore$ f is discontinuous at $x = 4$

$$, \because \lim_{x \to 4} f(x) = \lim_{x \to 4} \frac{2x^2 - 7x - 4}{x - 4}$$

$$= \lim_{x \to 4} \frac{(2x+1)(x-4)}{(x-4)} = \lim_{x \to 4} (2x + 1) = 9$$

To make the function continuous at $x = 4$, it is necessary

that : $f(4) = \lim_{x \to 4} f(x) = 9$

$\therefore$ We can redefine the function as the following :

$$f(x) = \begin{cases} \dfrac{2x^2 - 7x - 4}{x - 4} & , \ x \neq 4 \\ 9 & , \ x = 4 \end{cases}$$

Example 6

Can we redefine the function f :

$$f(x) = \begin{cases} \dfrac{|x - 2|}{x - 2} & , \ x \neq 2 \\ 0 & , \ x = 2 \end{cases}$$ **, to become continuous at $x = 2$?**

Solution

$$\because |x - 2| = \begin{cases} x - 2 & , \ x \geq 2 \\ -(x - 2) & , \ x < 2 \end{cases}$$

$$\therefore f(x) = \begin{cases} \dfrac{x - 2}{x - 2} & , \ x > 2 \\ 0 & , \ x = 2 \\ \dfrac{-(x - 2)}{x - 2} & , \ x < 2 \end{cases} \qquad \therefore f(x) = \begin{cases} 1 & , \ x > 2 \\ 0 & , \ x = 2 \\ -1 & , \ x < 2 \end{cases}$$

$\therefore f(2) = 0 , f(2^+) = 1 , f(2^-) = -1$ $\qquad$ $\therefore f(2^+) \neq f(2^-)$

$\therefore \lim_{x \to 2} f(x)$ does not exist.

$\therefore$ We cannot redefine the function at $x = 2$ to become continuous.

Example 7

If the function $f : f(x) = \begin{cases} \dfrac{x \cos 5x}{\sin 3x + \tan 4x} & , \quad 0 < x < \dfrac{\pi}{8} \\ 2b & , \quad x = 0 \\ 3a - \dfrac{2}{7}\cos x & , \quad x < 0 \end{cases}$

is continuous at $x = 0$, find the values of a and b

Solution

$\because f(0^+) = \lim\limits_{x \to 0^+} \dfrac{x \cos 5x}{\sin 3x + \tan 4x} = \lim\limits_{x \to 0^+} \dfrac{\cos 5x}{\dfrac{\sin 3x}{x} + \dfrac{\tan 4x}{x}} = \dfrac{1}{3+4} = \dfrac{1}{7}$

$, f(0^-) = \lim\limits_{x \to 0^-} \left(3a - \dfrac{2}{7}\cos x\right) = 3a - \dfrac{2}{7}$

$, \because f(0) = 2b$ $, \because f$ is continuous at $x = 0$

$\therefore f(0^+) = f(0^-) = f(0)$ $\therefore 3a - \dfrac{2}{7} = \dfrac{1}{7}$

$\therefore 3a = \dfrac{3}{7}$ $\therefore a = \dfrac{1}{7}$

$, 2b = \dfrac{1}{7}$ $\therefore b = \dfrac{1}{14}$

Example 8

If the function $f : f(x) = \begin{cases} x^2 + ax - 2 & , \quad x > 2 \\ 4 & , \quad x = 2 \\ 5a + bx & , \quad x < 2 \end{cases}$

is continuous at $x = 2$, find the value of : b

Solution

$\because f(2) = 4 , f(2^+) = \lim\limits_{x \to 2^+} (x^2 + ax - 2) = 2 + 2a$

$, \because f(2^-) = \lim\limits_{x \to 2^-} (5a + bx) = 5a + 2b$

$, \because f$ is continuous at $x = 2$ $\therefore f(2^+) = f(2^-) = f(2)$

$\therefore 2 + 2a = 4$ $\therefore a = 1$

$, 5a + 2b = 4$ $\therefore 5 + 2b = 4$

$\therefore 2b = -1$ $\therefore b = \dfrac{-1}{2}$

Example 9

If $f(x) = \begin{cases} x^2 - 2x + 2 & , \ x \le 2 \\ |x - 3| & , \ x > 2 \end{cases}$, discuss the continuity of f at $x = 2$ and $x = 3$

Solution

At $2 < x < 3$, then $|x - 3| = 3 - x$

At $x \ge 3$, then $|x - 3| = x - 3$

$$\therefore f(x) = \begin{cases} x^2 - 2x + 2 & , \ x \le 2 \\ 3 - x & , \ 2 < x < 3 \\ x - 3 & , \ x \ge 3 \end{cases}$$

*** At $x = 2$:**

$f(2) = 2^2 - 2(2) + 2 = 2$, $f(2^-) = \lim\limits_{x \to 2^-} (x^2 - 2x + 2) = 2$

, $f(2^+) = \lim\limits_{x \to 2^+} (3 - x) = 1$ $\qquad \therefore f(2^+) \ne f(2^-)$

$\therefore \lim\limits_{x \to 2} f(x)$ does not exist. $\qquad \therefore f$ is discontinuous at $x = 2$

*** At $x = 3$:**

$f(3) = 3 - 3 = 0$, $f(3^-) = \lim\limits_{x \to 3^-} (3 - x) = 0$

, $f(3^+) = \lim\limits_{x \to 3^+} (x - 3) = 0$

$\therefore f(3) = f(3^+) = f(3^-)$ $\qquad \therefore f$ is continuous at $x = 3$

Example 10

Redefine the function $f : f(x) = \dfrac{4 \cos x}{\pi - 2x}$ to become continuous at $x = \dfrac{\pi}{2}$

Solution

$\because$ The function f is undefined at $x = \dfrac{\pi}{2}$

$$, \lim_{x \to \frac{\pi}{2}} f(x) = \lim_{x \to \frac{\pi}{2}} \frac{4 \cos x}{\pi - 2x} = \lim_{\left(\frac{\pi}{2} - x\right) \to 0} \frac{4 \sin\left(\frac{\pi}{2} - x\right)}{2\left(\frac{\pi}{2} - x\right)} = \frac{4}{2} = 2$$

To make the function continuous at $x = \dfrac{\pi}{2}$, it is necessary that : $f\left(\dfrac{\pi}{2}\right) = \lim\limits_{x \to \frac{\pi}{2}} f(x) = 2$

$\therefore$ We can redefine the function as the following :

$$f : f(x) = \begin{cases} \dfrac{4 \cos x}{\pi - 2x} & , \ x \ne \dfrac{\pi}{2} \\ 2 & , \ x = \dfrac{\pi}{2} \end{cases}$$

Second Continuity of a function on an interval

Definition

(1) If the function f is defined on the open interval $I =]a , b[$, then f is continuous on I if it is continuous at each point belongs to this interval.

(2) If the function f is defined on the closed interval $I = [a , b]$, then the function f is continuous on I if :

* f is continuous on $]a , b[$
* f is continuous from the right at a
  i.e. $f(a) = \lim_{X \to a^+} f(X)$
* f is continuous from the left at b
  i.e. $f(b) = \lim_{X \to b^-} f(X)$

Some types of continuous functions

From our studying for the graph of algebraic functions , we can deduce types of some continuous functions as :

1 The polynomial function

$f(X) = a_0 + a_1 X + a_2 X^2 + \cdots + a_n X^n$ is continuous on $\mathbb{R}$ or any interval subset of $\mathbb{R}$

2 The rational function

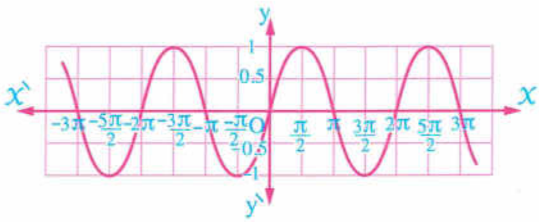
$f(X) = \dfrac{a_0 + a_1 X + a_2 X^2 + \cdots + a_n X^n}{b_0 + b_1 X + b_2 X^2 + \cdots + b_n X^n}$ is continuous on $\mathbb{R}$ – {zeroes of the denominator}

or any interval subset of $\mathbb{R}$ except zeroes of the denominator.

3 Trigonometric functions

(a) The sine function :

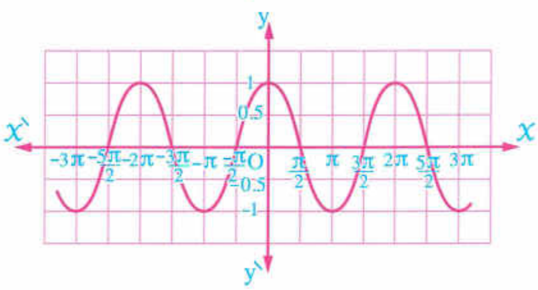
$f(X) = \sin X$ is continuous on $\mathbb{R}$ or any interval subset of $\mathbb{R}$ and the graph shows that as in the opposite figure :



(b) The cosine function :

$f(X) = \cos X$ is continuous on $\mathbb{R}$ or any interval subset of $\mathbb{R}$

and the graph shows that as in the opposite figure :

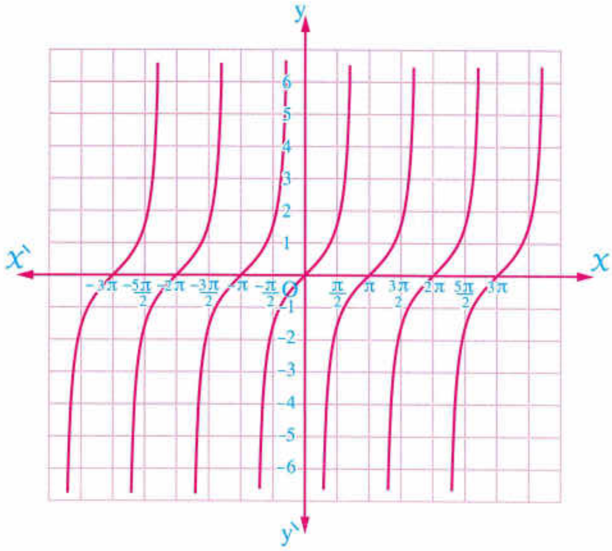


(c) The tangent function :

$f(x) = \tan x$ is continuous on $\mathbb{R}$ except the points :

$\ldots, -\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

i.e. The function is continuous on $\mathbb{R} - \left\{x : x = \frac{\pi}{2} + n\pi, n \in \mathbb{Z}\right\}$ and the graph shows that as in the opposite figure :



Enrich your knowledge

Each of the following functions is continuous on any open interval subset of its domain :

(1) The function $f : f(x) = |h(x)|$ where $h(x)$ is a polynomial function and its domain is $\mathbb{R}$

(2) The function $f : f(x) = \sqrt[n]{h(x)}$ where $n \in \mathbb{Z}^+$ and greater than 1 , $h(x)$ is a polynomial function.

* Its domain = $\mathbb{R}$ if n is an odd number greater than 1
* Its domain is the interval which satisfies $h(x) \geq 0$ if n is an even number.

Example 11

Discuss the continuity of each of the functions defined by the following rules on $\mathbb{R}$:

(1) $f(x) = x^3 + 4x - 5$

(2) $f(x) = 3$

(3) $f(x) = \frac{x - 4}{x^2 - 5x + 6}$

(4) $f(x) = \frac{x}{x^2 + 25}$

Solution

(1) $f(x) = x^3 + 4x - 5$ (polynomial function)

$\therefore f$ is continuous on $\mathbb{R}$

(2) $f(x) = 3$ (polynomial function)

$\therefore f$ is continuous on $\mathbb{R}$

(3) $f(x) = \dfrac{x-4}{x^2-5x+6}$ (Rational function)

$\because x^2 - 5x + 6 = 0$ when $(x-2)(x-3) = 0$

i.e. $x = 2$ or $x = 3$

$\therefore f$ is continuous on $\mathbb{R} - \{2, 3\}$

(4) $f(x) = \dfrac{x}{x^2+25}$ (Rational function)

$\because x^2 + 25 > 0$ for all values of $x \in \mathbb{R}$

$\therefore$ There are no zeroes for the denominator

$\therefore f$ is continuous on $\mathbb{R}$

Remark

If the two functions f_1 and f_2 are defined on the interval $I =]a, b[$ and they are continuous on the interval I, then each of the following functions is continuous on the interval I :

(1) $f_1 \pm f_2$

(2) $f_1 \cdot f_2$

(3) $\dfrac{f_1}{f_2}$ where $f_2(x) \neq 0$

Example 12

Discuss the continuity of each of the functions defined by the following rules :

(1) $f(x) = (x+2)\sin x$

(2) $f(x) = (x+1)^2 - \sin x$

(3) $f(x) = \dfrac{\cos x + \sin x}{x+3}$

(4) $f(x) = x^3 + 5 + 3x^{-2}$

(5) $f(x) = \dfrac{\tan x}{x^2-1}$

Solution

(1) $\because$ Each of $(x+2)$ and $\sin x$ is continuous on $\mathbb{R}$

$\therefore f$ is continuous on $\mathbb{R}$

(2) $\because$ Each of $(x+1)^2$ and $\sin x$ is continuous on $\mathbb{R}$

$\therefore f$ is continuous on $\mathbb{R}$

(3) $\because$ Each of $\cos x$, $\sin x$ and $(x+3)$ is continuous on $\mathbb{R}$

$\therefore f$ is continuous on $\mathbb{R} - \{\text{zeroes of the denominator}\}$

$\therefore f$ is continuous on $\mathbb{R} - \{-3\}$

(4) $\because f(x) = x^3 + 5 + \frac{3}{x^2}$

$\therefore f$ is continuous on $\mathbb{R} - \{0\}$

(5) $\because \tan x$ is continuous on $\mathbb{R} - \left\{x : x = \frac{\pi}{2} + n\pi , n \in \mathbb{Z}\right\}$ and $(x^2 - 1)$ is continuous on $\mathbb{R}$

, $\because x^2 - 1 = 0$ at $x = \pm 1$

$\therefore$ The function f is continuous on $\mathbb{R} - \left\{1 , -1 , \frac{\pi}{2} + n\pi\right\}$, where $n \in \mathbb{Z}$

Example 13

Discuss the continuity of the function $f : f(x) = \begin{cases} 3x + 2 & , -3 \le x < 2 \\ x^2 + 4 & , 2 \le x \le 5 \end{cases}$

Solution

$\because f$ is defined on the interval $[-3 , 5]$, to discuss its continuity on this interval , we discuss :

(1) Its continuity on each of the two intervals $]-3 , 2[$ and $]2 , 5[$

(2) Its continuity at the point $x = 2$ where the function rule changes at it.

(3) Its continuity from the right at $x = -3$ and its continuity from the left at $x = 5$

First :

* For $x \in]-3 , 2[$ $\therefore f(x) = 3x + 2$ (polynomial function)

$\therefore f$ is continuous on $]-3 , 2[$

* For $x \in]2 , 5[$ $\therefore f(x) = x^2 + 4$ (polynomial function)

$\therefore f$ is continuous on $]2 , 5[$

Second :

$\because f(2) = 2^2 + 4 = 8$, $f(2^-) = \lim_{x \to 2^-} (3x + 2) = 8$

, $f(2^+) = \lim_{x \to 2^+} (x^2 + 4) = 8$

$\therefore f(2^-) = f(2^+) = f(2)$ $\therefore f$ is continuous at $x = 2$

Third :

$\because f(-3) = 3(-3) + 2 = -7$, $f(-3^+) = \lim_{x \to -3^+} (3x + 2) = -7$

$\therefore f(-3^+) = f(-3)$

$\therefore f$ is continuous from the right at $x = -3$

Also , $\because f(5) = 5^2 + 4 = 29$

, $f(5^-) = \lim_{x \to 5^-} (x^2 + 4) = 29$ $\therefore f(5^-) = f(5)$

$\therefore f$ is continuous from the left at $x = 5$

From first , second and third , we deduce that : f is continuous on $[-3 , 5]$

Example 14

Discuss the continuity of each of the functions defined by the following rules on its domain :

(1) $f(x) = \sqrt{x+2}$

(2) $f(x) = \sqrt[3]{2-x^2}$

(3) $f(x) = \frac{1}{\sqrt[3]{x^2-25}}$

Solution

(1) $\because f(x) = \sqrt{x+2}$ (index of the root = 2 (even))

$\therefore f$ is defined if $x + 2 \geq 0$

$\therefore x \geq -2$

$\therefore$ The domain of $f = [-2, \infty[$

Let $a \in]-2, \infty[$

$, \because \lim_{x \to a} f(x) = \lim_{x \to a} \sqrt{x+2} = \sqrt{a+2} = f(a)$

$\therefore$ The function f is continuous for all values of a

$\therefore f$ is continuous on the interval $]-2, \infty[$

$, \because f(-2) = 0, f(-2^+) = \lim_{x \to -2^+} \left(\sqrt{x+2}\right) = 0$

$\therefore f(-2^+) = f(-2)$

$\therefore f$ is continuous from the right at $x = -2$

$\therefore f$ is continuous on $[-2, \infty[$

(2) $\because f(x) = \sqrt[3]{2-x^2}$ (index of the root = 3 (odd))

$\therefore f$ is continuous on $\mathbb{R}$

(3) $\because f(x) = \frac{1}{\sqrt[3]{x^2-25}}$ (index of the root = 3 (odd))

$\therefore f$ is continuous on $\mathbb{R} - \{$zeroes of the denominator$\}$

$, \because x^2 - 25 = 0$ when $x = \pm 5$

$\therefore f$ is continuous on $\mathbb{R} - \{5, -5\}$

Example 15

Discuss the continuity of the function $f : f(x) = \begin{cases} \sin x - \cos x & , \quad 0 \le x \le \pi \\ 2\cos^2 x - 1 & , \quad x > \pi \end{cases}$

Solution

$\because$ Each of $\sin x$ and $\cos x$ is continuous on the interval $]0 , \pi[$

$\therefore f(x) = \sin x - \cos x$ is continuous on the interval $]0 , \pi[$

similarly $f(x) = 2\cos^2 x - 1$ is continuous on the interval $]\pi , \infty[$ (1)

, $\because f(\pi) = \sin \pi - \cos \pi = 1$

, $f(\pi^-) = \lim\limits_{x \to \pi^-} (\sin x - \cos x) = \sin \pi - \cos \pi = 1$

, $f(\pi^+) = \lim\limits_{x \to \pi^+} (2\cos^2 x - 1) = 2\cos^2 \pi - 1 = 1$

$\therefore f(\pi^-) = f(\pi^+) = f(\pi)$

$\therefore f$ is continuous at $x = \pi$ (2)

, $\because f(0) = \sin 0 - \cos 0 = -1$

, $f(0^+) = \lim\limits_{x \to 0^+} (\sin x - \cos x) = -1$

$\therefore f(0^+) = f(0)$

$\therefore f$ is continuous from the right at $x = 0$ (3)

From (1) , (2) and (3) , we deduce that :

f is continuous on $[0 , \infty[$

Example 16

Discuss the continuity of each of the functions defined by the following rules on $\mathbb{R}$:

(1) $f(x) = \dfrac{x - 1}{\cos x}$ (2) $f(x) = \dfrac{x^2 + 3}{1 + \sin x}$

Solution

(1) $\because$ Each of $(x - 1)$ and $\cos x$ is continuous on $\mathbb{R}$

, $\because \cos x = 0$

when $x = \dfrac{\pi}{2} + \pi n$

$\therefore f$ is continuous on

$\mathbb{R} - \left\{x : x = \dfrac{\pi}{2} + \pi n , n \in \mathbb{Z}\right\}$

Remember that

The equation	The general solution
• $\sin x = 0$	• $x = \pi n$
• $\sin x = 1$	• $x = \dfrac{\pi}{2} + 2\pi n$
• $\sin x = -1$	• $x = \dfrac{3\pi}{2} + 2\pi n$
• $\cos x = 0$	• $x = \dfrac{\pi}{2} + \pi n$
• $\cos x = 1$	• $x = 2\pi n$
• $\cos x = -1$	• $x = \pi + 2\pi n$

(2) $\because$ Each of $(x^2 + 3)$ and $(1 + \sin x)$ is continuous on $\mathbb{R}$

, $\because$ $1 + \sin x = 0$

i.e. $\sin x = -1$ when $x = \frac{3\pi}{2} + 2\pi n$

$\therefore$ f is continuous on $\mathbb{R} - \{x : x = \frac{3\pi}{2} + 2\pi n , n \in \mathbb{Z}\}$

Example 17

If the function $f : f(x) = \begin{cases} 3x - 2 & , x \le -2 \\ a x + b & , -2 < x < 5 \\ x^2 - 12 & , x \ge 5 \end{cases}$

is continuous on $\mathbb{R}$, find the value of each of : a and b

Solution

$\because$ f is continuous on $\mathbb{R}$

$\therefore$ f is continuous at $x = -2$

$\therefore$ $f(-2^-) = f(-2^+) = f(-2)$

$\because$ $\lim\limits_{x \to -2^+} (a x + b) = 3(-2) - 2$

$\therefore$ $-2a + b = -8$ (1)

, $\because$ f is continuous at $x = 5$

$\therefore$ $f(5^-) = f(5^+) = f(5)$

$\therefore$ $\lim\limits_{x \to 5^-} (a x + b) = (5)^2 - 12$

$\therefore$ $5a + b = 13$ (2)

From (1) and (2) :

$\therefore$ $a = 3$, $b = -2$

UNIT 4

Trigonometry

Unit Lessons



Revision on the most important rules have been studied before

Radian and degree measures of an angle

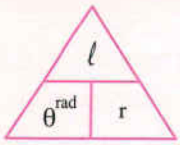
- The radian measure of a central angle in a circle

$$= \frac{\text{The length of the arc which the central angle subtends}}{\text{The length of the radius of this circle}}$$

i.e.

$\theta^{rad} = \frac{\ell}{r}$ and from it

$\ell = \theta^{rad}\, r$, $r = \frac{\ell}{\theta^{rad}}$



- The converting between the radian measure and the degree measure :

$\frac{x^\circ}{180^\circ} = \frac{\theta^{rad}}{\pi}$ and from it $\theta^{rad} = x^\circ \times \frac{\pi}{180^\circ}$, $x^\circ = \theta^{rad} \times \frac{180^\circ}{\pi}$

Notice that :

π in radians is equivalent to 180° in degrees.

The basic trigonometric identities

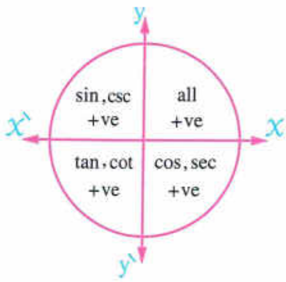
(1) $\cos^2\theta + \sin^2\theta = 1$

(2) $1 + \tan^2\theta = \sec^2\theta$

(3) $1 + \cot^2\theta = \csc^2\theta$

(4) $\sin\theta \csc\theta = 1$, $\cos\theta \sec\theta = 1$, $\tan\theta \cot\theta = 1$

(5) $\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\cot\theta = \frac{\cos\theta}{\sin\theta}$

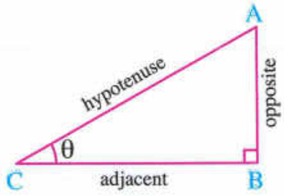


Remember the following relations

(1) $\sin\theta = \frac{opp.}{hyp.} = \frac{AB}{AC}$

(2) $\cos\theta = \frac{adj.}{hyp.} = \frac{BC}{AC}$

(3) $\tan\theta = \frac{opp.}{adj.} = \frac{AB}{BC}$



(4) If the terminal side of the directed angle of measure θ in the standard position intersects the unit circle at the point (x, y) , then :

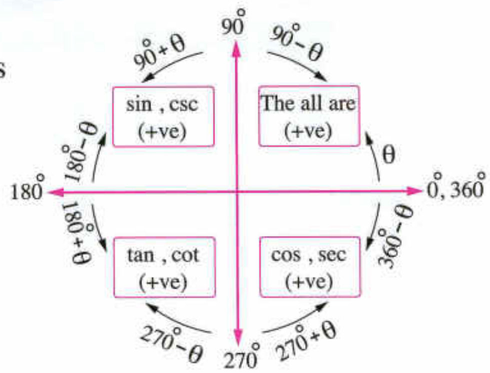
$x = \cos\theta$, $y = \sin\theta$ and $x^2 + y^2 = 1$

(5) The relations between the trigonometric functions of the related angles are identities :

For example :

$\sin(90° + \theta) = \cos\theta$

, $\tan(360° - \theta) = -\tan\theta$, ... each one of them is a trigonometric identity.



Areas of some geometric figures

* The area of $\Delta ABC = \frac{1}{2}\, a\, b \sin C = \frac{1}{2}\, b\, c \sin A = \frac{1}{2}\, a\, c \sin B$

* The area of $\Delta ABC = \sqrt{S(S-a)(S-b)(S-c)}$ where $S = \frac{a+b+c}{2}$

* The area of the quadrilateral = $\frac{1}{2}$ the product of the lengths of its diagonals × sine of the included angle between them.

* The area of the regular polygon whose number of its sides is n sides and the length of its side is $x = \frac{1}{4}\, n\, x^2 \cot\frac{\pi}{n}$

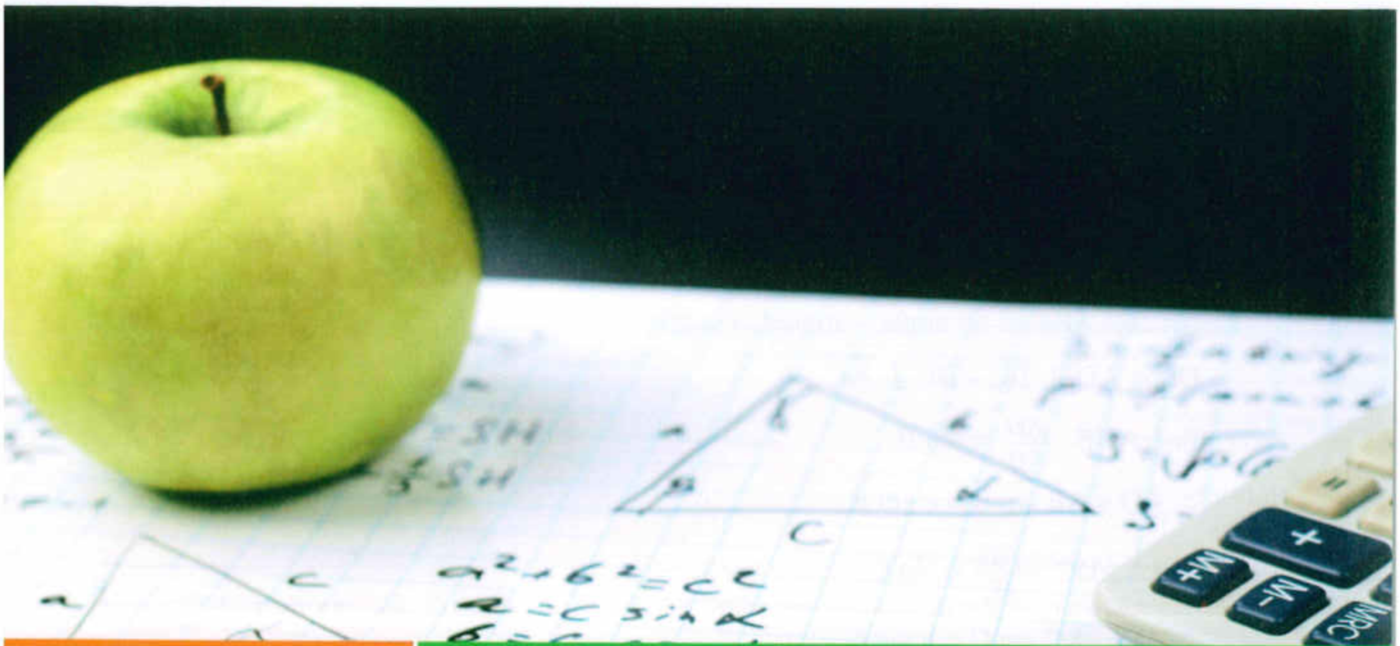
* The area of the circle = πr^2

  , the circumference of the circle = $2\pi r$

* The area of the circular sector = $\frac{1}{2}\,\ell\, r = \frac{1}{2}\,\theta^{rad} r^2$

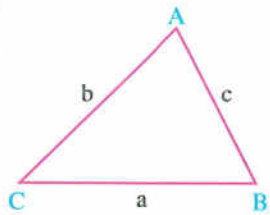
  , the perimeter of the circular sector = $2r + \ell$

* The area of the circular segment = $\frac{1}{2}\, r^2 (\theta^{rad} - \sin\theta)$



Lesson One — The sine rule

"In any triangle the lengths of the sides are proportional to the sines of the opposite angles"



i.e. In Δ ABC : $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Where the symbols A , B and C represent the measures of the angles of Δ ABC and the symbols a , b and c represent the lengths of the sides $\overline{BC}$, $\overline{AC}$ and $\overline{AB}$ respectively.

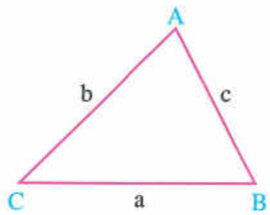
Proof

$\because$ The area of the triangle = $\frac{1}{2}$ the product of the lengths of any two sides × sine of their included angle.

$\therefore$ The area of Δ ABC = $\frac{1}{2}$ ac sin B (1)

= $\frac{1}{2}$ cb sin A (2)

= $\frac{1}{2}$ ab sin C (3)



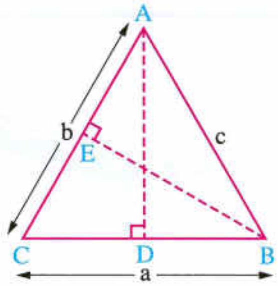
From (1) , (2) , (3) : $\therefore$ cb sin A = ac sin B = ab sin C

, dividing by abc : $\therefore \dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

$\therefore \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ (Q.E.D.)

Another proof

First : If Δ ABC is an acute - angled triangle :



Draw $\overline{AD} \perp \overline{BC}$, $\overline{BE} \perp \overline{AC}$

In Δ ADB : $\frac{AD}{AB} = \sin B$

$\therefore AD = AB \sin B = c \sin B$

, in Δ ADC : $\frac{AD}{AC} = \sin C$

$\therefore AD = AC \sin C = b \sin C$

$\therefore c \sin B = b \sin C \qquad \therefore \boxed{\frac{b}{\sin B} = \frac{c}{\sin C}} \qquad (1)$

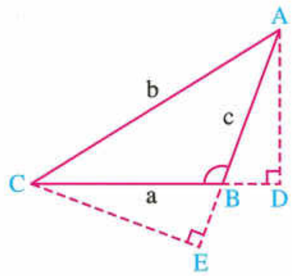
Similarly in Δ BCE : $\frac{BE}{BC} = \sin C \qquad \therefore BE = BC \sin C = a \sin C$

, in Δ BEA : $\frac{BE}{AB} = \sin A \qquad \therefore BE = AB \sin A = c \sin A$

$\therefore a \sin C = c \sin A \qquad \therefore \boxed{\frac{a}{\sin A} = \frac{c}{\sin C}} \qquad (2)$

From (1) , (2) : $\qquad \therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ (Q.E.D.)

Second : If Δ ABC is an obtuse - angled triangle at B :



Draw $\overline{AD} \perp \overrightarrow{CB}$, $\overline{CE} \perp \overrightarrow{AB}$

In Δ ADB : $\frac{AD}{AB} = \sin(\angle ABD)$

$\therefore AD = AB \sin(180° - B) = c \sin B$

, in Δ ADC : $\frac{AD}{AC} = \sin C \qquad \therefore AD = AC \sin C = b \sin C$

$\therefore c \sin B = b \sin C \qquad \therefore \boxed{\frac{b}{\sin B} = \frac{c}{\sin C}} \qquad (1)$

, in Δ CEB : $\frac{CE}{BC} = \sin(\angle CBE)$

$\therefore CE = BC \sin(180° - B) = a \sin B$

, in Δ CEA : $\frac{CE}{AC} = \sin A \qquad \therefore CE = AC \sin A = b \sin A$

$\therefore a \sin B = b \sin A \qquad \therefore \boxed{\frac{a}{\sin A} = \frac{b}{\sin B}} \qquad (2)$

From (1) , (2) : $\qquad \therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ (Q.E.D.)

∗ **Notice that :** The sine rule is also true in case of the right-angled triangle.

Example 1

In ΔABC , if : $a = 10$ cm. , $m(\angle A) = 30°$, $m(\angle B) = 45°$ Find using the calculator each of b and c to the nearest one decimal , find also the area of ΔABC to the nearest whole number.

Solution

$\because m(\angle C) = 180° - (30° + 45°) = 105°$

, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$\therefore \frac{10}{\sin 30°} = \frac{b}{\sin 45°} = \frac{c}{\sin 105°}$

$\therefore b = \frac{10 \sin 45°}{\sin 30°} \approx 14.1$ cm. , $c = \frac{10 \sin 105°}{\sin 30°} \approx 19.3$ cm.

, the area of $\Delta ABC = \frac{1}{2} bc \sin A = \frac{1}{2} \times 14.1 \times 19.3 \sin 30° \approx 68$ cm^2.

Example 2

ABCD is a parallelogram in which :

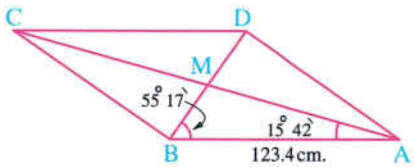
$AB = 123.4$ cm. , $m(\angle CAB) = 15°\ 42'$, $m(\angle DBA) = 55°\ 17'$ Find :

(1) The length of each of the two diagonals $\overline{BD}$, $\overline{AC}$

(2) The area of ▱ ABCD

Solution

Let $\overline{AC} \cap \overline{BD} = \{M\}$



$\therefore$ In ΔMAB :

$m(\angle AMB) = 180° - (15°\ 42' + 55°\ 17') = 109°\ 1'$

$\therefore \frac{123.4}{\sin 109°\ 1'} = \frac{BM}{\sin 15°\ 42'} = \frac{AM}{\sin 55°\ 17'}$

$\therefore BM = \frac{123.4 \sin 15°\ 42'}{\sin 109°\ 1'} \approx 35.3$ cm.

$\therefore BD = 2\,BM = 70.6$ cm. , $AM = \frac{123.4 \sin 55°\ 17'}{\sin 109°\ 1'} \approx 107.3$ cm.

$\therefore AC = 2\,AM = 214.6$ cm.

, the area of ▱ ABCD = 4 the area of ΔMAB

$= 4 \times \frac{1}{2} \times BM \times AM \sin(\angle AMB)$

$= 4 \times \frac{1}{2} \times 35.3 \times 107.3 \sin 109°\ 1' \approx 7162$ cm^2.

Example 3

ABC is a triangle in which : 3 sin A = 4 sin B = 2 sin C

Find the lengths of its sides , given that its perimeter = 39 cm.

Solution

$\because$ 3 sin A = 4 sin B = 2 sin C by dividing by 12

$\therefore \frac{\sin A}{4} = \frac{\sin B}{3} = \frac{\sin C}{6}$ $\qquad\qquad$ $\therefore \frac{4}{\sin A} = \frac{3}{\sin B} = \frac{6}{\sin C}$

$\therefore$ a : b : c = 4 : 3 : 6

Let a = 4 k , b = 3 k , c = 6 k

, $\because$ the perimeter of Δ ABC = 39 cm.

$\therefore$ 4 k + 3 k + 6 k = 39 $\qquad\qquad$ $\therefore$ 13 k = 39

$\therefore$ k = 3 $\qquad\qquad$ $\therefore$ a = 12 cm. , b = 9 cm. , c = 18 cm.

Example 4

If the perimeter of Δ ABC = 24 cm. , m ($\angle$ B) = 30° , m ($\angle$ C) = 48° , find b

Solution

$\because$ m ($\angle$ A) = 180° – (30° + 48°) = 102°

$$\therefore \frac{a}{\sin 102°} = \frac{b}{\sin 30°} = \frac{c}{\sin 48°}$$

$$, \because \frac{\text{the sum of antecedents}}{\text{the sum of consequents}} = \text{one of the ratios}$$

$$\therefore \frac{a + b + c}{\sin 102° + \sin 30° + \sin 48°} = \frac{b}{\sin 30°}$$

$$\therefore \frac{24}{\sin 102° + \sin 30° + \sin 48°} = \frac{b}{\sin 30°}$$

$$\therefore b = \frac{24 \sin 30°}{\sin 102° + \sin 30° + \sin 48°} \approx 5.4 \text{ cm.}$$

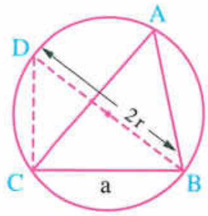
Well known problem

In any Δ ABC : $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2r$

Where r is the radius length of the circumcircle of the triangle ABC

Proof

Draw the circumcircle of Δ ABC , then draw the diameter $\overline{BD}$ and the chord $\overline{CD}$



First : If Δ ABC is an acute-angled triangle :

$\therefore m(\angle BCD) = 90°$ (drawn in a semicircle)

, $m(\angle A) = m(\angle D)$ (subtended by $\overset{\frown}{BC}$)

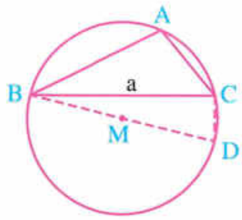
In Δ DBC : $\sin D = \frac{a}{BD} = \frac{a}{2r}$

$\therefore \sin A = \frac{a}{2r}$ $\quad \therefore \frac{a}{\sin A} = 2r$

Similarly : $\frac{b}{\sin B} = 2r$ $\quad , \frac{c}{\sin C} = 2r$

$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2r$ (Q.E.D.)

Second : If Δ ABC is an obtuse-angled triangle :



$\therefore m(\angle BDC) = 180° - m(\angle A)$

$\therefore \sin(180° - A) = \frac{BC}{BD}$

$\therefore \sin A = \frac{a}{2r}$

$\therefore \frac{a}{\sin A} = 2r$

$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2r$ (Q.E.D.)

Example 5

In Δ ABC , if : b = 7 cm. , m ($\angle$ B) = 30° , c = 9 cm. , calculate the radius length of the circumcircle of Δ ABC , calculate also m ($\angle$ A) to the nearest degree.

Solution

$\because \frac{b}{\sin B} = 2r$ $\quad \therefore \frac{7}{\sin 30°} = 2r$

$\therefore r = \frac{7}{2\sin 30°} = 7$ cm.

$, \because \frac{c}{\sin C} = 2r$ $\quad \therefore \frac{9}{\sin C} = 14$

$\therefore \sin C = \frac{9}{14}$ $\quad \therefore m(\angle C) \approx 40°$

$\therefore m(\angle A) = 180° - (30° + 40°) = 110°$

or $m(\angle C) \approx 140°$

$\therefore m(\angle A) = 180° - (30° + 140°) = 10°$

Notice that :

There are two triangles satisfy these givens and this case is called ambiguous case and we will study it at the lesson "solution of the triangle"

Example 6

In any ΔABC , prove that : The area of $\Delta ABC = 2r^2 \sin A \sin B \sin C$ where r is the radius length of the circumcircle of the triangle ABC

Solution

$\because$ The area of $\Delta ABC = \frac{1}{2} bc \sin A$ where :

$b = 2r \sin B$, $c = 2r \sin C$

$\therefore$ The area of $\Delta ABC = \frac{1}{2} \times 2r \sin B \times 2r \sin C \times \sin A$

$= 2r^2 \sin A \sin B \sin C$



Lesson Two — The cosine rule

In any triangle ABC :

$a^2 = b^2 + c^2 - 2\,bc \cos A$, hence $\cos A = \dfrac{b^2 + c^2 - a^2}{2\,bc}$

$b^2 = c^2 + a^2 - 2\,ca \cos B$, hence $\cos B = \dfrac{c^2 + a^2 - b^2}{2\,ca}$

$c^2 = a^2 + b^2 - 2\,ab \cos C$, hence $\cos C = \dfrac{a^2 + b^2 - c^2}{2\,ab}$

This rule is used if :
The lengths of two sides in Δ ABC and the measure of their included angle are given.

This rule is used if :
The lengths of the sides of Δ ABC or the ratio among these lengths are given.

Proof **To prove that : $a^2 = b^2 + c^2 - 2\,bc \cos A$**

First : If Δ ABC is an acute-angled triangle :

Draw $\overrightarrow{CD} \perp \overline{AB}$ to intersect it at D

From Δ ADC , we find that :

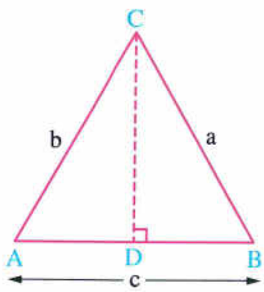
CD = b sin A , DA = b cos A

$\therefore$ BD = c − b cos A

In Δ CDB :

$\because (BC)^2 = (CD)^2 + (DB)^2$

i.e. $a^2 = (b \sin A)^2 + (c - b \cos A)^2$



$= b^2 \sin^2 A + c^2 - 2\,bc \cos A + b^2 \cos^2 A$

$= b^2 (\sin^2 A + \cos^2 A) + c^2 - 2\,bc \cos A$

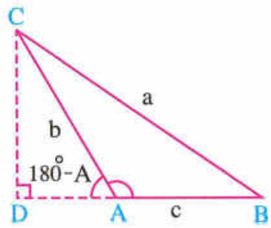
$\therefore a^2 = b^2 + c^2 - 2\,bc \cos A$

$\therefore \cos A = \dfrac{b^2 + c^2 - a^2}{2\,bc}$ (Q.E.D)

Remember that

$\sin^2 A + \cos^2 A = 1$

Second : If ΔABC is an obtuse-angled triangle at A :



Draw $\overrightarrow{CD} \perp \overrightarrow{BA}$ to intersect it at D

From ΔADC , we find that : $CD = b \sin (180° - A)$

, $AD = b \cos (180° - A)$

$\therefore BD = c + b \cos (180° - A)$

In ΔCDB : $\because (BC)^2 = (CD)^2 + (DB)^2$

i.e. $a^2 = (b \sin (180° - A))^2 + (c + b \cos (180° - A))^2$

$= (b \sin A)^2 + (c - b \cos A)^2$

$= b^2 \sin^2 A + c^2 - 2\,bc \cos A + b^2 \cos^2 A$

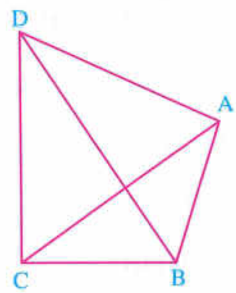
$= b^2 (\sin^2 A + \cos^2 A) + c^2 - 2\,bc \cos A$

$\therefore a^2 = b^2 + c^2 - 2\,bc \cos A$ $\quad \therefore \cos A = \dfrac{b^2 + c^2 - a^2}{2\,bc}$ (Q.E.D.)

* **Notice that :** The cosine rule is also true in case of the right-angled triangle [putting : $\cos A = \cos 90° = 0$]

Remarks

* To find the measure of an angle of a triangle , it is better to use the cosine rule , because it determines the type of the angle whether it is acute or obtuse.
* If $a : b : c = 2 : 3 : 4$, then we suppose : $a = 2\,k$, $b = 3\,k$, $c = 4\,k$ where $k \in \mathbb{R}^*$, then we substitute in the cosine rule to find the measures of the angles of ΔABC
* To prove that ABCD is a cyclic quadrilateral :



– We prove that there are two opposite supplementary angles :

$m (\angle A) + m (\angle C) = 180°$ *i.e.* $\cos A + \cos C =$ zero

or $m (\angle B) + m (\angle D) = 180°$ *i.e.* $\cos B + \cos D =$ zero

– We prove that the measures of two angles drawn on one base and on one side of it are equal : $m (\angle BAC) = m (\angle BDC)$

i.e. $\cos (\angle BAC) = \cos (\angle BDC)$

Example 1

In Δ ABC : If b = 30 cm. , c = 14 cm. , m ($\angle$ A) = 60° , find a

Solution

$\because a^2 = b^2 + c^2 - 2\,bc\cos A$

$\therefore a^2 = (30)^2 + (14)^2 - 2 \times 30 \times 14 \times \cos 60° = 676$ $\qquad \therefore a = \sqrt{676} = 26$ cm.

Example 2

XYZ is a triangle in which : x = 4 cm. , y = 5 cm. , z = 6 cm.
Calculate the measure of its greatest angle and its area.

Solution

$\because \angle Z$ is the greatest angle.

$\therefore \cos Z = \frac{x^2 + y^2 - z^2}{2xy} = \frac{4^2 + 5^2 - 6^2}{2 \times 4 \times 5} = 0.125$ $\qquad \therefore m(\angle Z) \approx 82° 49' 9''$

, the area of Δ XYZ $= \frac{1}{2} xy \sin Z = \frac{1}{2} \times 4 \times 5 \times \sin 82° 49' 9'' \approx 9.9$ cm.^{2}

Example 3

In Δ ABC : $\frac{1}{2}\sin A = \frac{1}{3}\sin B = \frac{1}{4}\sin C$, calculate m ($\angle$ C)

Solution

$\because \frac{\sin A}{2} = \frac{\sin B}{3} = \frac{\sin C}{4}$ $\qquad \therefore \frac{2}{\sin A} = \frac{3}{\sin B} = \frac{4}{\sin C}$

$\therefore a : b : c = 2 : 3 : 4$

Let a = 2 k , b = 3 k , c = 4 k

$\therefore \cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{4k^2 + 9k^2 - 16k^2}{2 \times 2k \times 3k} = \frac{-3k^2}{12k^2} = -\frac{1}{4}$

$\therefore m(\angle C) = 104° 28' 39''$

Example 4

In Δ ABC : a = 13 cm. , b = 14 cm. , c = 15 cm. Find the radius length of its circumcircle.

Solution

$\because \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{196 + 225 - 169}{2 \times 14 \times 15} = \frac{3}{5}$ $\qquad \therefore \sin A = \frac{4}{5}$

, $\because \frac{a}{\sin A} = 2r$ $\qquad \therefore \frac{13}{\frac{4}{5}} = 2r$ $\qquad \therefore r = \frac{13}{2 \times \frac{4}{5}} = 8\frac{1}{8}$ cm.

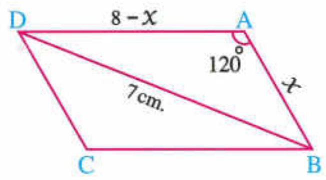
Example 5

ABCD is a parallelogram in which : m (∠ A) = 120° , its perimeter = 16 cm.

, the length of its greater diagonal = 7 cm.

Find the area of ▱ ABCD , given that : AB < BC

Solution



$\because \frac{1}{2}$ perimeter $= \frac{16}{2} = 8$ cm.

Let AB = x cm.

$\therefore$ AD = $(8 - x)$ cm.

In Δ ABD : $\because (BD)^2 = (AB)^2 + (AD)^2 - 2\,(AB)\,(AD)\cos 120°$

$\therefore 49 = x^2 + (8 - x)^2 - 2\,x\,(8 - x) \times \left(-\frac{1}{2}\right)$

$\therefore 49 = x^2 + 64 - 16\,x + x^2 + 8\,x - x^2$

$\therefore x^2 - 8\,x + 15 = 0$ $\qquad\therefore (x - 3)\,(x - 5) = 0$ $\qquad\therefore x = 3$ or $x = 5$

, $\because$ AB < BC $\qquad\therefore$ AB = 3 cm. , AD = 5 cm.

$\therefore$ The area of (▱ ABCD) = 2 the area of (Δ ABD)

$$= 2 \times \frac{1}{2} \times 3 \times 5 \sin 120° = \frac{15\sqrt{3}}{2} \text{ cm}^2$$

Example 6

ABCD is a parallelogram in which : AB = 8 cm. , BC = 11 cm. , BD = 9 cm.

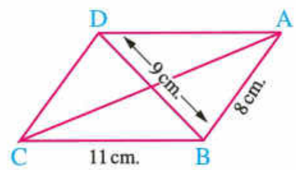
Find the length of $\overline{AC}$

Solution

In Δ ABD : $\cos A = \frac{(8)^2 + (11)^2 - (9)^2}{2 \times 8 \times 11} = \frac{13}{22}$

, $\because$ m (∠ A) + m (∠ B) = 180° (two consecutive angles in ▱ ABCD)

$\therefore \cos B = -\cos A = \frac{-13}{22}$



$\therefore$ In Δ ABC : $(AC)^2 = (8)^2 + (11)^2 - 2 \times 8 \times 11 \times \frac{-13}{22} = 289$

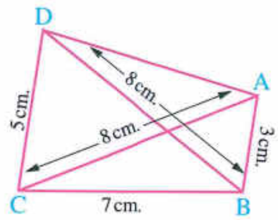
$\therefore$ AC = 17 cm.

Example 7

ABCD is a quadrilateral in which : AB = 3 cm. , BC = 7 cm. , CD = 5 cm.

, AC = BD = 8 cm. Prove that : ABCD is a cyclic quadrilateral.

Solution



In Δ BAC : $\cos(\angle BAC) = \frac{3^2 + 8^2 - 7^2}{2 \times 3 \times 8} = \frac{1}{2}$

, in Δ BDC : $\cos(\angle BDC) = \frac{5^2 + 8^2 - 7^2}{2 \times 5 \times 8} = \frac{1}{2}$

$\therefore \cos(\angle BAC) = \cos(\angle BDC)$

$\therefore m(\angle BAC) = m(\angle BDC)$ (and they are drawn on $\overline{BC}$ and on one side of it)

$\therefore$ ABCD is a cyclic quadrilateral.

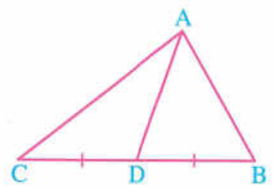
Example 8

ABC is a triangle in which D is the midpoint of $\overline{BC}$. Prove that :

$(AB)^2 + (AC)^2 = 2(AD)^2 + 2(BD)^2$ and if AB = 3 cm. , AC = 4 cm.

and AD = $\sqrt{3.5}$ cm. , find the length of $\overline{BC}$

Solution



In Δ ABD :

$(AB)^2 = (AD)^2 + (BD)^2 - 2(AD)(BD)\cos(\angle ADB)$ (1)

In Δ ACD : $(AC)^2 = (AD)^2 + (CD)^2 - 2(AD)(CD)\cos(\angle ADC)$

, $\because \cos(\angle ADB) = -\cos(\angle ADC)$, CD = BD

$\therefore (AC)^2 = (AD)^2 + (BD)^2 + 2(AD)(BD)\cos(\angle ADB)$ (2)

By adding (1) and (2) :

$\therefore (AB)^2 + (AC)^2 = 2(AD)^2 + 2(BD)^2$ (First req.)

$\therefore (3)^2 + (4)^2 = 2\left(\sqrt{3.5}\right)^2 + 2(BD)^2$

$\therefore 25 = 7 + 2(BD)^2$ $\therefore (BD)^2 = 9$

$\therefore$ BD = 3 cm. $\therefore$ BC = 6 cm. (Second req.)



Lesson Three — Solution of the triangle

WATCH VIDEO

Solution of the triangle means to find the lengths of its sides and the measures of its angles , if it is given three of these six elements (one of them at least is the length of one side).

There are four cases for solving a triangle :

WATCH VIDEO

First case — Solving the triangle given the length of one side and the measures of two angles :

In Δ ABC , if m ($\angle$ A) , m ($\angle$ B) , a are given :

(1) Use the relation : $m(\angle C) = 180° - [m(\angle A) + m(\angle B)]$ to find m ($\angle$ C)

(2) Use the law : $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ to find b , c

Example 1

Solve the triangle ABC in which : m ($\angle$ A) = 38° 52' , m ($\angle$ B) = 96° 51' , a = 22.3 cm.

Solution

$m(\angle C) = 180° - (38° 52' + 96° 51') = \boxed{44° 17'}$

$, \because \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ $\therefore \frac{22.3}{\sin 38° 52'} = \frac{b}{\sin 96° 51'} = \frac{c}{\sin 44° 17'}$

$\therefore b = \frac{22.3 \sin 96° 51'}{\sin 38° 52'} \approx \boxed{35.3 \text{ cm.}}$

$, c = \frac{22.3 \sin 44° 17'}{\sin 38° 52'} \approx \boxed{24.8 \text{ cm.}}$

Second case — Solving the triangle given the lengths of two sides and the measure of the included angle :

In Δ ABC , if a , b , m ($\angle$ C) are given :

(1) Use the law : $c^2 = a^2 + b^2 - 2\,ab\cos C$ to find c

(2) It's better to use the law : $\cos A = \frac{b^2 + c^2 - a^2}{2\,b\,c}$ to find m ($\angle$ A) because it determines the acute or obtuse angle (or you can use the sine law to find the measure of the angle opposite to the smaller of the two given sides)

(3) Use the relation : $m(\angle B) = 180^\circ - [m(\angle A) + m(\angle C)]$ to find m ($\angle$ B)

Example 2

Solve the triangle ABC in which : a = 8 cm. , b = 5 cm. , m ($\angle$ C) = $60^\circ\, 2'$

Solution

$\because c^2 = a^2 + b^2 - 2\,a\,b\cos C = 64 + 25 - 2 \times 8 \times 5 \cos 60^\circ\, 2' \approx 49.04$

$\therefore c \approx$ **7 cm.**

$\therefore \cos A = \frac{b^2 + c^2 - a^2}{2\,b\,c} = \frac{25 + 49 - 64}{2 \times 5 \times 7} = \frac{1}{7}$

$\therefore m(\angle A) \approx$ **$81^\circ\, 47'$**

$\therefore m(\angle B) = 180^\circ - (60^\circ\, 2' + 81^\circ\, 47') =$ **$38^\circ\, 11'$**

Another solution :

After finding c , you can find m ($\angle$ B) using the sine law because $\angle$ B is opposite to the smaller given side.

$\because \frac{c}{\sin C} = \frac{b}{\sin B}$ $\qquad \therefore \frac{7}{\sin 60^\circ\, 2'} = \frac{5}{\sin B}$

$\therefore \sin B = \frac{5 \sin 60^\circ\, 2'}{7}$ $\qquad \therefore m(\angle B) \approx 38^\circ\, 14'$ or $141^\circ\, 46'$

, $\because$ b is not the length of the longest side

$\therefore \angle$ B cannot be obtuse. $\qquad \therefore m(\angle B) = 38^\circ\, 14'$

$\therefore m(\angle A) = 180^\circ - (60^\circ\, 2' + 38^\circ\, 14') = 81^\circ\, 44'$

* **Notice that :** The differences in the measures of angles between the two solutions is due to approximation in calculators.

Third case Solving the triangle given the lengths of the three sides :

In ΔABC , if a , b , c are given :

(1) Use the law : $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ to find $m(\angle A)$

(2) Use the law : $\cos B = \frac{c^2 + a^2 - b^2}{2ca}$ to find $m(\angle B)$

(3) Use the relation : $m(\angle C) = 180° - [m(\angle A) + m(\angle B)]$ to find $m(\angle C)$

Example 3

Solve the triangle ABC in which : a = 5 cm. , b = 7 cm. , c = 11 cm.

Solution

$\because \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{49 + 121 - 25}{2 \times 7 \times 11} = \frac{145}{154}$

$\therefore m(\angle A) \approx 19° 41'$

, $\because \cos B = \frac{c^2 + a^2 - b^2}{2ca} = \frac{121 + 25 - 49}{2 \times 11 \times 5} = \frac{97}{110}$

$\therefore m(\angle B) \approx 28° 8'$

$\therefore m(\angle C) = 180° - (19° 41' + 28° 8') = 132° 11'$

Remember that

The sum of any two side lengths in a triangle is greater than the length of the third side.

For example :

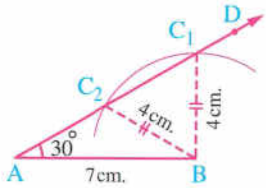
If a = 2 cm. , b = 5 cm. and c = 8 cm. , then these lengths cannot be side lengths of a triangle.

Fourth case Solving the triangle given the lengths of two sides and the measure of the opposite angle to one of them «Ambiguous case»

Illustrated Example

Using the geometric tools , draw ΔABC in which AB = 7 cm. , $m(\angle A) = 30°$ and BC = 4 cm. , then verify your answer using the sine rule.

Solution



* We draw a line segment $\overline{AB}$ of length 7 cm.
* We draw $\angle$ A of measure 30° with $\overline{AB}$ and it is $\angle$ BAD
* We place the sharp point of the compasses at the point B and adjust it with length 4 cm. , and draw an arc intersecting the straight line $\overleftrightarrow{AD}$ at C
* We notice that the point C has two positions , *i.e.* we can draw two triangles having the same previous conditions and they are ABC_1 and ABC_2 , by measuring we find that : m ($\angle$ C) ≈ 61° in ΔABC_1 or m ($\angle$ C) ≈ 119° in ΔABC_2

Verifying the answer by using the sine rule :

$\because \frac{a}{\sin A} = \frac{c}{\sin C}$ $\qquad \therefore \frac{4}{\sin 30°} = \frac{7}{\sin C}$

$\therefore \sin C = \frac{7 \sin 30°}{4} = \frac{7}{8}$ (positive)

$\therefore$ $\angle$ C lies in the first quadrant (acute) or in the second quadrant (obtuse)

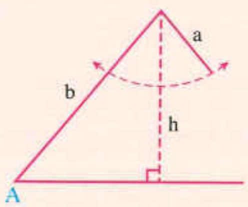
$\therefore$ m ($\angle$ C) ≈ 61° or m ($\angle$ C) ≈ 119°

Generally , by using the geometric solution , we can reach to the following :

* In Δ ABC , if a , b and m ($\angle$ A) are given , then we find h = b sin A , and to find the possible solutions of the triangle , we compare between the values a , b and h as follows :

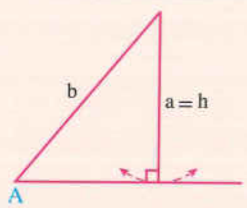
First : If $\angle$ A is acute and :

(1)



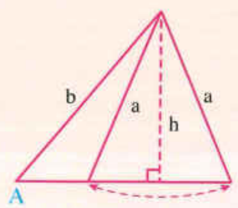
a < h , then we cannot draw the triangle.

(2)



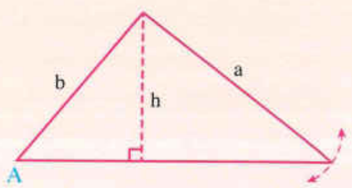
a = h , then we can draw a unique right-angled triangle.

(3)



h < a < b , then we can draw two triangles.

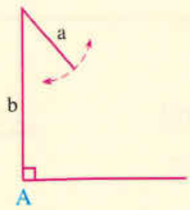
(4)



a ≥ b , then we can draw a unique triangle.

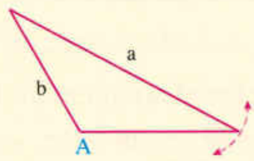
Second : If ∠ A is right or obtuse and :

(1)



$a \leq b$, then we cannot draw a triangle.

(2)



$a > b$, then we can draw a unique triangle

* In this case , we can solve the triangle using the sine rule directly without determining the number of possible triangles with considering the following :
  (1) ∠ A lies in the first quadrant (if it is acute) and lies in the second quadrant (if it is obtuse)
  (2) The range of the sine function is $[-1 , 1]$
  (3) If the triangle has an obtuse angle , then the other two angles must be acute angles.

Example 4

Show if the following conditions satisfy the existence of one triangle or more , or don't satisfy the existence of any triangle at all , then find the possible solutions :

WATCH VIDEO

(1) ABC is a triangle in which m (∠ A) = 112° , a = 7 cm. and b = 4 cm.
(2) ABC is a triangle in which m (∠ A) = 112° , a = 4 cm. and b = 7 cm.
(3) LMN is a triangle in which m (∠ L) = 50° , ℓ = 4 cm. and m = 7 cm.
(4) DEF is a triangle in which m (∠ D) = 60° , d = 7.5 cm. and e = $5\sqrt{3}$ cm.
(5) LMN is a triangle in which m (∠ L) = 30° , ℓ = 6 cm. and m = 9 cm.
(6) ABC is a triangle in which m (∠ A) = 40° , a = 8.5 cm. and b = 7 cm.

Solution

(1) ∵ ∠ A is obtuse , a > b

∴ The triangle has a unique solution.

, ∵ $\frac{a}{\sin A} = \frac{b}{\sin B}$

∴ $\frac{7}{\sin 112°} = \frac{4}{\sin B}$

∴ $\sin B = \frac{4 \sin 112°}{7} \approx 0.5298$

Notice that :

The triangle has one obtuse angle at most.
, ∵ ∠ A is obtuse.
∴ ∠ B must be acute.
∴ ∠ B lies in the first quadrant only.

$\therefore m(\angle B) \approx$ 32°

$\therefore m(\angle C) = 180° - (112° + 32°) =$ 36°

$, \because \frac{a}{\sin A} = \frac{c}{\sin C}$ $\therefore \frac{7}{\sin 112°} = \frac{c}{\sin 36°}$

$\therefore c \approx$ 4.4 cm.

(2) $\because \angle A$ is obtuse , $a < b$

$\therefore$ The conditions do not satisfy the existence of any triangle at all.

Notice that :

$\because \frac{4}{\sin 112°} = \frac{7}{\sin B}$ $\therefore \sin B = \frac{7 \sin 112°}{4} \approx 1.6$

and this is impossible because $\sin B \notin [-1 , 1]$

(3) $\because \angle L$ is acute , $h = m \sin L = 7 \sin 50° \approx 5.4$ cm.

$, \because \ell < h$

$\therefore$ The conditions do not satisfy the existence of any triangle at all.

(4) $\because \angle D$ is acute , $h = e \sin D = 5\sqrt{3} \sin 60° = 7.5$ cm.

$, \because d = h$

$\therefore$ There is a unique solution to the triangle which is right-angled at E

$\therefore m(\angle F) = 180° - (60° + 90°) =$ 30°

$, f = \sqrt{(5\sqrt{3})^2 - (7.5)^2} = \frac{5\sqrt{3}}{2}$ cm.

(5) $\because \angle L$ is acute , $h = m \sin L = 9 \sin 30° = 4.5$ cm.

$, \because 4.5 < 6 < 9$ *i.e.* $h < \ell < m$

$\therefore$ There are two solutions to the triangle

$\because \frac{\ell}{\sin L} = \frac{m}{\sin M}$ $\therefore \frac{6}{\sin 30°} = \frac{9}{\sin M}$

$\therefore \sin M = \frac{3}{4}$

$\therefore \angle M$ lies in the first or the second quadrant.

$\therefore m(\angle M) \approx \boxed{48^\circ\ 35'\ 25''}$

$\therefore m(\angle N)$

$= 180^\circ - (30^\circ + 48^\circ\ 35'\ 25'')$

$= \boxed{101^\circ\ 24'\ 35''}$

$, \because \frac{\ell}{\sin L} = \frac{n}{\sin N}$

$\therefore \frac{6}{\sin 30^\circ} = \frac{n}{\sin 101^\circ\ 24'\ 35''}$

$\therefore n \approx \boxed{11.76 \text{ cm.}}$

or

$\therefore m(\angle M) = 180^\circ - 48^\circ\ 35'\ 25''$

$= \boxed{131^\circ\ 24'\ 35''}$

$\therefore m(\angle N)$

$= 180^\circ - (30^\circ + 131^\circ\ 24'\ 35'')$

$= \boxed{18^\circ\ 35'\ 25''}$

$, \because \frac{6}{\sin 30^\circ} = \frac{n}{\sin 18^\circ\ 35'\ 25''}$

$\therefore n \approx \boxed{3.83 \text{ cm.}}$

(6) $\because \angle A$ is acute , $a > b$

$\therefore$ There is a unique solution to the triangle

$, \because \frac{a}{\sin A} = \frac{b}{\sin B}$

$\therefore \frac{8.5}{\sin 40^\circ} = \frac{7}{\sin B}$

$\therefore \sin B = \frac{7 \sin 40^\circ}{8.5} \approx 0.53$

$\therefore m(\angle B) \approx \boxed{32^\circ}$

$\therefore m(\angle C) = 180^\circ - (40^\circ + 32^\circ) = \boxed{108^\circ}$

$, \because \frac{a}{\sin A} = \frac{c}{\sin C}$

$\therefore \frac{8.5}{\sin 40^\circ} = \frac{c}{\sin 180^\circ}$

$\therefore c = \frac{8.5 \sin 108^\circ}{\sin 40^\circ} \approx \boxed{12.58 \text{ cm.}}$

Notice that :

If we consider $\angle B$ lies in the second quadrant.

$\therefore m(\angle B) = 180^\circ - 32^\circ = 148^\circ$

and this is impossible , because it is not possible that the sum of measures of two angles in a triangle $= 40^\circ + 148^\circ = 188^\circ$

i.e. Greater than 180°

Remark

In the ambiguous case , we can solve the triangle by using the cosine rule to find the length of the third side , then we get a quadratic equation and by solving it , the number of the triangles is the number of positive solutions of this equation.

Example 5

Using the previous remark , solve the triangle ABC in which a = 6 cm. , b = 8 cm. and m (∠ A) = 30°

Solution

$\because a^2 = b^2 + c^2 - 2\,bc\cos A$

$\therefore (6)^2 = (8)^2 + c^2 - 2 \times 8 \times c\cos 30^\circ$

$\therefore c^2 - 8\sqrt{3}\,c + 28 = 0$

$$\therefore c = \frac{8\sqrt{3} \pm \sqrt{\left(-8\sqrt{3}\right)^2 - 4\,(1)\,(28)}}{2\,(1)}$$

Remember that

The general formula of solving a quadratic equation in the form :

$a\,x^2 + bx + c = 0$ is

$$x = \frac{-b \pm \sqrt{b^2 - 4\,ac}}{2\,a}$$

$\therefore c \approx 11.4$ cm. or $c \approx 2.456$ cm.

, $\because$ each positive value of c is corresponding to one triangle.

$\therefore$ We have two triangles , then we find cos B from the relation : $\cos B = \frac{c^2 + a^2 - b^2}{2\,ca}$

At $c \approx 11.4$ cm.

$\therefore \cos B_1 = \frac{(11.4)^2 + (6)^2 - (8)^2}{2\,(11.4)\,(6)}$

$\therefore m(\angle B_1) \approx 41^\circ\, 49'$

$\therefore m(\angle C_1)$

$= 180^\circ - (30^\circ + 41^\circ\, 49')$

$= 108^\circ\, 11'$

At $c \approx 2.456$ cm.

$\therefore \cos B_2 = \frac{(2.456)^2 + (6)^2 - (8)^2}{2\,(2.456)\,(6)}$

$\therefore m(\angle B_2) \approx 138^\circ\, 12'$

$\therefore m(\angle C_2)$

$= 180^\circ - (30^\circ + 138^\circ\, 12')$

$= 11^\circ\, 48'$

$\therefore$ The first solution is c = 11.4 cm. , $m(\angle B) = 41^\circ\, 49'$ and $m(\angle C) = 108^\circ\, 11'$

, the second solution is c = 2.456 cm. , $m(\angle B) = 138^\circ\, 12'$ and $m(\angle C) = 11^\circ\, 48'$

* Try to solve this example by using the sine rule.

Example 6

Solve the triangle ABC in which : m (∠ A) = 40° , m (∠ C) = 35° , the radius length of its circumcircle = 6 cm.

Solution

$m(\angle B) = 180^\circ - (40^\circ + 35^\circ) = 105^\circ$

$$\therefore \frac{a}{\sin 40^\circ} = \frac{b}{\sin 105^\circ} = \frac{c}{\sin 35^\circ} = 12$$

$\therefore a \approx$ 7.7 cm. , $b \approx$ 11.6 cm. , $c \approx$ 6.9 cm.

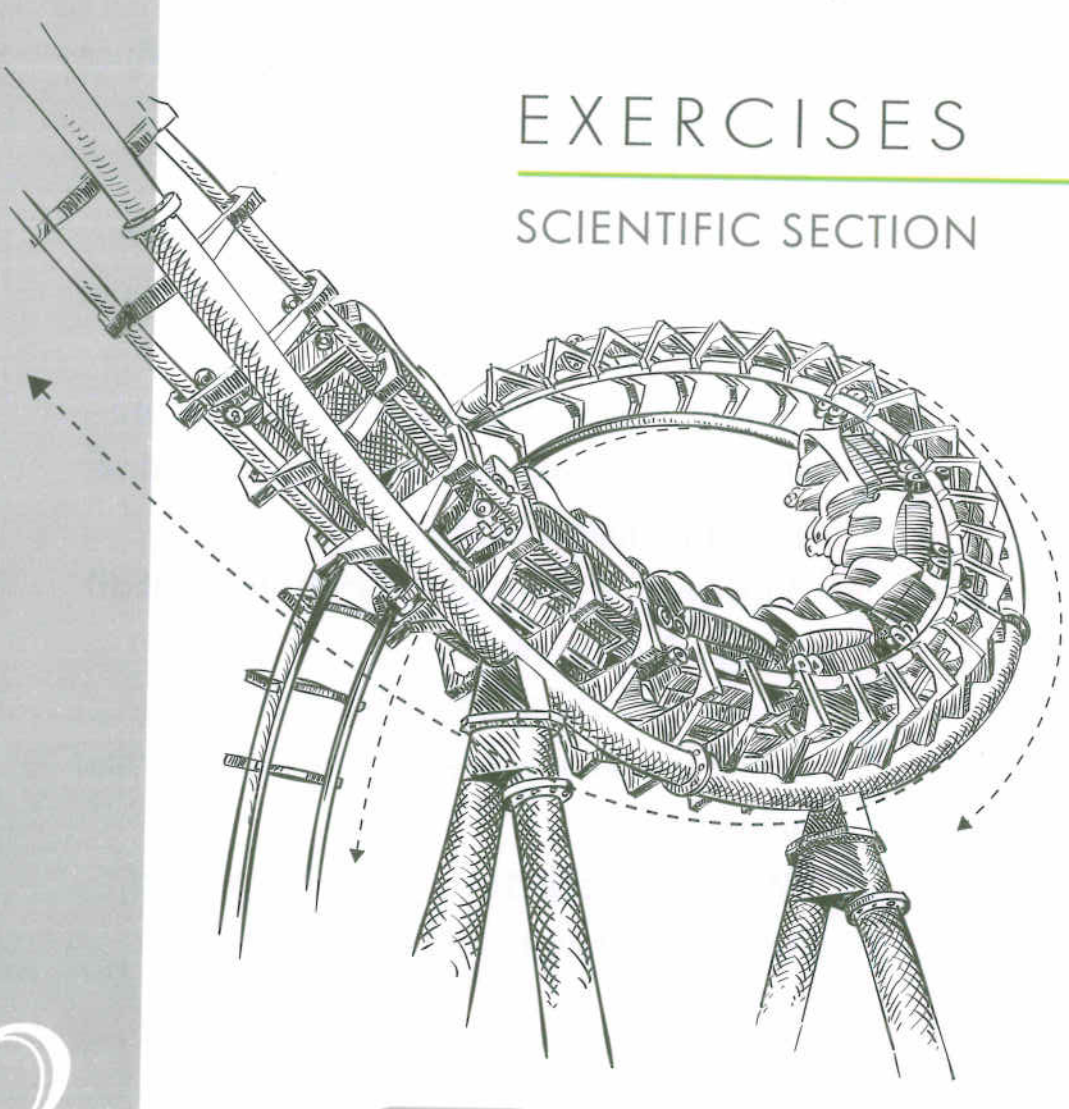
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UNIT 2 Exponents , logarithms and their applications.

Second Calculus and Trigonometry

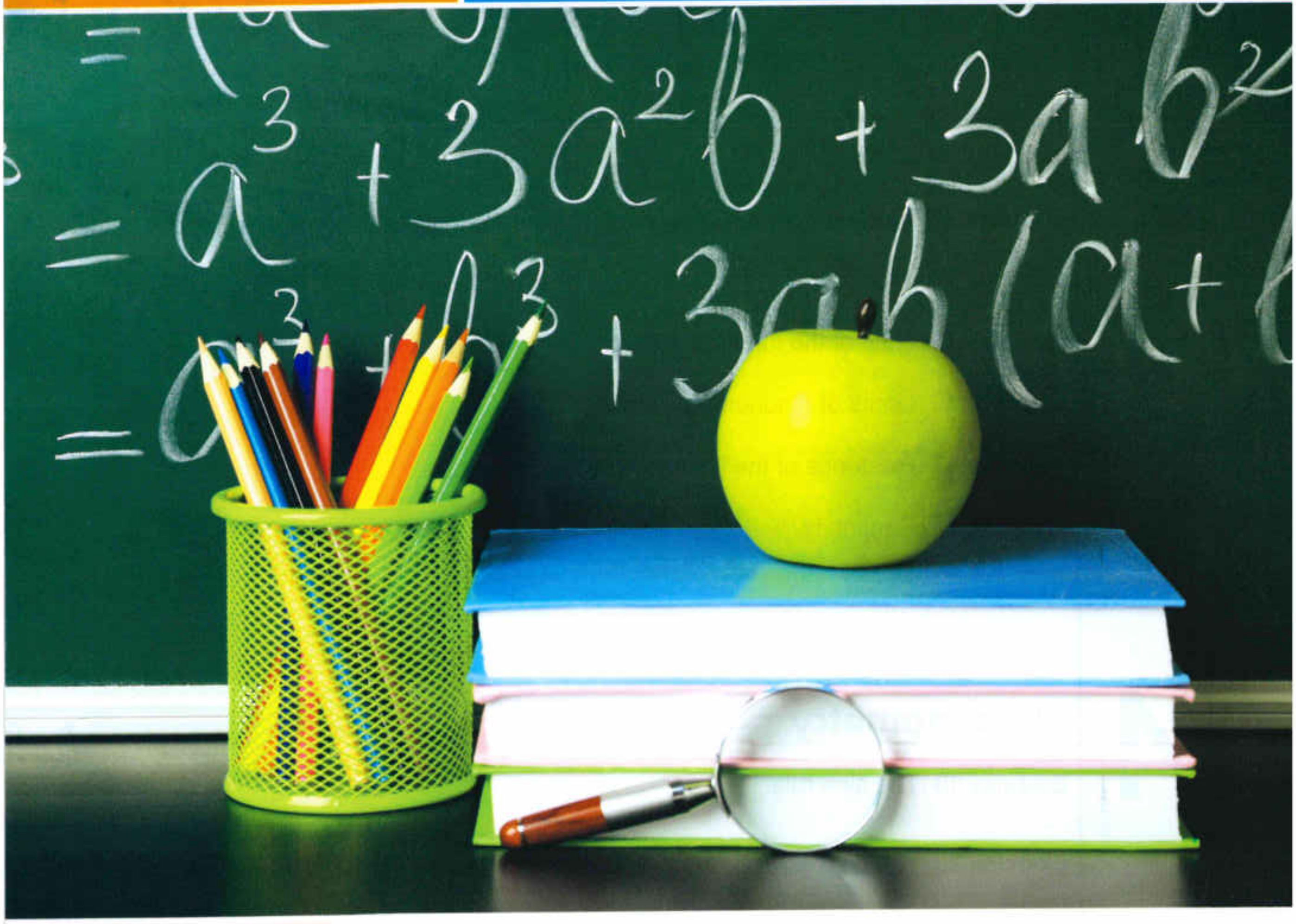
UNIT 3

Limits and continuity.

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First Algebra



UNIT 1 Functions of a real variable and drawing curves.

UNIT 2 Exponents , logarithms and their applications.

UNIT 1

Functions of a real variable and drawing curves

Unit Exercises

- Exercise on pre-requirements for unit one.

Exercise 1 Real functions.
(Determination the domain and range – Discuss the monotony).

Exercise 2 Operations on functions – Composition of functions.

Exercise 3 Some properties of functions (even and odd functions / one - to - one functions)

Exercise 4 Graphical representation of basic functions and graphing piecewise functions.

Exercise 5 Geometrical transformations of basic function curves.

Exercise 6 Solving absolute value equations and inequalities.

- At the end of the unit :
  life applications on unit one.

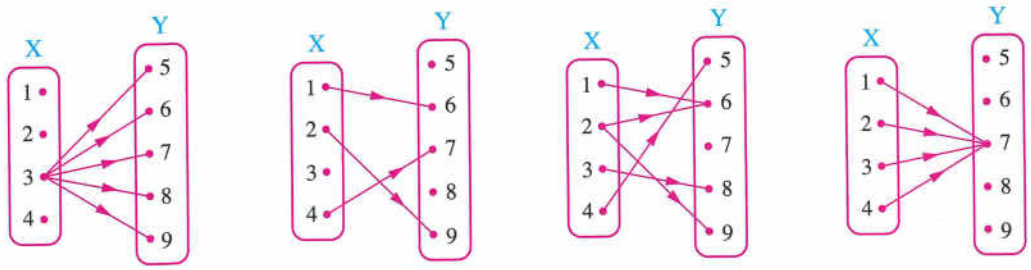
Exercise on pre-requirements

📖 From the school book

First Multiple choice questions

Choose the correct answer from the given ones :

(1) 📖 Which of the following arrow diagrams represents a function from X to Y ?



(a) (b) (c) (d)

(2) 📖 The relation shown by the set of the ordered pairs and does not represent a function is ....

(a) $\{(1 , 3) , (3 , 5) , (5 , 7) , (7 , 9)\}$

(b) $\{(2 , 3) , (3 , 4) , (2 , 1) , (3 , 5)\}$

(c) $\{(0 , 3) , (1 , 3) , (2 , 3) , (3 , 3)\}$

(d) $\{(-3 , 5) , (-1 , 5) , (0 , 5) , (2 , 5)\}$

(3) If $f : \mathbb{R} \longrightarrow \mathbb{R}$ and f maps a number to half its square added to 3 , then $f(2) =$ ....

(a) $\frac{1}{2}$ (b) 1 (c) 3 (d) 5

(4) If $\mathbb{N}$ is the set of the natural numbers , which of the following represents a function from $\mathbb{N} \longrightarrow \mathbb{N}$?

(a) $f(X) = \frac{2X}{3}$ (b) $g(X) = 1 - X$

(c) $h(X) = 2X + 3$ (d) $n(X) = \frac{1}{X - 2}$

(5) If $f : \{1 , 2 , 3 , 4 , 5\} \longrightarrow \mathbb{R}$ where $f(X + 2) = 3X + 1$, then $f(3) =$ ....

(a) 10 (b) 9 (c) 4 (d) 1

(6) The domain of the function f where $f(x) = \frac{x^3 - 8}{4}$ is ....

(a) $\mathbb{R}$ (b) $\mathbb{R} - \{8\}$ (c) $\mathbb{R} - \{2\}$ (d) $\mathbb{R} - \{4\}$

(7) If $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$, $f(x) = \frac{x^2 + 1}{x}$, then the domain of the function is ....

(a) $\mathbb{R}$ (b) $\mathbb{R} - \{0\}$ (c) $\mathbb{R}^+$ (d) $\mathbb{R} - \{-1\}$

Second Essay questions

1 If f , g are two polynomial functions where $f(x) = (a x + 5)^2$, $g(x) = 9x^2 + 30x + c - 4$ and $f(x) = g(x)$, find the values of a and c « 3 , 29 »

2 **Determine the values of a , b and c that make $f(x) = g(x)$ where :**

(1) $f(x) = (a + b) x^3 + 3x - 2$, $g(x) = 5x^3 + (a + c) x + b$

(2) $f(x) = (a + b) x^3 - 2$, $g(x) = x^3 + (a + c) x + b$

(3) $f(x) = (a + 2b) x^3 - c x + 4$, $g(x) = 7x^3 + 5x + (a - b)$



Exercise 1 Real functions

Test yourself

From the school book ● Understand ● Apply ● Higher Order Thinking Skills

First Multiple choice questions

Choose the correct answer from those given :

(1) In all the following relations , y is a function in x except ....

(a) $y = 3x + 1$ (b) $y = x^2 - 4$ (c) $x = y^2 - 2$ (d) $y = \sin x$

(2) In all the following relations , y is a function except ....

(a) $y = \cos x$ (b) $y = 2$ (c) $y = x^2 - 1$ (d) $y^2 = x^2 + 1$

(3) If $f\left(\sqrt{x}\right) = x^2$, then $f(2) =$ ....

(a) $\sqrt{2}$ (b) 2 (c) 4 (d) 16

(4) If $f : \mathbb{R} \longrightarrow \mathbb{R}$ where $f(x) = (a + 1)x + b - 2$ and $f(x)$ relates each real number to itself then $(a , b) =$ ....

(a) (1 , 3) (b) (0 , 3) (c) (0 , 2) (d) (0 , 0)

(5) The domain of the function $f : f(x) = 5$ is ....

(a) $\mathbb{R}$ (b) $\mathbb{R}^+$ (c) $\{5\}$ (d) $\{0 , 5\}$

(6) The domain of the function $f : f(x) = \frac{2x + 1}{x - 2}$ is ....

(a) $\mathbb{R}$ (b) $\mathbb{R} - \{-\frac{1}{2}\}$ (c) $\mathbb{R} - \{-\frac{1}{2} , 2\}$ (d) $\mathbb{R} - \{2\}$

(7) The domain of the function $f : f(x) = \frac{x + 5}{(x + 5)(x - 5)}$ is ....

(a) $\mathbb{R}$ (b) $\{5 , -5\}$ (c) $\mathbb{R} - \{5\}$ (d) $\mathbb{R} - \{5 , -5\}$

(8) The domain of the function $f : f(x) = \frac{x^2 + 1}{x^2 + 4x}$ is ....

(a) $\mathbb{R} - \{1 , -1\}$ (b) $\mathbb{R} - \{0 , -4\}$ (c) $\mathbb{R}$ (d) $\mathbb{R} - \{0 , 4\}$

(9) The domain of the function f where $f(x) = \frac{5+2x}{x^2+x+1}$ is ....

(a) $\mathbb{R}$ (b) $\mathbb{R} - \{5\}$ (c) $\mathbb{R} - \{2\}$ (d) $\mathbb{R} - \{-2, -5\}$

(10) The domain of the function f where $f(x) = \frac{7}{x^3 - x}$ is ....

(a) $\mathbb{R} - \{3\}$ (b) $\mathbb{R} - \{7\}$ (c) $\mathbb{R} - \{0, 1\}$ (d) $\mathbb{R} - \{0, 1, -1\}$

(11) The domain of the function f where $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$, $f(x) = \frac{x-1}{4x}$ is ....

(a) $\mathbb{R}$ (b) $\mathbb{R} - \{0\}$ (c) $\mathbb{R}^+$ (d) $\mathbb{R} - \{1\}$

(12) If the domain of the function $f : f(x) = \frac{2}{x^2 - 6x + k}$ is $\mathbb{R} - \{3\}$, then k = ...

(a) 3 (b) 9 (c) ± 9 (d) 18

(13) The domain of the function $f : f(x) = \sqrt{x-3}$ is ....

(a) $\mathbb{R}$ (b) $\mathbb{R} - \{3\}$ (c) $[3, \infty[$ (d) $]-\infty, 3[$

(14) The domain of the function f where $f(x) = \sqrt{4-x}$ is ....

(a) $[4, \infty[$ (b) $]-\infty, 4[$ (c) $]4, \infty[$ (d) $]-\infty, 4]$

(15) The domain of the function $f : f(x) = \sqrt[3]{x-5}$ is ....

(a) $[5, \infty[$ (b) $]-\infty, 5[$ (c) $\mathbb{R}$ (d) $\mathbb{R}^+$

(16) The domain of the function $f : f(x) = \sqrt[3]{9-x^2}$ is ....

(a) $]-3, 3[$ (b) $\mathbb{R}$ (c) $\mathbb{R} -]-3, 3[$ (d) $[-3, 3]$

(17) The domain of the function $f : f(x) = \sqrt{x+3}$ is ....

(a) $]-3, \infty[$ (b) $[-3, \infty[$ (c) $\{-3\}$ (d) $\{0\}$

(18) The domain of the function $f : f(x) = \sqrt{x^2+9}$ is ....

(a) $\mathbb{R}$ (b) $\mathbb{R} - \{-3, 3\}$ (c) $\mathbb{R} - [-3, 3]$ (d) $]-3, 3[$

(19) The domain of the function $f : f(x) = \frac{5}{\sqrt{x-4}}$ is ....

(a) $[4, \infty[$ (b) $]4, \infty[$ (c) $]-\infty, 4]$ (d) $]-\infty, 4[$

(20) The domain of the function $f : f(x) = \frac{1}{\sqrt{9-x^2}}$ is ....

(a) $\mathbb{R}$ (b) $\mathbb{R} - [-3, 3]$ (c) $\mathbb{R} - \{-3, 3\}$ (d) $]-3, 3[$

(21) The domain of the function f where $f(x) = \sqrt[4]{x^2+4}$ is ....

(a) $\mathbb{R}$ (b) $\mathbb{R} - \{4\}$ (c) $\mathbb{R} - \{0\}$ (d) $\mathbb{R} - \{-2, 2\}$

(22) The domain of the function f where $f(x) = \frac{1}{\sqrt[3]{x^2 - 5x - 6}}$ is ....

(a) $\mathbb{R} - \{5\}$ (b) $\mathbb{R} - \{6\}$ (c) $\mathbb{R} - \{1, -6\}$ (d) $\mathbb{R} - \{-1, 6\}$

(23) The domain of the function $f : f(x) = \frac{1}{\sqrt[3]{x^2 + 9x + 8}}$ is ....

(a) $\mathbb{R} - \{-8, -1\}$ (b) $\mathbb{R} - [-8, -1]$ (c) $]-8, -1[$ (d) $\{-8, -1\}$

(24) If A is the area of a circle and x is its radius length and the area is given as a function of x where $A = \pi x^2$, then its domain = ...

(a) $\mathbb{R}$ (b) $\mathbb{R} - \{0\}$ (c) $\mathbb{Z}^+$ (d) $\mathbb{R}^+$

(25) If the domain of the function $f : f(x) = \frac{1}{\sqrt{x - a}}$ is $]-3, \infty[$, then a = ...

(a) 3 (b) -3 (c) ± 3 (d) $\sqrt{3}$

(26) If the domain of the function $f : f(x) = \frac{1}{\sqrt{x^2 + a}}$ is $\mathbb{R} - [-5, 5]$, then a = ...

(a) 5 (b) -25 (c) -5 (d) 25

(27) If the domain of the function $f : f(x) = \frac{1}{\sqrt{x^2 + a}}$ is $\mathbb{R}$, then a can not be equal ....

(a) 5 (b) $\sqrt{4}$ (c) zero. (d) 9

(28) If : $f(x) = \begin{cases} -4x + 3 & , \quad x < 3 \\ -x^3 & , \quad 3 \le x \le 8 \\ 3x^2 + 1 & , \quad x > 8 \end{cases}$, then $f(10) =$ ....

(a) -37 (b) -1000 (c) 301 (d) 43

(29) The domain of the function f where $f(x) = \begin{cases} -2 & , \quad x < 2 \\ 3 & , \quad x > 2 \end{cases}$ is ....

(a) $\mathbb{R}$ (b) $\mathbb{R} - \{3\}$ (c) $\mathbb{R} - \{-2\}$ (d) $\mathbb{R} - \{2\}$

(30) The domain of the function f where $f(x) = \begin{cases} x & , \quad 0 \le x \le 1 \\ 2 - x & , \quad 1 < x \le 2 \end{cases}$ is ....

(a) $\mathbb{R} - \{1\}$ (b) $[0, 2]$ (c) $\mathbb{R} - \{0, 2\}$ (d) $]0, 2[$

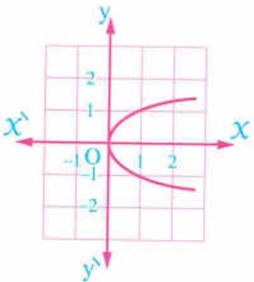
(31) The domain of the function $f : f(x) = \begin{cases} x^2 + 2 & , \quad -2 \le x < 0 \\ x - 1 & , \quad x > 0 \end{cases}$ is ....

(a) $\mathbb{R}$ (b) $[-2, 0[$ (c) $[-2, \infty[- \{0\}$ (d) $[-2, \infty[$

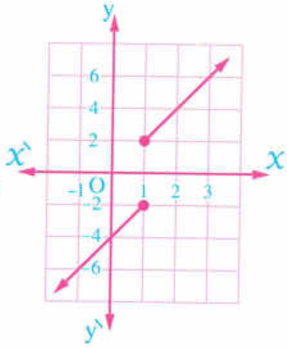
(32) The range of the function $f : f(X) = \begin{cases} 0 & , \quad X \le 0 \\ 1 & , \quad X > 0 \end{cases}$ is ....

(a) $\{1\}$ (b) $\{0\}$ (c) $\mathbb{R}$ (d) $\{0 , 1\}$

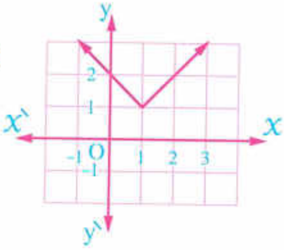
(33) The figure which represents y as a function in X is ....



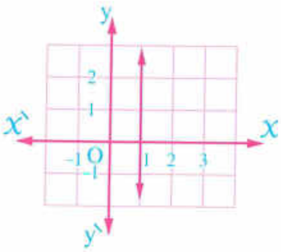
(a)



(b)

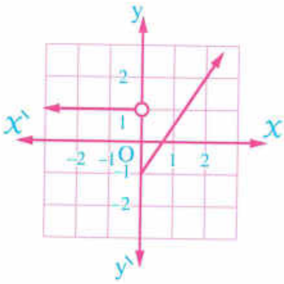


(c)

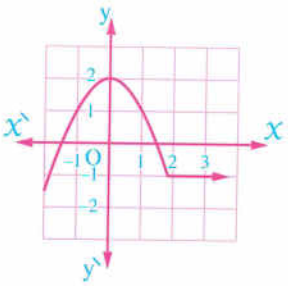


(d)

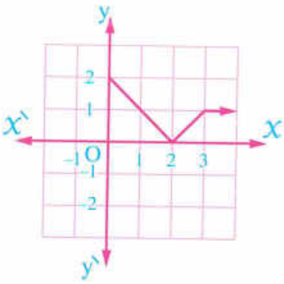
(34) Which of the following graphs does not represent a function ?



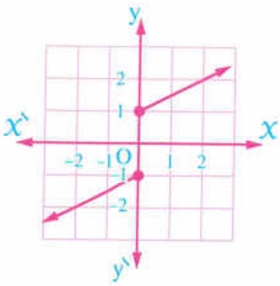
(a)



(b)

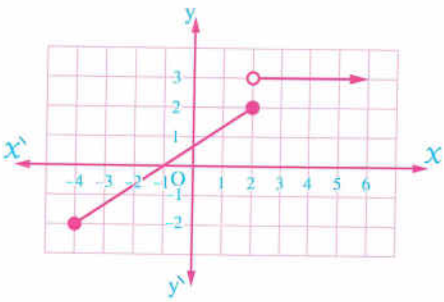


(c)



(d)

(35) The opposite figure represents ....



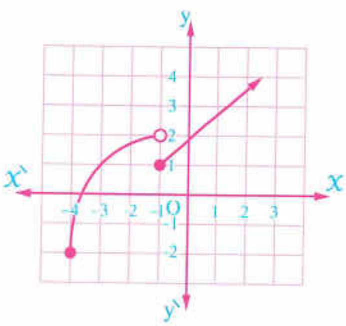
(a) function $f : [-4 , 2] \longrightarrow \mathbb{R}$

(b) function $f : [-4 , \infty[\longrightarrow \mathbb{R}$

(c) function $f : [-4 , 2] \longrightarrow [-2 , 3]$

(d) relation between X , y but not a function.

(36) The opposite figure represents the curve of function f , then its domain is ....



(a) $\mathbb{R} - \{-4 , -1\}$

(b) $]-4 , -1[$

(c) $[-4 , \infty[$

(d) $[-4 , \infty[- \{-1\}$

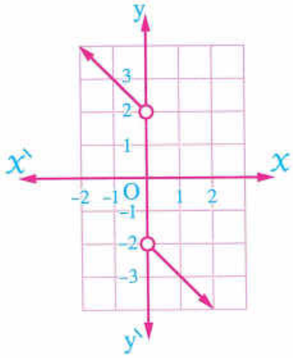
(37) The opposite figure represents function of x, its domain is ....

(a) $\mathbb{R}$

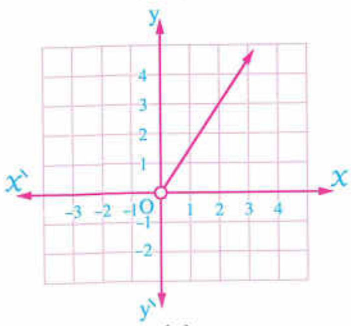
(b) $\mathbb{R} -]-2, 2[$

(c) $\mathbb{R} - [-2, 2]$

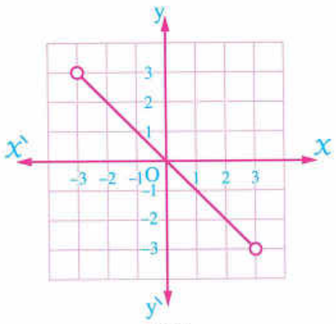
(d) $\mathbb{R} - \{0\}$



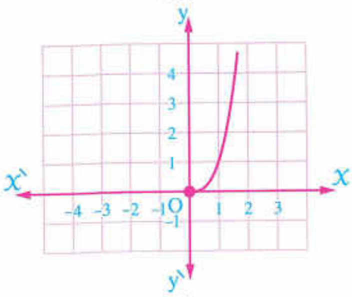
(38) Which of the following figures represents the curve of a function in which its range ≠ its domain ?



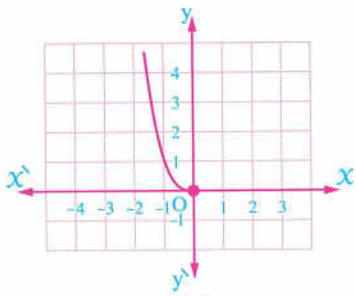
(a)



(b)



(c)



(d)

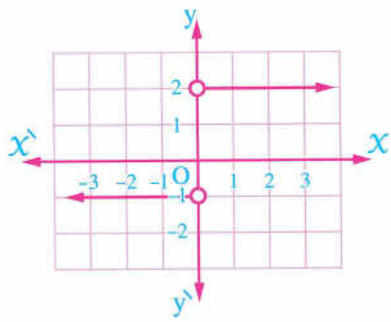
(39) The rainge of the function shown in the opposite figure is ....

(a) $\{-1, 2\}$

(b) $\{2\}$

(c) $\{-3, 3\}$

(d) $\{-1\}$



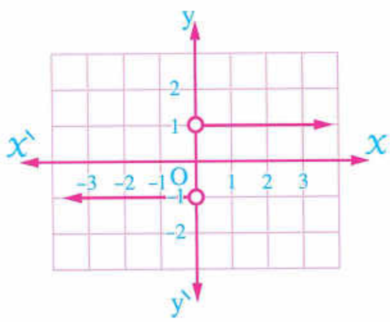
(40) The range of the function shown in the opposite figure is ....

(a) $\{1\}$

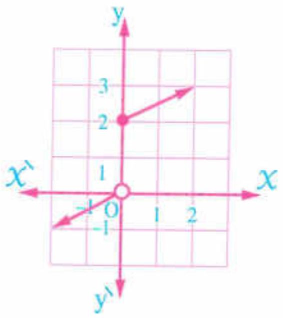
(b) $\{1, -1\}$

(c) $\{-1\}$

(d) $\mathbb{R} - \{0\}$



(41) The opposite figure represents function of x, its range is ....



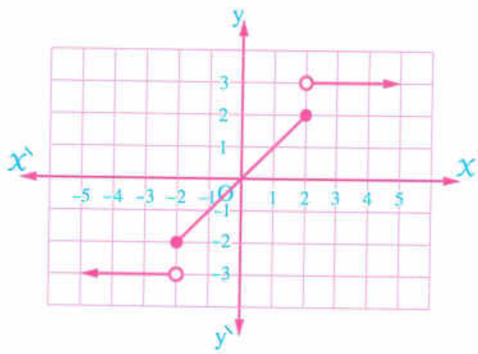
(a) $\mathbb{R} - [0, 2]$

(b) $\mathbb{R} - \{0\}$

(c) $\mathbb{R} - [0, 2[$

(d) $\mathbb{R} -]0, 2]$

(42) The opposite figure represents the function f, then its range is ....



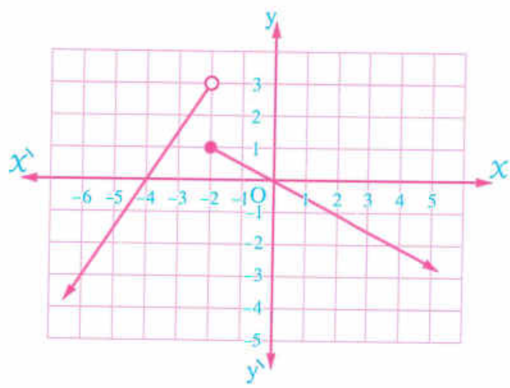
(a) $\mathbb{R}$

(b) $[-2, 2]$

(c) $[-2, 2] \cup \{-3, 3\}$

(d) $]-3, 3[$

(43) The opposite figure represents the curve of the function f, then its range is ....



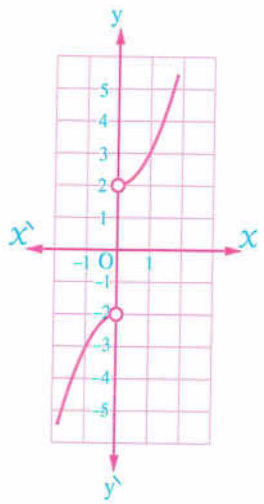
(a) $\mathbb{R}$

(b) $\mathbb{R} - \{2\}$

(c) $]-\infty, 3[$

(d) $]-\infty, 1] \cup \{3\}$

(44) In the opposite figure



first : The range of the function is ....

(a) $\mathbb{R} - \{0\}$ (b) $\mathbb{R} - [-2, 2]$

(c) $\mathbb{R}$ (d) $[-2, 2]$

Second : The function is increasing in ....

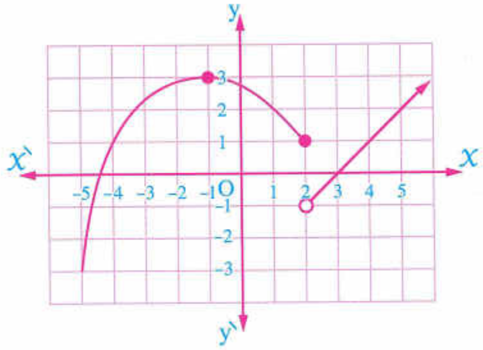
(a) $]-\infty, 0[$ only

(b) $]0, \infty[$ only

(c) $]-\infty, 0[$, $]0, \infty[$

(d) $\mathbb{R} - [-2, 2]$

(45) The opposite figure represents the curve of the function f which is increasing on ....



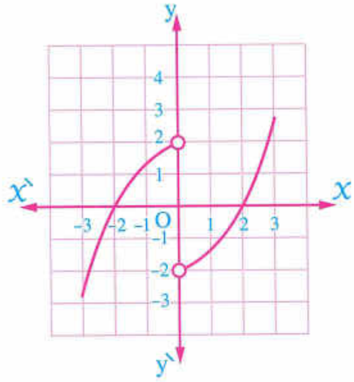
(a) $]-\infty, -1[$

(b) $]-\infty, -1[,]2, \infty[$

(c) $]2, \infty[$

(d) $]-1, 2[$

(46) In the opposite figure if the drawn curve shows the function f, which of the following statements is true ?



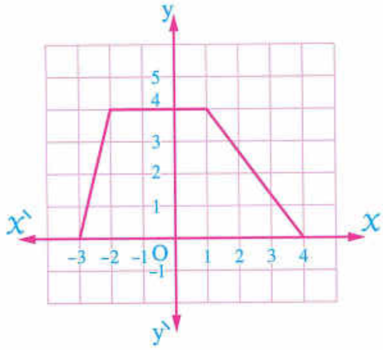
(a) The function increases on its domain.

(b) The function decreases on $]-\infty, -2[$ and increases on $]0, \infty[$

(c) The function increases on each $]-\infty, 2[,]-2, \infty[$

(d) The function increases on each $]-\infty, 0[,]0, \infty[$

(47) The opposite figure represents the curve of the function f which of the following statements is false ?



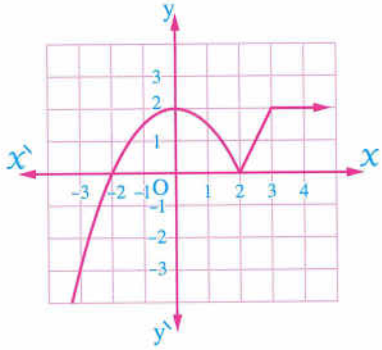
(a) f is constant on $]-2, 1[$

(b) f is decreasing on $]1, 4[$

(c) f is increasing on $]-3, -2[$

(d) f is constant on $]-3, 4[$

(48) In the opposite figure if the function decreases on $]0, a[$ and constant on $]b, \infty[$, then $a - b =$ ....



(a) 5

(b) 1

(c) -1

(d) 3

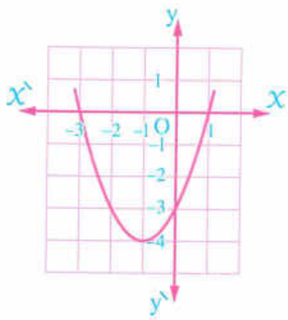
Second Essay questions

1 If x and y are two real variables , then determine which of the following relations represents a rule of a function in x :

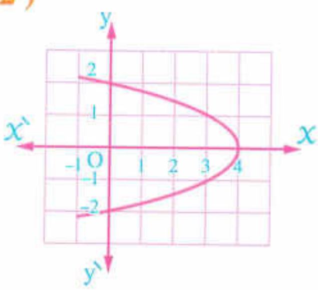
(1) $y = 2x + 5$
(2) $y^2 = x + 4$
(3) $y = \sqrt{x^2 + 4}$
(4) $(x - y)^2 = 5$
(5) $y^3 - x^2 = 2$
(6) $y^4 = x^2 - 4x + 4$
(7) $y = 2$
(8) $x = 3$
(9) $y = 3 \sin x + \cos 2x$
(10) $xy + y = 2x + 2$

2 In each of the following graphs , show if y is a function in x or not :

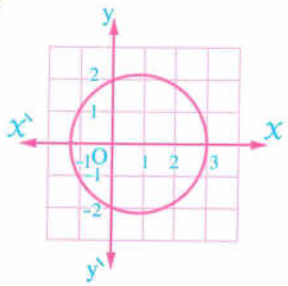
(1)



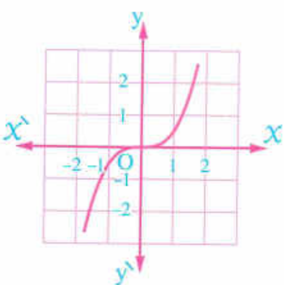
(2)



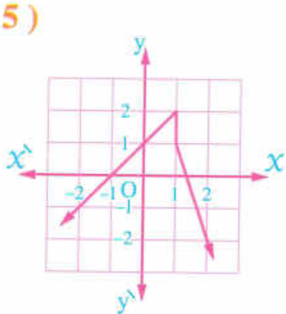
(3)



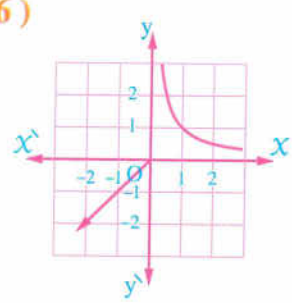
(4)



(5)



(6)



3 Determine the domain of each of the real functions defined by the following rules :

(1) $f(x) = \frac{x + 3}{x^2 - 9}$
(2) $f(x) = \frac{2x + 3}{x^2 - 3x + 2}$
(3) $f(x) = \frac{x^2 - 1}{3x^3 + 12x}$
(4) $f(x) = \frac{8}{x^2 - 6x + 9}$
(5) $f(x) = \frac{x + 3}{3x^2 - x - 2}$
(6) $f(x) = \frac{x + 1}{x^3 + 1}$

4 Determine the domain of each of the real functions defined by the following rules :

(1) $f(x) = \dfrac{4}{\sqrt[3]{2x-5}}$

(2) $f(x) = \dfrac{3}{\sqrt{x-3}}$

(3) $f(x) = \dfrac{2}{\sqrt{1-x}}$

(4) $f(x) = \sqrt{x^2-16}$

(5) $f(x) = \sqrt{4-x^2}$

(6) $f(x) = \dfrac{5}{\sqrt{9-x^2}}$

(7) $f(x) = \sqrt{x^2+2x+5}$

(8) $f(x) = \dfrac{1}{\sqrt{x^2-4x+4}}$

(9) $f(x) = \dfrac{5}{\sqrt{x}-1}$

5 Determine the domain of each of the real functions defined by the following rules :

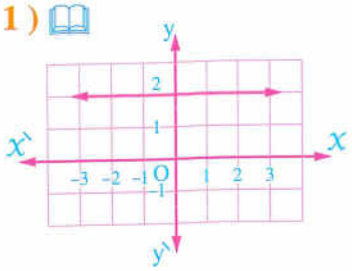
(1) $f(x) = \begin{cases} -3 & , \quad x < 3 \\ 5-x & , \quad x \geq 3 \end{cases}$

(2) $f(x) = \begin{cases} x^2-1 & , \quad x \leq 2 \\ -5 & , \quad 2 < x < 4 \end{cases}$

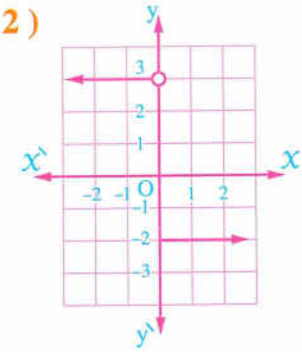
(3) $f(x) = \begin{cases} 3x & , \quad x \in [0, 2] \\ 6 & , \quad x \in]2, 4[\\ x+2 & , \quad x \in [4, 6] \end{cases}$

6 Determine the domain and range, then discuss the monotony of each of the functions represented by the following graphs :

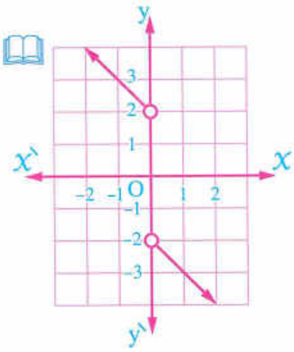
(1)



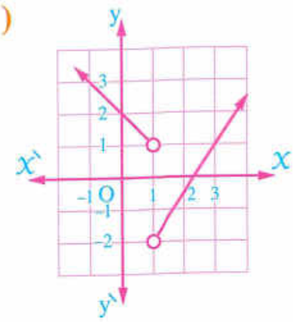
(2)



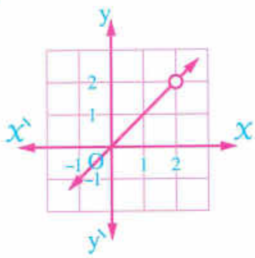
(3)



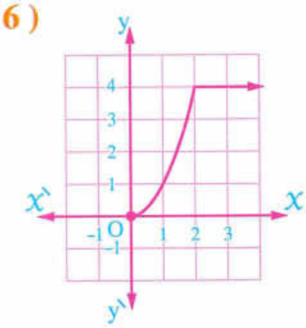
(4)



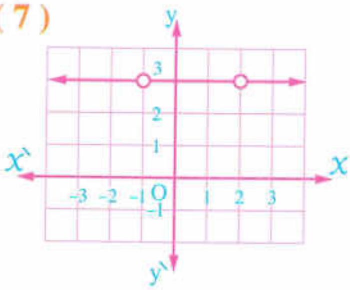
(5)



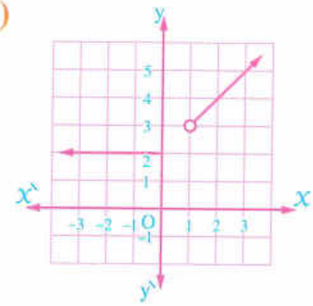
(6)



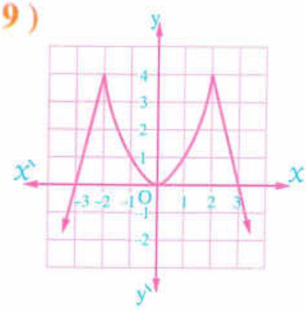
(7)



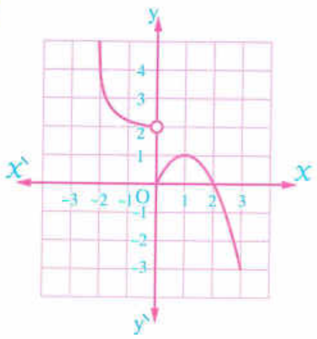
(8)



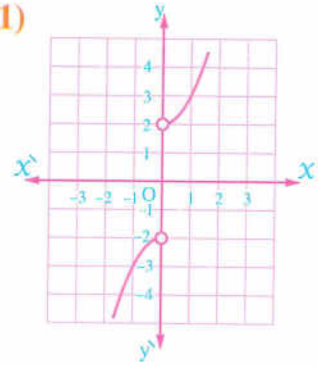
(9)



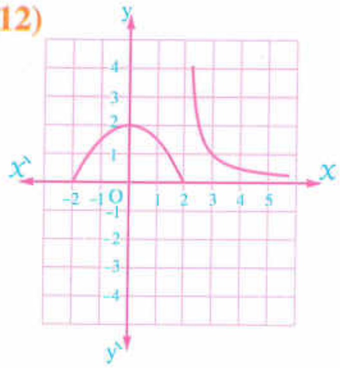
(10)



(11)



(12)



Third Higher skills

Choose the correct answer from those given :

(1) If the relation between the sum (y) of the interior angle measures of a polygon and the number of its sides (X) is $y = \pi (X - 2)$, then the domain of this function is ....

(a) $\mathbb{R}^+$ (b) $\mathbb{R} - \{2\}$ (c) $\mathbb{Z}^+$ (d) $\mathbb{Z}^+ - \{1 , 2\}$

(2) The domain of the function $f : f(X) = \frac{X}{\sqrt[3]{X} - 2}$ is ....

(a) $\mathbb{R}$ (b) $\mathbb{R} - \{2\}$ (c) $\mathbb{R} - \{0 , 2\}$ (d) $\mathbb{R} - \{8\}$

(3) The domain of the function $f : f(X) = \frac{X}{\sqrt{3X} - X}$ is ....

(a) $]0 , \infty[$ (b) $]-\infty , 0[$ (c) $[0 , \infty[- \{1\}$ (d) $]0 , \infty[- \{3\}$

(4) The domain of the function $f : f(X) = \frac{5}{\sqrt{X - 1} - 3}$ is ....

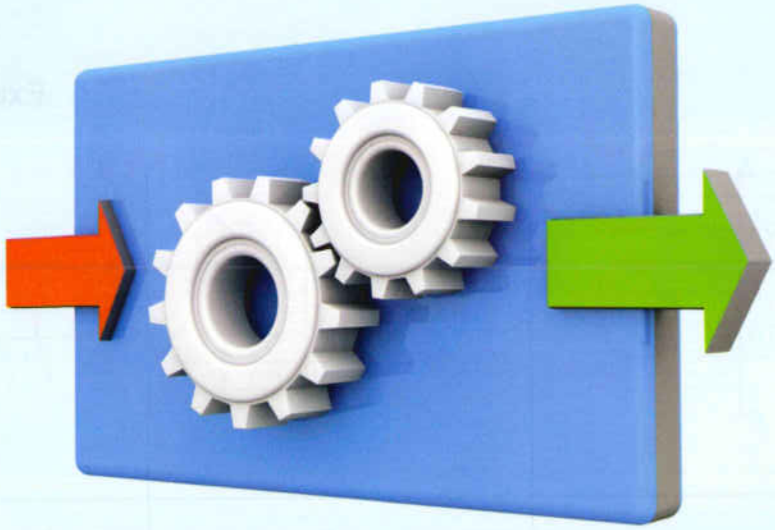
(a) $[1 , \infty[$ (b) $[1 , \infty[- \{3\}$ (c) $[1 , \infty[- \{10\}$ (d) $[-3 , \infty[$

(5) The domain of the function $f : f(X) = \frac{7}{\sqrt{X - 1} + 3}$ is ....

(a) $[1 , \infty[$ (b) $[1 , \infty[- \{-3\}$ (c) $[1 , \infty[- \{10\}$ (d) $[3 , \infty[$

(6) The domain of the function $f : f(X) = \sqrt{6X - 9 - X^2}$ is ....

(a) $\mathbb{R}$ (b) $\mathbb{R} - \{3\}$ (c) $[3 , \infty[$ (d) $\{3\}$



Exercise 2 — Operations on functions – Composition of functions

Test yourself

From the school book · Understand · Apply · Higher Order Thinking Skills

First — Multiple choice questions

Choose the correct answer from those given :

(1) The domain of the function $f : f(x) = \frac{x-1}{x^2-1} + \frac{1}{x+2}$ is ……………

(a) $\mathbb{R} - \{1 , -1 , -2\}$ (b) $\mathbb{R} - \{1 , -2\}$

(c) $\mathbb{R} - \{1 , -1 , 2\}$ (d) $\{1 , 2\}$

(2) The domain of the function $f : f(x) = \sqrt{x-2} - \sqrt{7-x}$ is ……………

(a) $[2 , 7]$ (b) $]2 , 7[$ (c) $\mathbb{R} - [2 , 7]$ (d) $\mathbb{R} -]2 , 7[$

(3) If $f(x) = \sqrt[3]{x-1}$, $g(x) = \sqrt{x-1}$, then the domain of $(f - g)$ is ……………

(a) $[-1 , \infty[$ (b) $[1 , \infty[$ (c) $\mathbb{R}$ (d) $]-\infty , 1]$

(4) The domain of the function $f : f(x) = \sqrt{x-1} + \sqrt{x+2}$ is ……………

(a) $[-2 , \infty[$ (b) $[1 , \infty[$ (c) $]1 , \infty[$ (d) $]-2 , \infty[$

(5) The domain of the function $f : f(x) = \sqrt{x+2} - \sqrt{3-x}$ is ……………

(a) $\mathbb{R}$ (b) $\mathbb{R} - [-2 , 3]$ (c) $[-2 , 3]$ (d) $]-2 , 3[$

(6) If f is a function where $f = \{(1 , 2) , (2 , 4) , (3 , 6) , (4 , 8)\}$ and g is a function where $g = \{(1 , 3) , (3 , 5) , (5 , 7)\}$, then $f + g =$ ……………

(a) $\{(1 , 2) , (2 , 4) , (3 , 6) , (4 , 8) , (1 , 3) , (3 , 5) , (5 , 7)\}$

(b) $\{(1 , 2) , (2 , 4) , (3 , 6) , (4 , 8) , (5 , 7)\}$

(c) $\{(1 , 5) , (3 , 11)\}$

(d) $\{(1 , 5) , (2 , 4) , (3 , 11) , (4 , 8) , (5 , 7)\}$

(7) If f , g : $\mathbb{R} \longrightarrow \mathbb{R}$ where $f(x) = 3x + 1$, and $(f + g)(x) = x^3 + 2x - 1$
, then $g(-1) =$ ....

(a) -2 (b) -1 (c) zero (d) 2

(8) If $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ where $f(x) = x^2 - 7$, $g : [-3 , 2] \longrightarrow \mathbb{R}$ where $g(x) = x^2 + 3$
, then $(f - g)(-5) =$ ....

(a) -5 (b) -10 (c) -20 (d) undefined

(9) The domain of the function $(f . g)$ where $f(x) = x^2 - 4$, $g(x) = x + 1$ is ....

(a) $\mathbb{R} - \{2 , -2 , -1\}$ (b) $\{2 , -2 , -1\}$

(c) $\mathbb{R}$ (d) $\mathbb{R} - \{4 , 1\}$

(10) If $f(x) = \frac{1}{x}$, $g(x) = \sqrt{x}$, then the domain of $(f . g) =$ ....

(a) $\mathbb{R} - \{0\}$ (b) $\mathbb{R}$ (c) $\mathbb{R}^+$ (d) $[0 , \infty[$

(11) The domain of the function $f : f(x) = \sqrt{x - 4}\ \sqrt[3]{x - 2}$ is ....

(a) $]4 , \infty[$ (b) $[2 , \infty[$ (c) $\mathbb{R}$ (d) $[4 , \infty[$

(12) If $f : \mathbb{R} \longrightarrow \mathbb{R}$ where $f(x) = x^2 - x$, $g : \mathbb{R} \longrightarrow \mathbb{R}$ where $g(x) = 3x - 2$
, then $(f . g - 6g)(2) =$ ....

(a) 24 (b) 40 (c) 16 (d) -16

(13) If $f(x) = 2x + 1$, $g(x) = -x - 2$, $h(x) = f(x) + g(x)$, $k(x) = f(x) - g(x)$
, then $h(2) . k(1) =$ ....

(a) 3 (b) 1 (c) 6 (d) zero

(14) If $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ where $f(x) = x - 5$, $g : [-1 , 5] \longrightarrow \mathbb{R}$ where $g(x) = x - 2$
, then $\left(\frac{f}{g}\right)(1)$ ....

(a) 1 (b) 2 (c) 4 (d) 8

(15) If f , g are two real functions where $f(x) = \frac{x - 2}{x^2 - 3x + 2}$, $g(x) = x - 3$
, then $\left(\frac{f}{g}\right)(3) =$ ....

(a) $\frac{1}{2}$ (b) 1 (c) zero (d) undefined.

(16) The domain of the function $f : f(x) = \frac{x}{\sqrt[3]{x}}$ is ....

(a) $\mathbb{R}$ (b) $\mathbb{R} - \{0\}$ (c) $\mathbb{R}^+$ (d) $\mathbb{R}^+ \cup \{0\}$

(17) The domain of the function $f : f(x) = \frac{x - 5}{\sqrt{2x - 3}}$ is ....

(a) $\mathbb{R} - \{5\}$ (b) $]\frac{3}{2} , \infty[$ (c) $[\frac{3}{2} , \infty[- \{5\}$ (d) $\mathbb{R} - \{\frac{3}{2}\}$

(18) The domain of the function $f : f(x) = \frac{\sqrt{x-2}}{x-3}$ is ....

(a) $\mathbb{R}$ (b) $\{3\}$ (c) $[2, \infty[$ (d) $[2, \infty[- \{3\}$

(19) If $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ where $f(x) = x - 5$, $g : [-1, 5] \longrightarrow \mathbb{R}$ where $g(x) = x - 2$, then the domain of $\left(\frac{f}{g}\right)$ is ....

(a) $]0, 5]$ (b) $]0, 5] - \{2\}$ (c) $[-1, \infty[$ (d) $\mathbb{R}$

(20) If $f_1 : [-2, 3] \longrightarrow \mathbb{R}$, where $f_1(x) = x$, $f_2 : [0, 4[\longrightarrow \mathbb{R}$, where $f_2(x) = x^2$, then the domain of $\left(\frac{f_1}{f_2}\right) =$ ....

(a) $[-2, 4[$ (b) $[-2, 4[- \{0\}$ (c) $]0, 3]$ (d) $[0, 3]$

(21) The domain of the function f where $f(x) = \frac{x+1}{\sqrt[3]{x-1}}$ is ....

(a) $\mathbb{R} - \{1\}$ (b) $\mathbb{R} - \{-1\}$ (c) $[-1, \infty[$ (d) $\mathbb{R}$

(22) If $f(x) = \sqrt[3]{x-3}$, $g(x) = \sqrt{3-x}$, then the domain of $\left(\frac{f}{g}\right)$ is ....

(a) $]3, \infty[$ (b) $[3, \infty[$ (c) $]-\infty, 3[$ (d) $]-\infty, 3]$

(23) The domain of the function $f : f(x) = \frac{\sqrt{x-4}}{(x-3)(x-5)}$ is ....

(a) $\mathbb{R}$ (b) $\mathbb{R} - \{3, 5\}$ (c) $[4, \infty[$ (d) $[4, \infty[- \{5\}$

(24) The domain of the function f where $f(x) = \frac{\sqrt{x-3}}{\sqrt{5-x}}$ is ....

(a) $]3, 5]$ (b) $\mathbb{R} - \{3, 5\}$ (c) $[3, 5[$ (d) $\{-3, -5\}$

(25) The domain of the function $f : f(x) \sqrt{\frac{3-x}{5-x}}$ is ....

(a) $]3, 5]$ (b) $\mathbb{R} -]3, 5]$ (c) $[3, 5]$ (d) $\mathbb{R} - [3, 5[$

(26) If : $f(x) = \sqrt{x+5}$, $g(x) = x^2$, then $(f \circ g)(2) =$ ....

(a) 7 (b) 3 (c) 4 (d) 9

(27) If $f(1) = 4$, $g(4) = 7$, then $(g \circ f)(1) =$ ....

(a) 1 (b) 4 (c) 7 (d) 11

(28) If $f(x) = x^2 + 6$, $g(x) = 3x$, then : $(f \circ g)(3) =$ ....

(a) 75 (b) 87 (c) 90 (d) 100

(29) If $f(x) = 5x + 4$, $g(x) = 2 - x$, then $(f \circ g)(-2) + (f \circ g)(5) =$ ....

(a) 24 (b) – 27 (c) 49 (d) – 3

(30) If $g(x) = \frac{x}{x-1}$, then $(g \circ g)(1) =$ ....

(a) 1 (b) zero (c) – 1 (d) undefined.

(31) If $f(x) = \sqrt[3]{x}$, $g(x) = x^3$, then $(f \circ g)(x) =$ ....

(a) x (b) x^3 (c) $\sqrt[3]{x}$ (d) $|x|$

(32) If $f(x) = x - 3$, $g(x) = x^2$, then $(f \circ g)(x) =$ ....

(a) $(x-3)^2$ (b) $x^2 - 3$ (c) $x^2 + 3$ (d) $\sqrt{x-3}$

(33) If $f(x) = \frac{3x+2}{2x-3}$, then $(f \circ f)(x) =$ ....

(a) x (b) $-x$ (c) $f(x)$ (d) $-f(x)$

(34) If $f(x) = 2x + 5$ and $g(x) = x^2$, then $f\left(g(x)\right) =$ ....

(a) $(2x+5)^2$ (b) $2x^2 + 5$ (c) $(2x^2+5)^2$ (d) $2x^2 + 10$

(35) If $f(x) = 3x + 2$, $g(x) = 2x + k$ and $(f \circ g)(x) = (g \circ f)(x)$, then $k =$ ....

(a) 1 (b) 2 (c) 3 (d) 4

(36) If $f(x) = x + 1$, $g(x) = x^2 - 1$, then the solution set of the equation $g\left(f(x)\right) =$ zero is ....

(a) $\{0, 2\}$ (b) $\{1, -1\}$ (c) $\{0, -2\}$ (d) $\{2, -2\}$

(37) If $f(x) = 2x$, $g(x) = \frac{1}{2x}$, $f\left(g(x)\right) = 6$, then $x =$ ....

(a) $\frac{1}{6}$ (b) 6 (c) 4 (d) $\frac{1}{12}$

(38) If $f(x) = \frac{1}{x}$, $g(x) = x^2 - 1$, then the domain of $(f \circ g) =$ ....

(a) $\{0, 1, -1\}$ (b) $\mathbb{R} - \{0, 1, -1\}$ (c) $\mathbb{R} - \{1, -1\}$ (d) $[-1, 1]$

(39) If $f(x) = \sqrt{x}$, $g(x) = x^2$, then the domain of $(g \circ f) =$ ....

(a) $]-\infty, 0]$ (b) $[0, \infty[$ (c) $\mathbb{R}^+$ (d) $\mathbb{R}$

(40) If $f(x) = \sqrt{x}$, $g(x) = x^2$, then the domain of $(f \circ g) =$ ....

(a) $]-\infty, 0]$ (b) $[0, \infty[$ (c) $\mathbb{R}^+$ (d) $\mathbb{R}$

(41) The domain of the function $(f \circ g)$ where :

$f(x) = \sqrt{x-2}$, $g(x) = \sqrt{6-x}$ is ....

(a) $\mathbb{R}$ (b) $[2, \infty[$ (c) $]-\infty, 2]$ (d) $]-\infty, 2[$

(42) If the relation between x , $f(x)$, $g(x)$ is as given in the opposite table , then $g(f(6)) =$ ……………

x	$f(x)$	x	$g(x)$
-3	4	-4	3
-2	3	-1	1
3	1	1	-4
6	-1	4	-7

(a) 3 (b) 1

(c) -4 (d) -7

(43) If the relation between x , $f(x)$ and $g(x)$ for some values of x are shown in the given table. Which of following is true ?

x	$f(x)$	$g(x)$
zero	-2	2
2	4	zero
4	8	-2

(1) $f(0) = g(4)$ **(2)** $f\left(g(2)\right) = -2$

(3) $g\left(f(2)\right) = -2$

(a) only (1) (b) both (1) , (2)

(c) both (1) , (3) (d) All the previous.

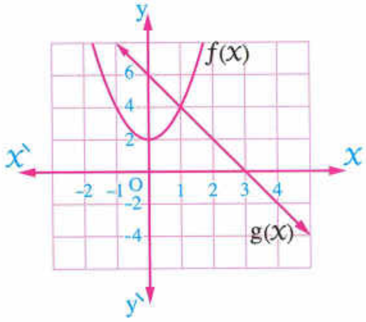
(44) If the opposite table shows two functions f and g , then $(f \circ g)(1) =$ ……………

x	1	2	3	4
$f(x)$	3	1	4	2
$g(x)$	4	3	2	1

(a) 1 (b) 2

(c) 3 (d) 4

(45) The opposite figure shows the curves of two functions f and g , then $(g \circ f)(1) =$ ……………



(a) 5 (b) -2

(c) -4 (d) 6

Second Essay questions

1 Let f be a function such that : $f(x) = x^2 - x - 12$ and its domain is $[-4 , 8]$, g is another function such that : $g(x) = x - 4$ and its domain is $[-7 , 4]$, find each of the following functions and determine the domain of each of them :

(1) $(f + g)(x)$ (2) $(f - g)(x)$ (3) $(f \cdot g)(x)$

(4) $\left(\frac{f}{g}\right)(x)$ (5) $\left(\frac{g}{f}\right)(x)$

2 **If f , g are two real functions where $f(x) = x^2 - 4$, $g(x) = \sqrt{x-1}$, find :**

(1) The domain of each of the following functions : $(f + g)$, $(f . g)$, $\left(\frac{f}{g}\right)$, $\left(\frac{g}{f}\right)$

(2) The numerical value – if possible – for each of : $(f + g)(5)$, $(f . g)(2)$, $\left(\frac{f}{g}\right)(3)$, $\left(\frac{g}{f}\right)(-2)$

---

3 **If $f(x) = x^2 - 4x$, $g(x) = \sqrt{x+2}$, $h(x) = \sqrt{4-x}$**

First : Find the rule and the domain of each of the following functions :

(1) $(f + g)$ (2) $(g - h)$ (3) $(f . h)$ (4) $\left(\frac{h}{f}\right)$

Second : Evaluate the numerical value – if possible – for each of :

(1) $(g - h)(1)$ (2) $(f . h)(5)$ (3) $\left(\frac{h}{f}\right)(3)$

---

4 **Determine the domain of each of the real functions defined by the following rules :**

(1) $f(x) = \frac{1}{x} + \frac{1}{x+2}$

(2) $f(x) = \frac{x-1}{x^2-1} + \frac{1}{x+1}$

(3) $f(x) = \frac{x}{\sqrt{1-x}}$

(4) $f(x) = \frac{3x}{\sqrt{2x-1}}$

(5) $f(x) = \sqrt[3]{x-4} + \sqrt{x-2}$

(6) $f(x) = \frac{\sqrt{x-4}}{x-6}$

(7) $f(x) = \frac{\sqrt{x-3}}{x^2-4}$

(8) $f(x) = \frac{\sqrt{x-3}}{\sqrt{x-5}}$

---

5 **If f , g are two real functions where :**

$f(x) = \begin{cases} x-2 & , x \geq 2 \\ -x+2 & , x < 2 \end{cases}$, **$g(x) = x$, then find each of the following :**

(1) $(f . g)(x)$ (2) $\left(\frac{f}{g}\right)(x)$ (3) $\left(\frac{g}{f}\right)(x)$

determining the domain of each function.

---

6 **Determine the domain of each of the real functions defined by the following rules :**

(1) $f(x) = \sqrt{\frac{x-3}{x-5}}$

(2) $f(x) = \frac{5x+2}{\sqrt{x-2} - 1}$

(3) $f(x) = \frac{\sqrt{x+2}}{x^2 - 6x - 7}$

(4) $f(x) = \frac{\frac{3}{x-1}}{\frac{1}{x-2} + 4}$

7 If $f(x) = 3x + 1$, $g(x) = x^2 - 5$, $k(x) = x^3$, find :

(1) $(f \circ g)(2)$

(2) $(g \circ f)(-3)$

(3) $(g \circ k)(1)$

(4) $(k \circ f)(-2)$

« −2 , 59 , −4 , −125 »

8 If $f(x) = \frac{1}{x}$, $g(x) = x + 3$, find :

(1) $(f \circ g)(x)$

(2) $(g \circ f)(x)$

and deduce the domain of each function.

9 If $f(x) = \frac{1}{x-1}$, $g(x) = x^2$, then find in the simplest form , showing the domain :

(1) $(f \circ g)(x)$

(2) $(g \circ f)(x)$

10 If $f(x) = x^2 + 6$, $g(x) = 3x$

(1) Find : $(f \circ g)(3)$

(2) Determine the values of x that make $(f \circ g)(x) = 42$

« 87 , ±2 »

11 If $f(x) = x^2 - 3$, $g(x) = \sqrt{x-2}$, find $(f \circ g)(x)$ in the simplest form , determining its domain , then find $(f \circ g)(3)$

« −2 »

12 If $f(x) = \sqrt{x+1}$, $g(x) = \frac{2}{x-3}$, then find the domain of $(f \circ g)$

13 If $f(x) = \sqrt{x-2}$, $g(x) = \sqrt{4-x}$, then find each of the following functions showing the domain :

(1) $f \circ g$

(2) $g \circ f$

14 If $f(x) = \sqrt{x-1}$, $g(x) = \sqrt{x^2-4}$, then find each of the following functions :

(1) $f \circ g$

(2) $g \circ f$

15 If $h(x) = \sqrt{x^3-4}$, then find the two functions f and g where : $h(x) = (f \circ g)(x)$

16 If f is a linear function and $(f \circ f)(x) = 16x + 15$, find : $f(x)$

Third Higher skills

1 Choose the correct answer from those given :

(1) If $f(x) = \sqrt{x-1}$, $g(x) = \sqrt{1-x}$, then the domain of $(f + g)$ is ....

(a) $[1 , \infty[$

(b) $]-\infty , 1]$

(c) $[-1 , \infty[$

(d) $\{1\}$

(2) If $f(x)=\begin{cases}-x+2 & , x>0\\ 3x & , x\le 0\end{cases}$, $g(x)=\begin{cases}2x-1 & , x\ge 1\\ x+2 & , x<1\end{cases}$

, then $(f+g)(x)=$ .... where $0<x<1$

(a) $4x+2$ (b) $x+1$ (c) $5x-1$ (d) 4

(3) If $f(x)=\begin{cases}x^2+1 & , x<0\\ 3x+1 & , x\ge 0\end{cases}$, $g(x)=\begin{cases}4x-2 & , x<2\\ x^2+2 & , x\ge 2\end{cases}$

, then $(g\circ f)(-1)=$ ....

(a) 37 (b) -17 (c) 10 (d) 6

(4) If $f(x)=x^2$, $g(x)=2x+3$, then the solution set of the equation $f(g(x))=f(x)$ is ....

(a) $\{1,3\}$ (b) $\{-3,1\}$ (c) $\{-1,-3\}$ (d) $\{-1,3\}$

(5) If $f(x)=2x+1$ and $(f\circ g)(x)=3x+2$, then $g(x)=$ ....

(a) $x+1$ (b) $5x+3$ (c) $\frac{3}{2}x+\frac{1}{2}$ (d) $2x+3$

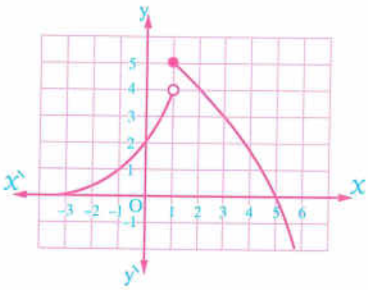
(6) If $f:\mathbb{R}\longrightarrow\mathbb{R}$ and $f(x)=2x-3$, $(g\circ f)(x)=x+2$, then $g(x)=$ ....

(a) $2x^2+x-6$ (b) $\frac{x+2}{2x-3}$ (c) $\frac{x+3}{2}$ (d) $\frac{x+7}{2}$

(7) If $f(x)=x+2$, $g(x)=\frac{x-k}{3x-1}$ and $(g\circ f)(0)=(f\circ g)(1)$, then $k=$ ....

(a) 7 (b) 5 (c) 1 (d) $\frac{7}{2}$

(8) The opposite figure represents the curve $y=f(x)$, then $(f\circ f\circ f)(1)=$ ....



(a) -5 (b) -1 (c) 1 (d) 2

(9) If $f:\mathbb{N}\longrightarrow\mathbb{N}$ where $f(x)=2x$, $h:\mathbb{N}\longrightarrow\mathbb{N}$ where

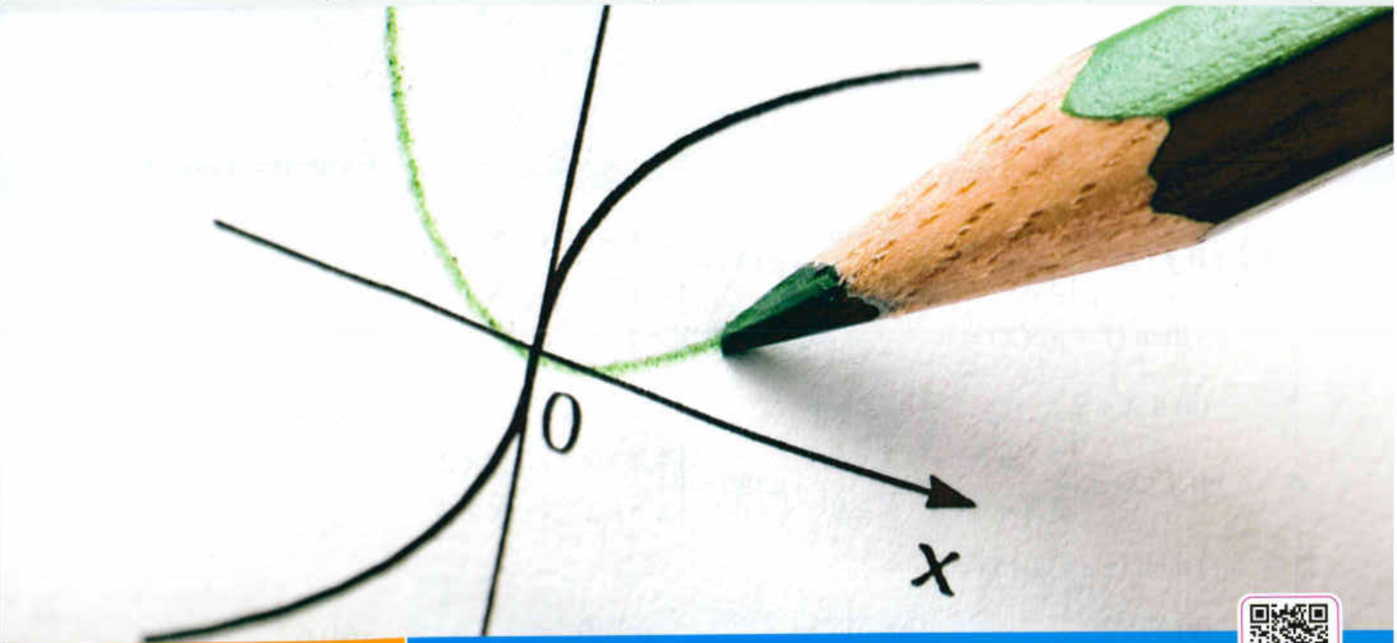
$h(x)=\begin{cases}\frac{x}{2} & , x \text{ is even}\\ \frac{x+1}{2} & , x \text{ is odd}\end{cases}$, then $(f\circ h)(3)-(f\circ h)(8)=$ ....

(a) 4 (b) 8 (c) -4 (d) -5

---

2 **If $f:\mathbb{R}-\{0\}\longrightarrow\mathbb{R}$ and $f(x)=\frac{1}{x}$, find :**

(1) $(f\circ f)(x)$ **(2)** $(f\circ f\circ f)(x)$ **(3)** $(f\circ f\circ f\circ$ .... to n times$)(x)$



Exercise 3 Some properties of functions (even and odd functions / one - to - one functions)

Test yourself

From the school book ● Understand ● Apply ● Higher Order Thinking Skills

First Multiple choice questions

Choose the correct answer from those given :

(1) The even function from the functions that are defined by the following rules is ……………

(a) $f(x) = x^3$ (b) $f(x) = \sin x$

(c) $f(x) = x \cos x$ (d) $f(x) = x \sin x$

(2) The odd function from the functions that are defined by the following rules is ……………

(a) $f(x) = x^2 \sin x$ (b) $f(x) = \tan^2 x$

(c) $f(x) = \cos x$ (d) $f(x) = 1$

(3) Which of the following rules is defined an odd function ?

(1) $f(x) = \frac{1}{x} + x + \frac{1}{x^3}$

(2) $f(x) = x^2 \tan x$

(3) $f(x) = x \sin x$

(a) both (1) , (2) (b) only (1)

(c) both (2) , (3) (d) All the previous.

(4) The type of the function $f : f(x) = \frac{\sin x}{x}$ is ……………

(a) even. (b) odd.

(c) neither even nor odd. (d) one - to - one.

(5) The function $f : f(x) = x \cos x$ is ……………

(a) even. (b) odd.

(c) neither even nor odd. (d) one - to - one.

(6) The following functions are even except ....

(a) $f(x) = \sin x$ (b) $f(x) = \cos x$ (c) $f(x) = x^2 - 1$ (d) $f(x) = 1$

(7) Which of the following functions is not even ?

(a) $y = \frac{1}{x^2}$ (b) $y = \sec x$

(c) $y = x^2 + \sin x$ (d) $y = 3x^4 - 2x^2 + 27$

(8) If $f(x) = \frac{1}{\sin x}$, then ....

(a) $f(x) = \frac{1}{f(x)}$ (b) $f(x) = -f(-x)$

(c) $f(x) = f(-x)$ (d) $f(-x) = f\left(\frac{1}{x}\right)$

(9) If f is an odd function , $f(1) = 2$, then which of the following points lies on the curve of f ?

(a) $(-1, 2)$ (b) $(-1, -2)$ (c) $(1, -2)$ (d) $(-1, 0)$

(10) If f is an odd function , $a \in$ the domain of f , then $f(a) + f(-a) =$ ....

(a) zero. (b) $2f(a)$ (c) $2a$ (d) $f(a)$

(11) If f is an odd function , then $f(a) - f(-a) =$ ....

(a) zero. (b) $f(a)$ (c) $2f(a)$ (d) $f(2a)$

(12) If f is an even function , then $f(a) - f(-a) =$ ....

(a) zero. (b) $f(a)$ (c) $2f(a)$ (d) $f(2a)$

(13) If f is an even function , $2 \in$ the domain of f , then $f(2) + f(-2) =$ ....

(a) zero. (b) 4 (c) 2 (d) $2f(2)$

(14) The function $f : f(x) = (x + \sin x)$ (...............) is even.

(a) $x^3 + \tan x$ (b) $1 + \sin x$ (c) $4 - x^2$ (d) $x - \cos x$

(15) If f is an even function and the curve of the function passes through point $(-3, 2m + 1)$, and $f(3) = 5$, then $m =$ ....

(a) -1 (b) zero (c) 1 (d) 2

(16) 📖 If the function f is an even over $[a, b]$, then $b =$ ....

(a) a (b) $-a$ (c) $2a$ (d) a^3

(17) If f is an odd function in $[a, b]$, $a < 0$ then : $b =$ ....

(a) a (b) $-a$ (c) $2a$ (d) a^2

(18) If f is an even function and $f(5) = 1$, $f(-5) = 3 - k$, then $k =$ ....

(a) 1 (b) 5 (c) 3 (d) 2

(19) If f is a function where $f :]-5 , 5] \longrightarrow \mathbb{R}$, $f(x) = x^2$, then the function $f(x)$ is ....

(a) even.
(b) odd.
(c) one - to - one.
(d) neither odd nor even.

(20) If the domain of an even function f is $]-5 , 5[$, then the solution set of the equation $f(x) = f(x-2)$ is ....

(a) $\{2\}$ (b) $\{1\}$ (c) $\{0\}$ (d) $\{-1\}$

(21) If $f(x) = a x^3 + b x + c$ is an odd function , then $c =$ ....

(a) 2 (b) 1 (c) zero. (d) -1

(22) If $f : f(x) = x^2 + a x + 9$ is an even function , then $a =$ ....

(a) 6 (b) 3 (c) zero. (d) -6

(23) If $f(x) = x^3 - x$, then $|f(x) + f(-x)| =$ ....

(a) zero. (b) 1 (c) 2 (d) 4

(24) If $f : f(x)$ is an even function and $f(x) + x^2 f(-x) = 3$, then $f(1) =$ ....

(a) 1 (b) $\frac{1}{4}$ (c) $\frac{3}{2}$ (d) 2

(25) If f is an odd function then : $\frac{2 f(x) + 8 f(-x)}{3 f(x)} =$ ....

(a) zero (b) -2 (c) -4 (d) 4

(26) If f is an odd function and $x f(x) + x^3 f(-x) = 2$, then $f(2) =$ ....

(a) -2 (b) 2 (c) $\frac{-1}{3}$ (d) $\frac{1}{10}$

(27) If $f : f(x) = a x^3 + b$ is an odd function and the curve of the function passes through the point $(2 , 8)$, then $a + b^2 =$ ....

(a) zero (b) -1 (c) 1 (d) 5

(28) The one - to - one function from the functions that are defined by the following rules is ....

(a) $f_1(x) = \cos x$
(b) $f_2(x) = x^2$
(c) $f_3(x) = x^3$
(d) $f_4(x) = x^4 + x^2$

(29) The one - to - one function from those defined by the following rules is ....

(a) $f(X) = 3 - X^2$ (b) $f(X) = X^3 + 3$

(c) $f(X) = \sin X \tan X$ (d) $f(X) = X^2 + X$

(30) All the functions defined by the following rules are one - to - one except ....

(a) $f(X) = X$ (b) $f(X) = X + 6$

(c) $f(X) = X^3$ (d) $f(X) = (X - 2)^2$

(31) The functions defined by the following rules are one - to - one except ....

(a) $f(X) = X^3$ (b) $g(X) = 3X$ (c) $h(X) = \frac{1}{X}$ (d) $n(X) = X^2$

(32) All functions defined by the following rules are not one - to - one except ....

(a) $f(X) = X^2$ (b) $f(X) = 5$ (c) $f(X) = X + 2$ (d) $f(X) = \sin X$

(33) If f , g are two functions where $f(X) = X^3$, $g(X) = X + 2$, then $(g \circ f)$ is .... function.

(a) a one - to - one (b) an odd (c) an even (d) a linear

(34) If the function is continuous and decreasing for all values of $X \in$ the domain of the function, then the function is ....

(a) even. (b) odd. (c) one - to - one. (d) not one - to - one.

(35) If a function is increasing for all real values of $X \in$ the domain of the function , then the function is ....

(a) one - to - one. (b) an even function.

(c) an odd function. (d) not one - to - one.

(36) The function f where $f(X) = 5X$ is symmetric about ....

(a) the point (2 , 1) (b) the straight line y = 1

(c) the point (5 , 0) (d) the point (0 , 0)

(37) The function $f : f(X) = X^3 + 5X$ is symmetric about ....

(a) the X-axis. (b) the y-axis.

(c) the origin. (d) can not be determined.

(38) The function $f : f(X) = X^2 + X^4 + 1$ is symmetric about ....

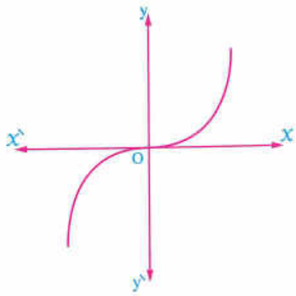
(a) the origin. (b) the X-axis.

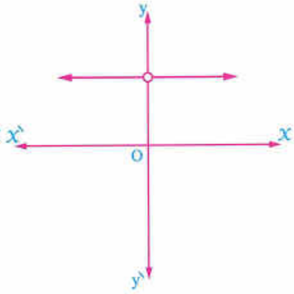
(c) the y-axis. (d) it has neither symmetric point nor symmetric line.

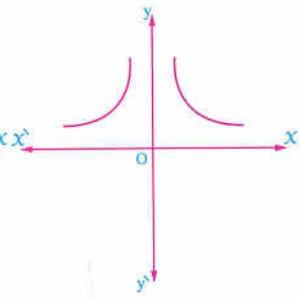
(39) The function $f : f(X) = \sin 3X$ is symmetric about the point ....

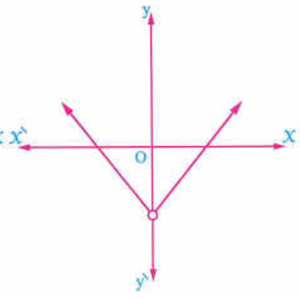
(a) (0 , 0) (b) (3 , 0) (c) (– 3 , 0) (d) (– 3 , 3)

(40) Which of the following functions is not even ?

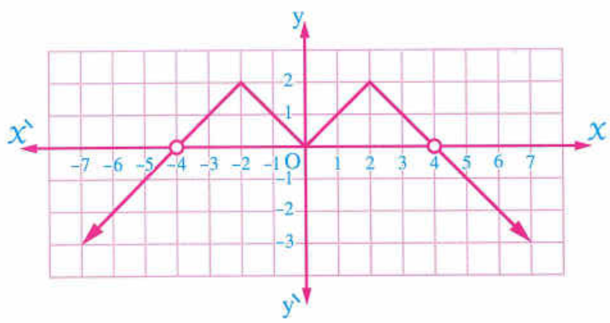

(a)


(b)


(c)

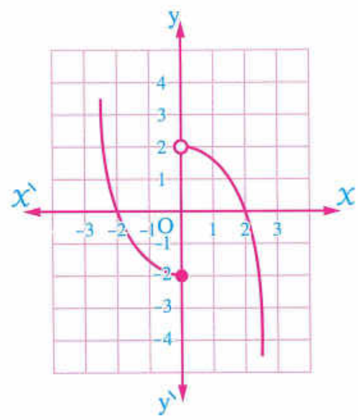

(d)

(41) The opposite figure represents the curve of the function f , then f is ....



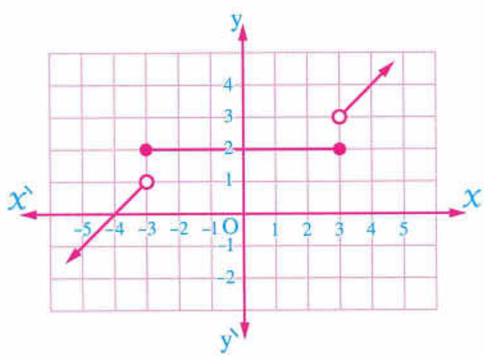
(a) one - to - one.
(b) an even function.
(c) an odd function.
(d) neither odd nor even.

(42) The opposite figure represents the curve of the function f , then f is ....



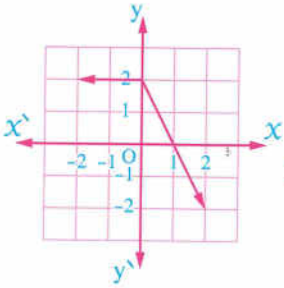
(a) one - to - one.
(b) an even function.
(c) an odd function.
(d) neither odd nor even.

(43) The opposite figure represents the curve of the function f , then f is ....

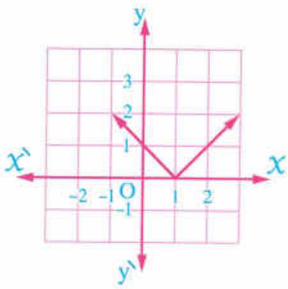


(a) one - to - one function.
(b) an even function.
(c) an odd function.
(d) neither odd nor even.

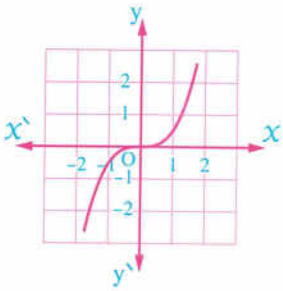
(44) Which of the following functions is even ?



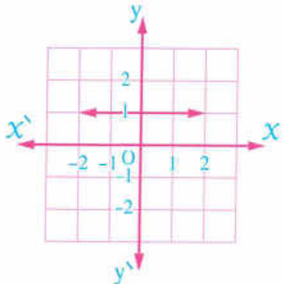
(a)



(b)

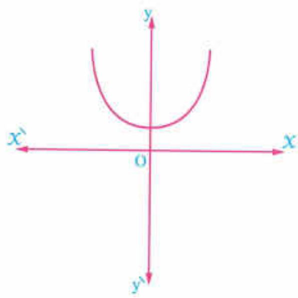


(c)

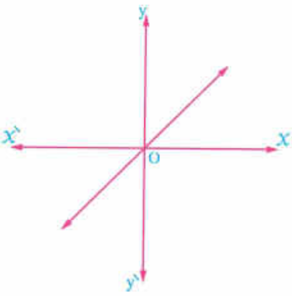


(d)

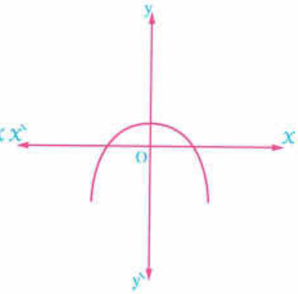
(45) Which of the following represents one - to - one function ?



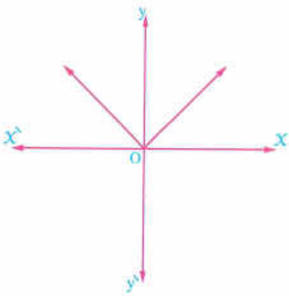
(a)



(b)

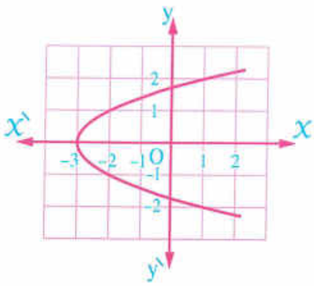


(c)



(d)

(46) The curve represented in the opposite figure is symmetric about the straight line whose equation is ....



(a) $x = 0$

(b) $y = 0$

(c) $y = -2$

(d) $x = 2$

Second Essay questions

1 In each of the following figures , mention the curve which is symmetric about the x-axis , the y-axis or the origin point :

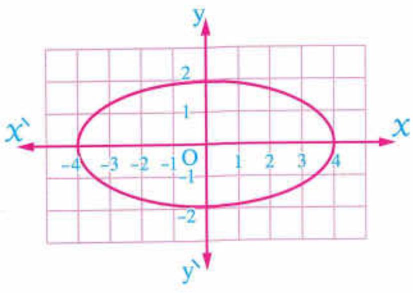


Fig. (1)

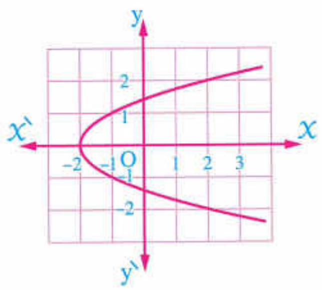


Fig. (2)

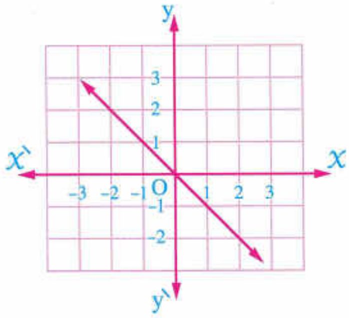


Fig. (3)

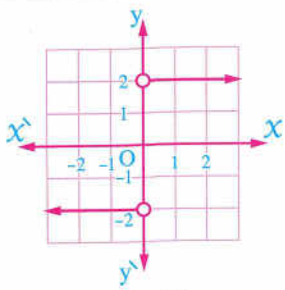


Fig. (4)

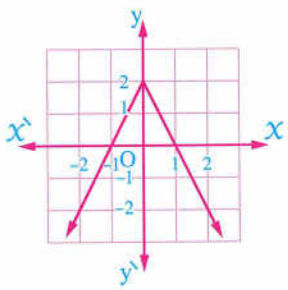


Fig. (5)

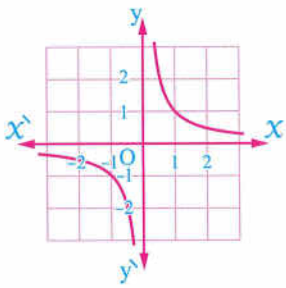
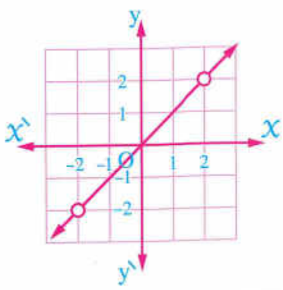


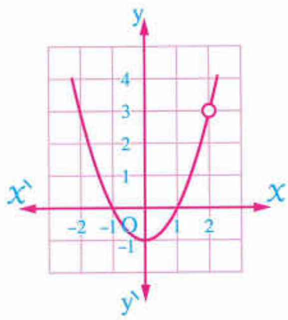
Fig. (6)

2 Determine which of the functions represented by the following graphs is even , odd or neither even nor odd :

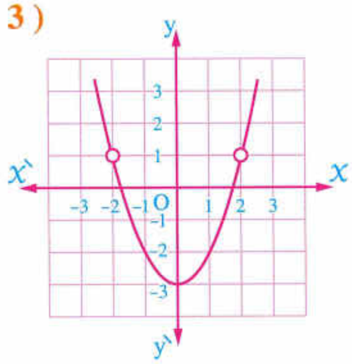
(1)



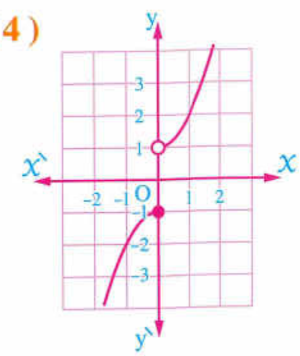
(2)



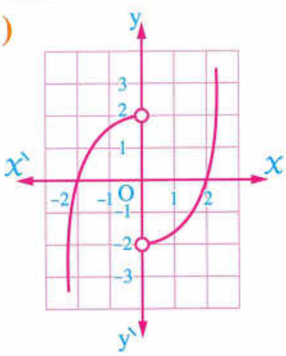
(3)



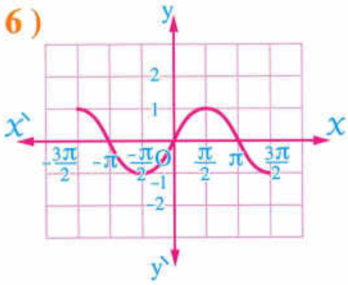
(4)



(5)



(6)



3 Find the range of each function and mention its type whether it is even , odd or otherwise :

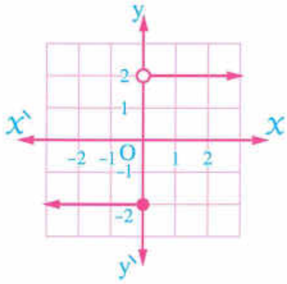


Fig. (1)

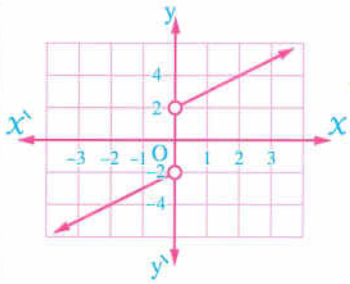


Fig. (2)

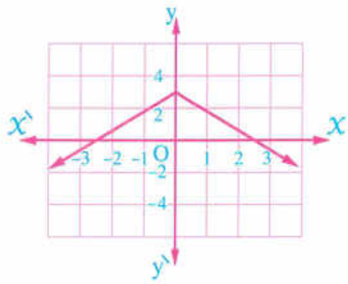


Fig. (3)

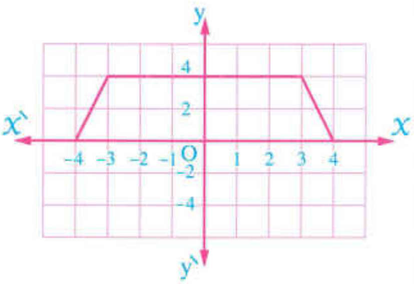


Fig. (4)

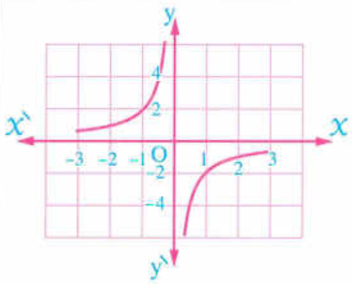


Fig. (5)

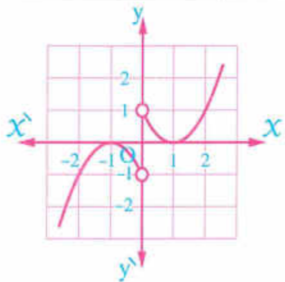


Fig. (6)

4 Each of the following graphs represents the curve of the function f , determine whether the function f is even , odd or otherwise verifying your answers algebraically :

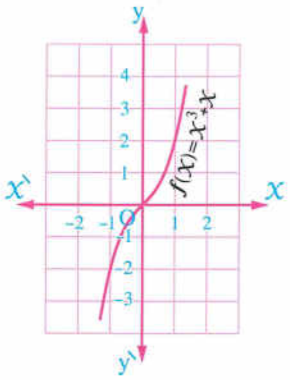


Fig. (1)

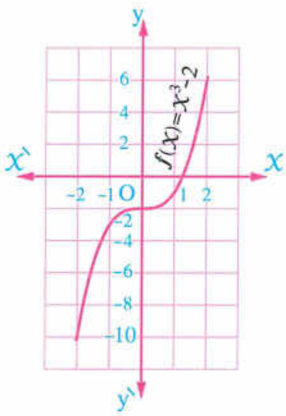


Fig. (2)

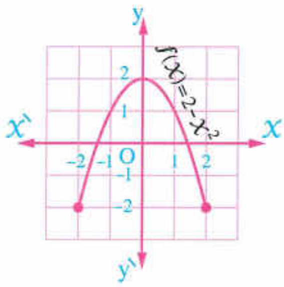


Fig. (3)

5 Use the following figures to answer the following questions :

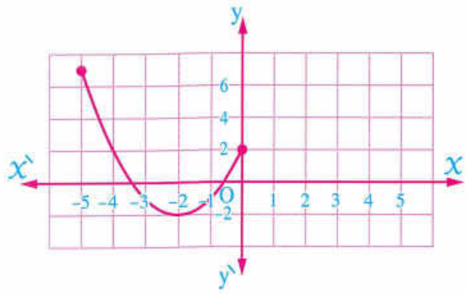


Fig. (1)

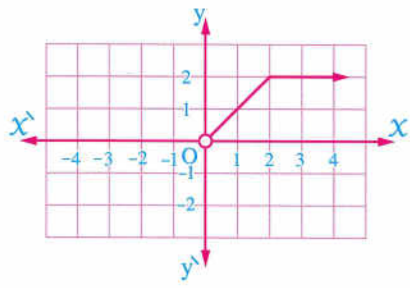


Fig. (2)

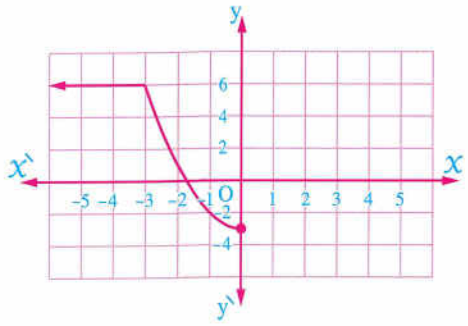


Fig. (3)

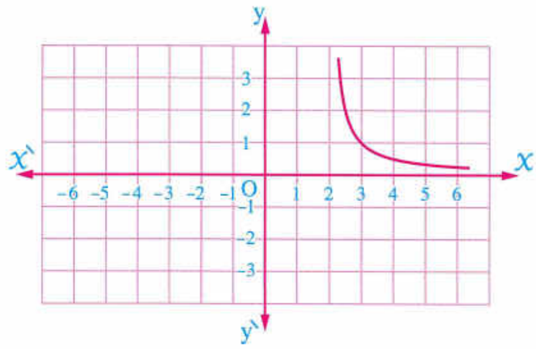


Fig. (4)

First : Complete the curve in each of fig. (1) and fig. (3) in your notebook to get an even function over its domain.

Second : Complete the curve in each of fig. (2) and fig. (4) in your notebook to get an odd function over its domain.

Third : Determine the domain and the range of the function in each case , then show which graph represents a one - to - one function.

6 Determine which of the functions defined by the following rules is even , which is odd and which is neither even nor odd :

(1) $f(x) = 5$

(2) $f(x) = x^4 + x^2 - 1$

(3) $f(x) = 3x - 4x^3$

(4) $f(x) = x^2 - 3x + 4$

(5) $f(x) = x^3 (x^2 - 1)$

(6) $f(x) = (x - 3)^2 - 7$

(7) $f(x) = \frac{x^2 + 2}{x - 3}$

(8) $f(x) = \frac{2x^3 - x^5}{x}$

(9) $f(x) = \frac{x^2}{1 + x}$

(10) $f(x) = \sqrt{x + 3}$

(11) $f(x) = (x^2 + 1)^3$

(12) $f(x) = \sqrt{x^2 + 6}$

(13) $f(x) = \sqrt[3]{x^3 + x}$

(14) 📖 $f(x) = x^3 - \frac{1}{x}$

(15) $f(x) = \left(x - \frac{2}{x}\right)^3$

(16) $f(x) = \left(\frac{x^2}{3} - \frac{5}{x^4}\right)^5$

(17) $f(x) = \left(\frac{2 - x}{3 + x}\right)^2 + \left(\frac{2 + x}{3 - x}\right)^2$

(18) $f(x) = \left(\frac{1 - x}{1 + x}\right)^2 - \left(\frac{1 + x}{1 - x}\right)^2$

(19) $f(x) = \left(\frac{x - 1}{x + 1}\right)^5 + \left(\frac{x + 1}{x - 1}\right)^5$

(20) $f(x) = (x^3 + 1)^4 - (x^3 - 1)^4$

(21) 📖 $f(x) = x \cos x$

(22) $f(x) = \frac{3x}{\tan x}$

(23) $f(x) = \frac{x^3 \sin 3x}{1 + x^4}$

(24) $f(x) = \frac{3x^2 - \cos x}{x^3 - 6x}$

(25) $f(x) = x^2 \sin^3 x$

(26) $f(x) = x \sin x^3$

(27) $f(x) = \frac{x^2 + \tan x}{x^4 + \sin x}$

(28) $f(x) = \sin x^2 - \sin^2 x$

(29) $f(x) = x^4 + \sin^6 x$

(30) $f(x) = x^7 + \tan^5 x$

(31) $f(x) = \frac{\sin 3x \cos 2x}{\sec x}$

(32) $f(x) = 4^x + \cos x + 2^{-2x}$

(33) $f(x) = (\cos x + \sin x - 1)(\cos x + \sin x + 1)$

(34) $f(x) = \begin{cases} 2x & , x \geq 0 \\ -2x & , x < 0 \end{cases}$

(35) $f(x) = \begin{cases} 2x + x^2 & , x \leq 0 \\ 2x - x^2 & , x > 0 \end{cases}$

7 **Each of the following graphs represents the curve of the function f , show from the graph that the function f is one - to - one verifying that algebraically :**

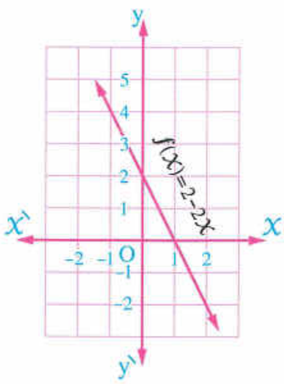


Fig. (1)

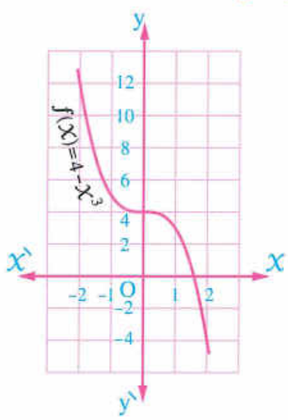


Fig. (2)

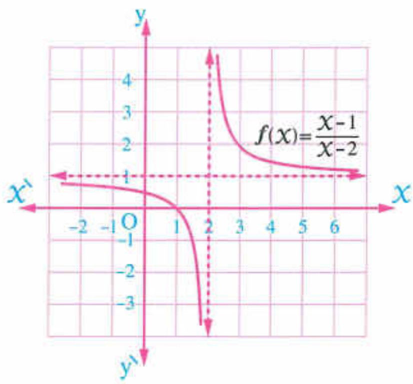


Fig. (3)

8 **Prove that the functions that are defined by the following rules are one - to - one :**

(1) $f(x) = 2x - 3$
(2) $f(x) = 4 - x^3$
(3) $f(x) = \frac{3}{2x + 5}$
(4) $f(x) = \frac{3x - 5}{4x + 3}$

9 **Prove that the functions that are defined by the following rules are not one - to - one :**

(1) $f(x) = 3$
(2) $f(x) = (x + 3)^2$
(3) $f(x) = x^2 - 5x + 6$
(4) $f(x) = \frac{1}{x^2 - 4}$

10 **In each of the functions that are defined by the following rules determine whether the function is one - to - one or not giving reason :**

(1) $f(x) = x^3 + 1$
(2) $f(x) = x^2 - 1$
(3) $f(x) = 2x^2 - x - 3$
(4) $f(x) = x^4 + 2x^2 + 1$
(5) $f(x) = \frac{2x + 1}{x - 2}$
(6) $f(x) = \frac{3 - x}{x + 2}$

11 **Determine which of the following functions is even , odd or otherwise :**

(1) $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x + 2$
(2) $f : [-3 , 3[\longrightarrow \mathbb{R}$, $f(x) = 3x^2$
(3) $f(x) = x^2$, $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$
(4) $f : f(x) = x^2$, $x \in \mathbb{R} - \{3\}$

12 **Let f be a function whose domain is $\mathbb{R}$ find the value of $\frac{7f(-5) + 3f(5)}{2f(-5)}$ if :**

(1) f is an odd function.
(2) f is an even function.

« 2 , 5 »

13 **If f and g are two real functions such that :**

$f(x) = (1 - x)^2$ and $g(x) = (1 + x)^2$, then determine which of the following functions is even , odd or otherwise :

(1) $f + g$
(2) $f - g$
(3) $f \cdot g$
(4) $\frac{f}{g}$

14 **Let f_1 , f_2 , g_1 and g_2 be real functions such that :**

$f_1(x) = x^4$, $f_2(x) = \cos^5 x$, $g_1(x) = 2x^3$ and $g_2(x) = \sin^3 x$

Determine which of the following functions is even , odd or otherwise :

(1) $f_1 + g_2$
(2) $f_1 - f_2$
(3) $g_1 + g_2$
(4) $f_1 \cdot g_2$
(5) $g_1 \cdot g_2$
(6) $\frac{f_2}{f_1}$

15 **Creative thinking : Represent graphically the curve which satisfies each of the following conditions :**

(1) Passes through the points $(0 , -2)$, $(2 , 2)$, $(3 , 7)$ and represents an even function.

(2) Passes through the points $(0 , 0)$, $(-2 , 1)$, $(-3 , 5)$ and represents an odd function.

Third Higher skills

Choose the correct answer from those given :

(1) If f is a one - to - one function and the point $(2 , 3)$ belongs to the function f , which of the following points can be belong to f ?

(a) $(5 , 3)$ (b) $(2 , -1)$ (c) $(3 , 2)$ (d) all the previous

(2) If f is a one - to - one function and the two points $(2 , a)$, $(3 , b)$ belong to the function f , which of the following is always true ?

(a) $a > b$ (b) $a = b$ (c) $a \neq b$ (d) $a + b = 0$

(3) If f is an even function and $f(X) + X^2 f(-X) = 3$, then $f(1) =$ ....

(a) $\frac{1}{4}$ (b) 1 (c) $1\frac{1}{2}$ (d) 2

(4) If f is an odd function and $f(1) = k$ and $f(X + 2) = f(X) + f(2)$, then $f(3) =$ ....

(a) zero (b) 3 k (c) 6 k (d) 9 k

(5) If $f(X) = \frac{1 + X}{1 - X}$ and $g(X) = \frac{1 - X}{1 + X}$, then each of the sum of these two functions and their product is .... function.

(a) an even (b) an odd

(c) a one - to - one (d) neither even nor odd

(6) If f is a real function and $X , -X \in$ the domain of the function , then the function $g(X) = f(X) + f(-X)$ is always ....

(a) odd. (b) even.

(c) neither even nor odd. (d) one - to - one.

(7) If f is defined on $\mathbb{R}$ and $3 f(X) + 2 f(-X) = X^3 - \sin X$, then f is ....

(a) odd. (b) even.

(c) neither even nor odd. (d) not one - to - one.

(8) If $f(X) = X^3$, $g(X) = X^2 + 1$, which of the following is an odd function ?

(1) $(f \times g)$ (2) $(f \circ g)$ (3) $(g \circ f)$

(a) (1) only (b) (2) , (3) (c) (1) , (2) (d) (1) , (3)



Exercise 4 Graphical representation of basic functions and graphing piecewise functions

Test yourself

From the school book ● Understand ● Apply ● Higher Order Thinking Skills

First Multiple choice questions

Choose the correct answer from those given :

(1) If $f(X) = 5$, then the domain of the function f is ……………

(a) $\mathbb{R}$ (b) $\mathbb{R}^+$ (c) $\{5\}$ (d) $\mathbb{R} - \{5\}$

(2) If $f(X) = 7$, then the range of the function f is ……………

(a) $\mathbb{R}$ (b) $\mathbb{R}^+$ (c) $\{7\}$ (d) $\mathbb{R} - \{7\}$

(3) The range of the function $f : f(X) = \begin{cases} 0 & \text{when } X \le 0 \\ 1 & \text{when } X > 0 \end{cases}$ is ……………

(a) $\{1\}$ (b) $\{0\}$ (c) $\mathbb{R}$ (d) $\{0 , 1\}$

(4) In the opposite figure :

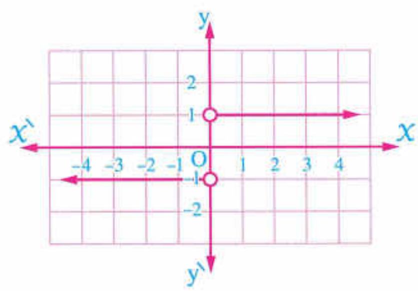
The range of the function is ……………

(a) $\{1\}$

(b) $\{1 , -1\}$

(c) $\{-1\}$

(d) $\mathbb{R}$



(5) The range of the function $f : f(X) = \frac{3X^2 - 3}{X^2 - 1}$ is ……………

(a) $\mathbb{R} - \{1 , -1\}$ (b) $\mathbb{R} - \{3 , -3\}$ (c) $\{3 , -3\}$ (d) $\{3\}$

(6) The range of the function $f : f(X) = \frac{X^2 - 3X}{X}$ where $X \ne 0$ is ……………

(a) $\mathbb{R}$ (b) $\mathbb{R} - \{0\}$ (c) $\{0\}$ (d) $\mathbb{R} - \{-3\}$

(7) The range of the function $f : f(x) = \frac{2x^3 - 2x}{x^2 - 1}$ is ....

(a) $\mathbb{R} - \{1, -1\}$ (b) $\mathbb{R} - \{2, -2\}$ (c) $\mathbb{R}^+$ (d) $\mathbb{R}^+ - \{2\}$

(8) The range of the function $f : [-2, 3[\longrightarrow \mathbb{R}, f(x) = x^2$ is ....

(a) $[4, 9[$ (b) $\mathbb{R}^+$ (c) $[0, 9[$ (d) $[0, 4]$

(9) The range of the function $f : f(x) = \begin{cases} x &, x > 0 \\ -2 &, x \leq 0 \end{cases}$ is ....

(a) $\mathbb{R}^+$ (b) $\mathbb{R}^+ - \{-2\}$ (c) $\mathbb{R}^+ \cup \{-2\}$ (d) $\mathbb{R}$

(10) The function f where $f(x) = \begin{cases} 2 &, x > 0 \\ -2 &, x < 0 \end{cases}$ is symmetric about the point ....

(a) $(2, 0)$ (b) $(-2, 0)$ (c) $(0, 0)$ (d) $(2, -2)$

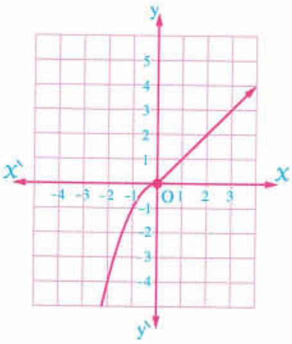
(11) The axis of symmetry for the function $f : f(x) = x^2$ is the straight line ....

(a) $y = 0$ (b) $y = x$ (c) $y = -x$ (d) $x = 0$

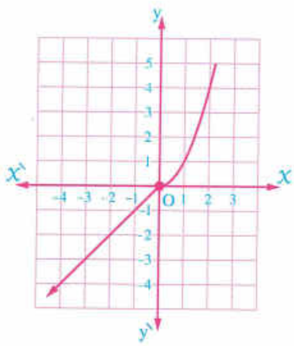
(12) The function $f : f(x) = \begin{cases} -x^2 &, x < 0 \\ \frac{1}{x} &, x > 0 \end{cases}$ is increasing on ....

(a) $\mathbb{R}$ (b) $\mathbb{R}^-$ (c) $\mathbb{R}^+$ (d) $\mathbb{R} - \{0\}$

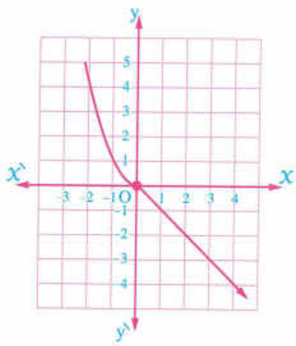
(13) The curve of the function $f : f(x) = \begin{cases} x^2 &, x > 0 \\ x &, x \leq 0 \end{cases}$ is ....



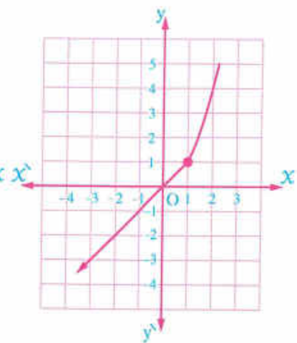
(a)



(b)

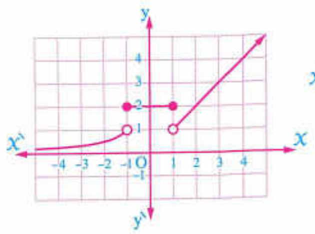


(c)

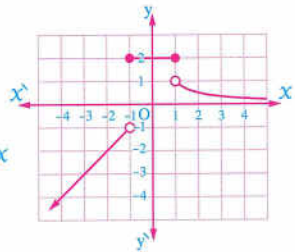


(d)

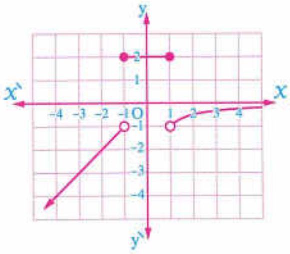
(14) The curve of the function $f : f(x) = \begin{cases} x & , \quad x < -1 \\ 2 & , \quad -1 \le x \le 1 \\ \frac{1}{x} & , \quad x > 1 \end{cases}$ is ....



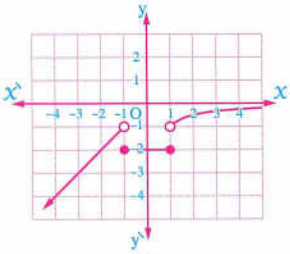
(a)



(b)

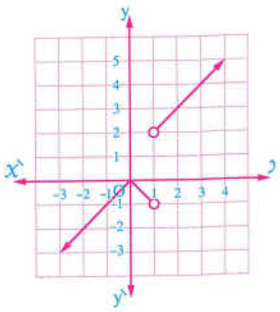


(c)

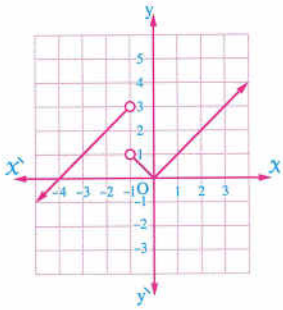


(d)

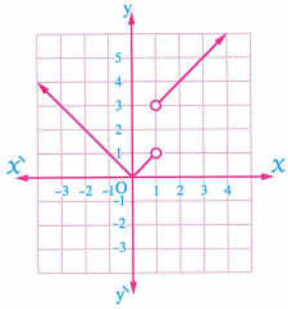
(15) The curve of the function $f : f(x) = \begin{cases} |x| & , \quad x < 1 \\ x + 2 & , \quad x > 1 \end{cases}$ is ....



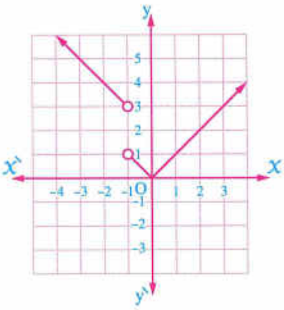
(a)



(b)



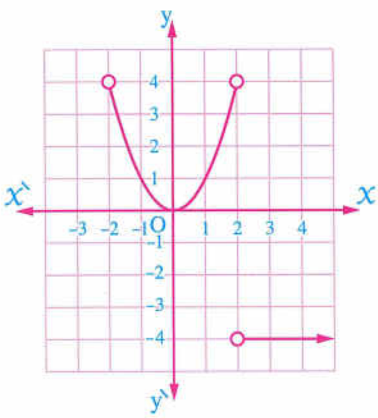
(c)



(d)

(16) In the opposite figure :

The curve of the function f defined by the rule $f(x) =$ ....



(a) $\begin{cases} x^2 & , \quad -2 < x < 2 \\ -4 & , \quad x < -2 \end{cases}$

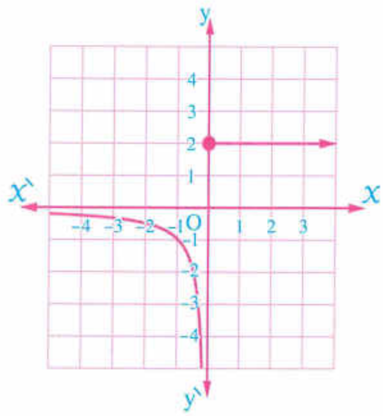
(b) $\begin{cases} x^2 & , \quad -2 < x < 2 \\ -4 & , \quad x > 2 \end{cases}$

(c) $\begin{cases} x^2 & , \quad -2 \le x \le 2 \\ -4 & , \quad x < 2 \end{cases}$

(d) $\begin{cases} x^2 & , \quad -2 < x < 2 \\ -4 & , \quad x < 2 \end{cases}$

(17) In the opposite figure :

The curve of the function f defined by the rule $f(x) = $ ....



(a) $\begin{cases} 2 & , x > 0 \\ \frac{1}{x} & , x < 0 \end{cases}$

(b) $\begin{cases} 2 & , x \geq 0 \\ \frac{1}{x} & , x < 0 \end{cases}$

(c) $\begin{cases} 2 & , x < 0 \\ \frac{1}{x} & , x > 0 \end{cases}$

(d) $\begin{cases} 2 & , x \geq 2 \\ \frac{1}{x} & , x < 2 \end{cases}$

Second Essay questions

1 Graph each of the following functions and determine its range :

(1) $f : \{-3, -1, 1, 2\} \longrightarrow [-3, 7]$, $f(x) = 2x + 3$

(2) 📖 $g : [1, 5[\longrightarrow \mathbb{R}$, $g(x) = x + 1$

(3) 📖 $g :]-\infty, -1[\longrightarrow \mathbb{R}$, $g(x) = 1 - x$

(4) $f : f(x) = -3x + 7$ for every $x \in \mathbb{R}$

(5) $f : \mathbb{R}^- \longrightarrow \mathbb{R}$, $f(x) = x^2$

(6) $f : f(x) = x^2 - 5$, $x \geq 0$

(7) $f : f(x) = x^2 - 3x$, $x \in [-3, 2]$

---

2 📖 If $f : [-2, 6] \longrightarrow \mathbb{R}$ where $f(x) = \begin{cases} 4 - x & , -2 \leq x < 1 \\ x & , 1 \leq x \leq 6 \end{cases}$

(1) Graph the function f , and from the graph deduce its range , and discuss its monotonicity.

(2) Is f one - to - one ? Explain your answer.

3 **Graph each of the functions defined by the following rules and from the graph. Find the domain and the range of each function and discuss its monotonicity and its type whether the function is even , odd or otherwise showing its symmetry :**

(1) $f(x) = \frac{3x^2 - 3}{x^2 - 1}$

(2) $f(x) = \frac{4 - x^2}{x + 2}$

(3) $f(x) = \frac{x^3 - x}{x^2 - 1}$

(4) $f(x) = \frac{x^4 - x^2}{x^2 - 1}$

4 **Represent graphically each of the functions that are defined by the following rules , from the graph find the domain and the range of each function and discuss its monotonicity and its type whether it is even , odd or otherwise and show its symmetry :**

(1) $f :]-\infty , 3[\longrightarrow \mathbb{R}$ where $f(x) = 2$

(2) $f(x) = \begin{cases} 2 & , \quad x \le 0 \\ -3 & , \quad x > 0 \end{cases}$

(3) $f(x) = \begin{cases} 2 & , \quad x > 1 \\ x - 2 & , \quad x \le 1 \end{cases}$

(4) $f(x) = \begin{cases} x + 2 & , \quad x \in [-2 , 1] \\ -x + 4 & , \quad x \in]1 , 4] \end{cases}$

(5) 📖 $f(x) = \begin{cases} 4 & , \quad x < -2 \\ x^2 & , \quad x \ge -2 \end{cases}$

(6) 📖 $f(x) = \begin{cases} x^2 & , \quad x < 0 \\ x & , \quad x \ge 0 \end{cases}$

(7) 📖 $f(x) = \begin{cases} x^3 & , \quad x < 1 \\ 1 & , \quad x > 1 \end{cases}$

(8) $f(x) = \begin{cases} x^3 & , \quad x < 1 \\ 2 - x & , \quad x \ge 1 \end{cases}$

(9) 📖 $f(x) = \begin{cases} |x| & , \quad x \le 0 \\ \frac{1}{x} & , \quad x > 0 \end{cases}$

(10) 📖 $f(x) = \begin{cases} |x| & , \quad x \le 0 \\ x^2 & , \quad x > 0 \end{cases}$

(11) $f(x) = \begin{cases} 3 & , \quad x \le -3 \\ |x| & , \quad -3 < x < 3 \\ 3 & , \quad x \ge 3 \end{cases}$

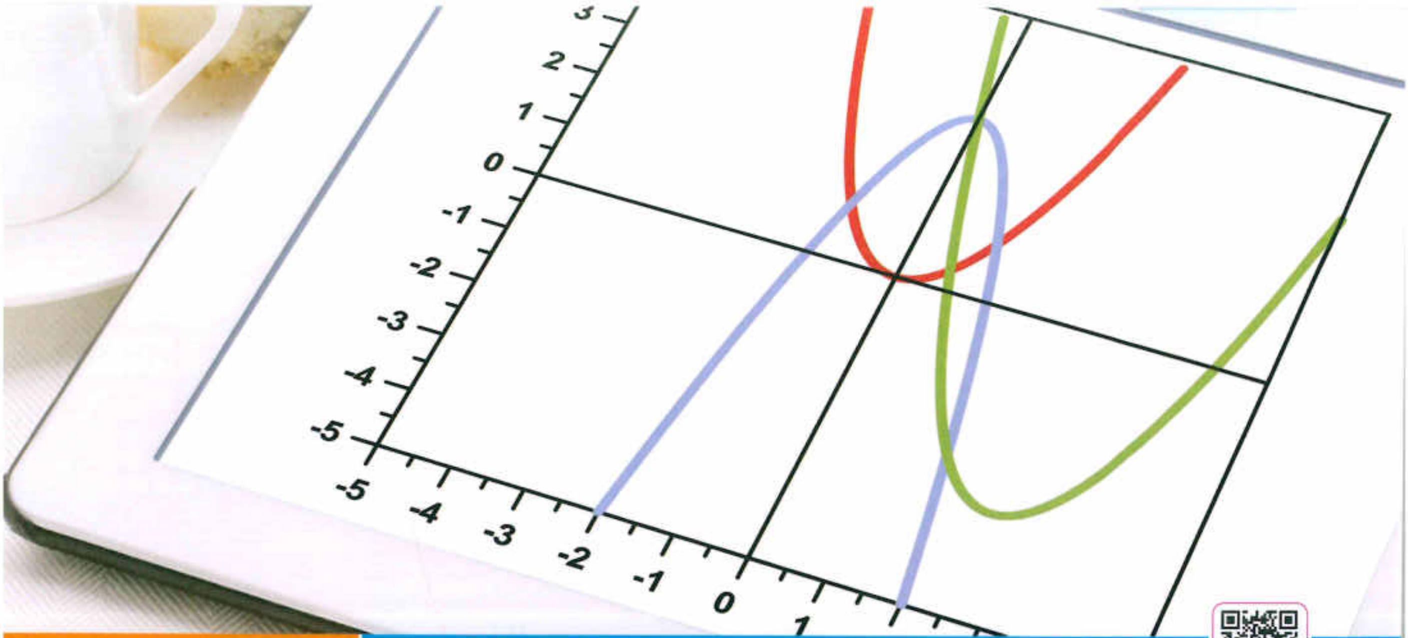
(12) $f(x) = \begin{cases} \frac{3x - 6}{2 - x} & , \quad x > 2 \\ 2 & , \quad x < 2 \end{cases}$

(13) $f(x) = \begin{cases} 2 & , \quad -3 \le x \le -1 \\ 0 & , \quad -1 < x < 1 \\ 2 & , \quad 1 \le x \le 3 \end{cases}$

(14) $f(x) = \begin{cases} -x - 1 & , \quad -4 \le x < -2 \\ 1 & , \quad -2 \le x \le 2 \\ x - 1 & , \quad 2 < x \le 4 \end{cases}$

5 📖 If $f_1 : \mathbb{R} \longrightarrow \mathbb{R}$ where $f_1(x) = 3x - 1$, $f_2 : [-2 , 3] \longrightarrow \mathbb{R}$ where $f_2(x) = 3 - 2x$, graph the function $(f_1 + f_2)$ showing its domain , then deduce its monotonicity.

6 If $f(x) = x^3 - 4x$, $g(x) = x^2 - 4$, determine the domain of the function $\frac{f}{g}$ and graph it and from the graph determine its range and mention whether it is even , odd or otherwise , and discuss its monotonicity , and mention whether it is one - to - one or not.



Exercise 5 — Geometrical transformations of basic function curves

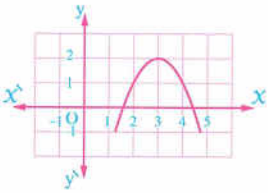
Test yourself

From the school book ● Understand ● Apply Higher Order Thinking Skills

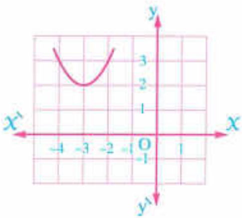
First — Multiple choice questions

Choose the correct answer from the given ones :

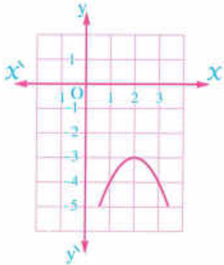
● **(1)** If $f(X) = -(X-3)^2 + 2$, then the graph that represents the function f is ....



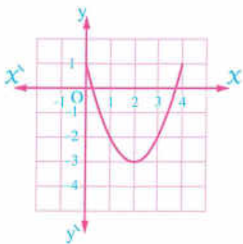
(a)



(b)

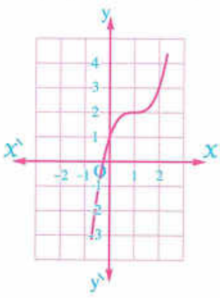


(c)

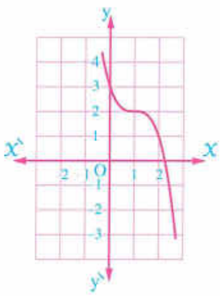


(d)

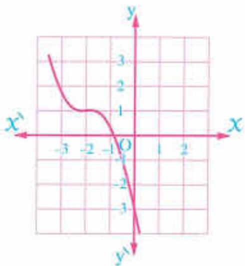
● **(2)** If $f(X) = 2-(X-1)^3$, then the graph that represents the function f is ....



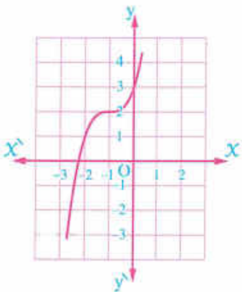
(a)



(b)

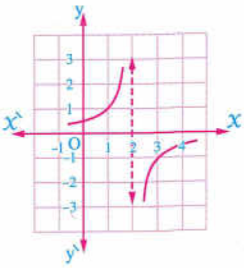


(c)

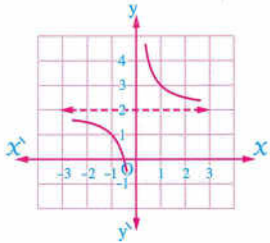


(d)

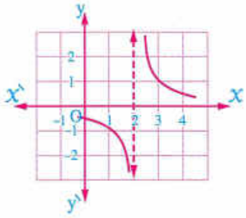
(3) If $f(X) = \dfrac{1}{X - 2}$, then the graph that represents the function f is ....



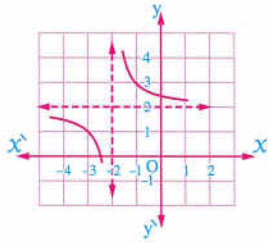
(a)



(b)

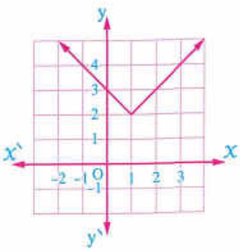


(c)

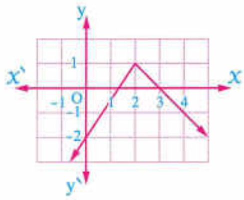


(d)

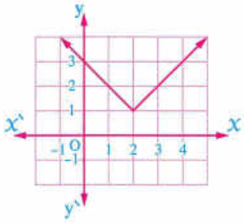
(4) If $f : f(X) = 1 - |X - 2|$ then the figure which represents the function f is ....



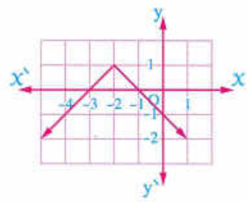
(a)



(b)

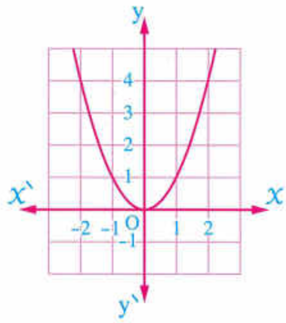


(c)



(d)

(5) The curve of the function in the given figure is translated two units in the positive directions of the two axes then the function represents this translation is ....



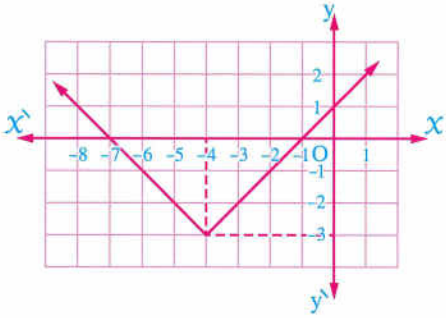
(a) $f(X) = (X + 2)^2 + 2$

(b) $f(X) = (X + 2)^2 - 2$

(c) $f(X) = (X - 2)^2 - 2$

(d) $f(X) = (X - 2)^2 + 2$

(6) Which of the following functions represents the curve in the given figure ?



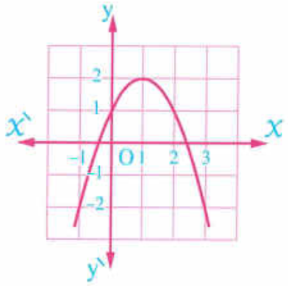
(a) $f(X) = |X - 4| - 3$

(b) $f(X) = |X - 4| + 3$

(c) $f(X) = |X + 4| - 3$

(d) $f(X) = |X + 4| + 3$

(7) Which of the following functions is represented in the given figure ?



(a) $f(X) = (X-1)^2 + 2$

(b) $f(X) = 1 - (X-2)^2$

(c) $f(X) = 2 - (X-1)^2$

(d) $f(X) = (X+1)^2 - 2$

(8) 📖 The point of the vertex of the curve of the function $f : f(X) = (2-X)^2 + 3$ is ....

(a) (2 , 3) (b) (2 , – 3) (c) (– 2 , 3) (d) (– 2 , – 3)

(9) The vertex of the curve of the function f where $f(X) = (1+X)^2 - 3$ is ....

(a) (1 , 3) (b) (1 , – 3) (c) (– 1 , 3) (d) (– 1 , – 3)

(10) The symmetric point of the function $f : f(X) = X^3 - 2$ is ....

(a) (0 , 2) (b) (0 , – 2) (c) (2 , 0) (d) (– 2 , 0)

(11) The symmetric point of the function $f : f(X) = 3 - (X+2)^2$ is ....

(a) (3 , 2) (b) (2 , 3) (c) (– 2 , 3) (d) (– 2 , – 3)

(12) The symmetric point of the function $f : f(X) = \frac{1}{X} + 2$ is ....

(a) (2 , 0) (b) (1 , 2) (c) (0 , 2) (d) (0 , 0)

(13) The point of symmetry of the function f where $f(X) = \frac{1}{X-5}$ is ....

(a) (5 , 0) (b) (– 5 , 0) (c) (0 , – 5) (d) (0 , 5)

(14) 📖 The point of symmetry of the curve of the function $f : f(X) = \frac{1}{X-3} + 4$ is ....

(a) (3 , – 4) (b) (– 3 , – 4) (c) (3 , 4) (d) (– 3 , 4)

(15) The symmetric point of the function $f : f(X) = \frac{X+1}{X}$ is ....

(a) (1 , 0) (b) (0 , 1) (c) (0 , 0) (d) (1 , – 1)

(16) 📖 If f is a function where $f(X) = \frac{1}{X}$, then the symmetric point of the function g : g $(X) = f(X+1)$ is ....

(a) (1 , 0) (b) (0 , 1) (c) (– 1 , 0) (d) (– 1 , 1)

(17) If $f(X) = \frac{a}{X-b} + c$ where a , b , c $\in \mathbb{R}$ has the symmetric point (3 , 3) , then $a^{b+c} =$ ....

(a) a^9 (b) 1 (c) a^6 (d) – 1

(18) The vertex of the curve of the function $f : f(X) = |X+3| - 2$ is ....

(a) (3 , 2) (b) (– 3 , – 2) (c) (– 3 , 2) (d) (3 , – 2)

(19) The curve of the function $f : f(X) = |X - 2|$ is symmetric about the straight line ....

(a) $X = 2$ (b) $X = -2$ (c) $y = 2$ (d) $y = -2$

(20) The axis of symmetry of the function $f : f(X) = X^2 - 1$ is the straight line ....

(a) $X = 1$ (b) $X = 0$ (c) $y = 1$ (d) $y = 0$

(21) If $f(X) = \frac{1}{|X|}$, then the equation of the axis of symmetry of the curve of the function f is ....

(a) $y = 0$ (b) $X = 0$ (c) $y = X$ (d) $y = -X$

(22) $f(X) = (X - 1)^2 + 2$ is increasing on the interval ....

(a) $\mathbb{R}$ (b) $]1, \infty[$ (c) $]-\infty, 1[$ (d) $]-1, 1[$

(23) The function f where $f(X) = \frac{2X - 1}{X - 1}$ is decreasing on the interval ....

(a) $]-\infty, 1]$ (b) $]-\infty, 1[,]1, \infty[$

(c) $[1, \infty[$ (d) $]-\infty, 2[,]2, \infty[$

(24) The area between the curve of the function $f : f(X) = |X + 2| - 2$ and the X-axis equals .... square units.

(a) 3 (b) 2 (c) 5 (d) 4

(25) The range of the function $f : f(X) = \frac{1}{|X|}$ is ....

(a) $\mathbb{R} - \{0\}$ (b) $]0, \infty[$ (c) $[0, \infty[$ (d) $\{0\}$

(26) The range of the function f where $f(X) = (X - 3)^2 + 4$ is ....

(a) $]-\infty, 3[$ (b) $[-3, 4]$ (c) $[4, \infty[$ (d) $]-\infty, 4]$

(27) The range of the function $f : f(X) = 3 - (2 - X)^2$ is ....

(a) $]-\infty, 2]$ (b) $[2, \infty[$ (c) $]-\infty, 3]$ (d) $[3, \infty[$

(28) The range of the function $f : f(X) = \frac{2X^2 - 2X}{X - 1}$ equals ....

(a) $\mathbb{R} - \{1\}$ (b) $\mathbb{R} - \{2\}$ (c) $\mathbb{R} - \{1, 2\}$ (d) $\mathbb{R} - \{0, 1\}$

(29) The range of the function $f : f(X) = 2 - \frac{3}{X - 1}$ is ....

(a) $\mathbb{R}$ (b) $\mathbb{R} - \{1\}$ (c) $\mathbb{R} - \{2\}$ (d) $\mathbb{R} - \{3\}$

(30) The range of the function $f : f(X) = |X - 2|$ is ....

(a) $]0, \infty[$ (b) $[2, \infty[$ (c) $[0, \infty[$ (d) $]2, \infty[$

(31) The range of the function $f : f(X) = 2 - |3 - 2X|$ is ....

(a) $]-\infty, 2]$ (b) $[-2, \infty[$ (c) $]\frac{3}{2}, \infty[$ (d) $]-\infty, -2]$

(32) The range of the function f where $f(X) = -|X^3|$ is ....

(a) $\mathbb{R}$ (b) $[0, \infty[$ (c) $]-\infty, 0]$ (d) $\mathbb{R} - \{0\}$

(33) The range of the function $f : f(x) = x|x|$ is ....

(a) $\mathbb{R}^+$ (b) $\mathbb{R}^-$ (c) $\mathbb{R}$ (d) $[0, \infty[$

(34) The range of the function $f : f(x) = \frac{|x|}{x}$ is ....

(a) $]0, \infty[$ (b) $]-\infty, 0[$ (c) $\mathbb{R} - \{0\}$ (d) $\{1, -1\}$

(35) The range of the function $f : f(x) = \begin{cases} -\frac{1}{x} & , x < -1 \\ |x| & , -1 \le x \le 1 \\ \frac{1}{x} & , x > 1 \end{cases}$

(a) $]0, 1]$ (b) $[0, 1]$ (c) $]0, \infty[$ (d) $\mathbb{R} - \{0\}$

(36) The range of the function $f : f(x) = \begin{cases} x - 1 & , x < 1 \\ 3 & , x > 1 \end{cases}$ is ....

(a) $\mathbb{R} - [0, 1]$ (b) $\mathbb{R} - \{1\}$

(c) $]-\infty, 0[\cup \{3\}$ (d) $]-\infty, 0[\cap \{3\}$

(37) The curve of the function $f : f(x) = \frac{1}{x - 3} + 4$ does not intersect the line ....

(a) $x = -3$ (b) $x = 3$ (c) $y = -4$ (d) $y = 3$

(38) If $y = f(x)$ is a real function, then its image by translation 3 units vertically upwards is $g(x) =$ ....

(a) $f(x - 3)$ (b) $f(x + 3)$ (c) $f(x) + 3$ (d) $(x) - 3$

(39) If the curve $y = f(x)$ represents a real function then its image by translation 5 units vertically downward is the same as $g(x) =$ ....

(a) $f(x - 5)$ (b) $f(x + 5)$ (c) $f(x) + 5$ (d) $f(x) - 5$

(40) 📖 The curve of the function $g : g(x) = x^2 + 4$ is the same curve of the function $f : f(x) = x^2$ by a translation of magnitude 4 units in the direction of ....

(a) $\overrightarrow{Ox}$ (b) $\overrightarrow{Ox'}$ (c) $\overrightarrow{Oy}$ (d) $\overrightarrow{Oy'}$

(41) The curve of the function g where $g(x) = |x| - 2$ is the same as the curve of the function $f : f(x) = |x|$ by translation two units in direction of ....

(a) $\overrightarrow{Ox}$ (b) $\overrightarrow{Ox'}$ (c) $\overrightarrow{Oy}$ (d) $\overrightarrow{Oy'}$

(42) If f is a real function whose domain is $[-3, 4]$, then the domain of $g : g(x) = f(x) + 2$ is ....

(a) $[-3, 4]$ (b) $[-1, 6]$ (c) $[-5, 2]$ (d) $\mathbb{R}$

(43) The curve of the function g : g $(X) = \frac{1}{|X|} + 2$ is the same as the curve of the function $f : f(X) = \frac{1}{|X|}$ by translation two units in direction of ....

(a) $\overrightarrow{OX}$ (b) $\overleftarrow{OX}$ (c) $\overrightarrow{Oy}$ (d) $\overleftarrow{Oy}$

(44) The curve of the function $f : f(X) = |X - 3|$ is the same as the curve of the function g : g $(X) = |X|$ by translation 3 units in direction of ....

(a) $\overrightarrow{OX}$ (b) $\overleftarrow{OX}$ (c) $\overrightarrow{Oy}$ (d) $\overleftarrow{Oy}$

(45) 📖 The curve of the function g : g $(X) = |X + 3|$ is the same curve of the function $f : f(X) = |X|$ by a translation of magnitude 3 units in the direction of ....

(a) $\overrightarrow{OX}$ (b) $\overleftarrow{OX}$ (c) $\overrightarrow{Oy}$ (d) $\overleftarrow{Oy}$

(46) If $y = f(X)$ is a real function , then its image by translation 4 units to the left is g $(X) =$ ....

(a) $f(X - 4)$ (b) $f(X + 4)$ (c) $f(X) + 4$ (d) $f(X) - 4$

(47) If f is a real function whose domain is $[-2 , 3]$, then the domain of g : g $(X) = f(X - 2)$ is ....

(a) $[-2 , 3]$ (b) $[-4 , 1]$ (c) $[0 , 5]$ (d) $\mathbb{R}$

(48) Lef $f(X) = -X^2$ move 3 units to the right and 2 units down , then resulted curve is g (X) , then g $(4) =$ ....

(a) -3 (b) -16 (c) 16 (d) -7

(49) If the curve $f(X) = -X^3$ moves 4 units to the left and 2 units upwards to become the curve g (X) , then g $(-2) =$ ....

(a) -218 (b) 214 (c) 6 (d) -6

(50) The curve of the function g : g $(X) = X$ is the same as the curve of the function $f : f(X) =$ .... by reflection in the X-axis.

(a) X (b) $-X$ (c) $X + 1$ (d) $-X + 1$

(51) The product of the slopes of the two straight lines $f(X) = aX + b$ and its image by reflection in X-axis equals ....

(a) 1 (b) -1 (c) a (d) $-a^2$

(52) The curve of the function g : g $(X) = 1 - |X|$ is the same curve of the function $f : f(X) = |X|$ by reflection in X-axis , then a translation of magnitude one unit in the direction of ....

(a) $\overrightarrow{OX}$ (b) $\overleftarrow{OX}$ (c) $\overrightarrow{Oy}$ (d) $\overleftarrow{Oy}$

Second Essay questions

1 Use the curve of the function f where $f(x) = x^2$ to represent each of the functions that are defined by the following rules , from the graph find the domain and the range of the function and discuss its monotonicity and its type whether it is even , odd or otherwise and write its axis of symmetry :

(1) $g(x) = x^2 - 3$

(2) $g(x) = x^2 + 1$

(3) $g(x) = 2 - x^2$

(4) $g(x) = (x - 2)^2$

(5) $g(x) = -(x - 3)^2$

(6) $g(x) = (x + 2)^2 - 4$

(7) $g(x) = 2 - (x + 3)^2$

(8) $g(x) = (2 - x)^2 - 1$

(9) $g(x) = \left(x + \frac{3}{2}\right)^2 - \frac{1}{2}$

(10) $g(x) = -\frac{1}{2}x^2$

(11) $g(x) = x^2 + 4x + 4$

(12) $g(x) = x^2 + 4x + 1$

2 Use the curve of the function f where $f(x) = x^3$ to represent each of the functions that are defined by the following rules , from the graph determine its domain , range , discuss its monotonicitly and its type whether it is even , odd or otherwise and write its point of symmetry :

(1) $g(x) = x^3 + 4$

(2) $g(x) = 4 - x^3$

(3) $g(x) = (x - 3)^3$

(4) $g(x) = (2 - x)^3$

(5) $g(x) = (x - 2)^3 + 1$

(6) $g(x) = 2 - (x - 1)^3$

(7) $g(x) = (3 - x)^3 + 1$

(8) $g(x) = 2x^3 - 1$

3 Use the curve of the function f where $f(x) = |x|$ to represent each of the functions that are defined by the following rules and from the graph determine its domain , range and discuss its monotonicity and its type whether it is even , odd or otherwise and write the equation of its axis of symmetry if exist :

(1) $g(x) = |x| + 1$

(2) $g(x) = |x| - 3$

(3) $g(x) = 2 - |x|$

(4) $g(x) = |x + 1|$

(5) $g(x) = -|x + 5|$

(6) $g(x) = |2 - x| + 1$

(7) $g(x) = 4 - |x - 2|$

(8) $g(x) = 2|x|$

(9) $g(x) = 2|x - 7| + 2$

(10) $g(x) = 5 - 2|x + 2|$

(11) $g(x) = 1 - \left|\frac{1}{2}x - 1\right|$

(12) $g(x) = \sqrt{x^2 - 8x + 16}$

4 **Use the curve of the function f where $f(x) = \frac{1}{x}$ to represent each of the functions that are defined by the following rules , from the graph determine its domain , range and discuss its monotonicity and its type whether it is even , odd or otherwise and write its point of symmetry :**

(1) $g(x) = \frac{1}{x} + 2$

(2) 📖 $g(x) = \frac{-1}{x+2}$

(3) $g(x) = \frac{1}{x-2} + 3$

(4) $g(x) = 3 - \frac{1}{x+2}$

(5) $g(x) = \frac{1}{4-x} - 3$

(6) 📖 $g(x) = \frac{x+1}{x}$

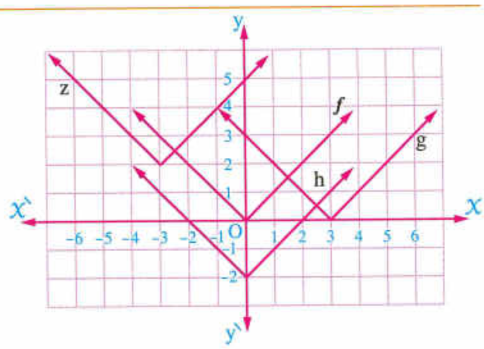
(7) $g(x) = \frac{x-3}{x-2}$

(8) $g(x) = \frac{2x}{x+1}$

(9) $g(x) = \frac{3x-5}{x-2}$

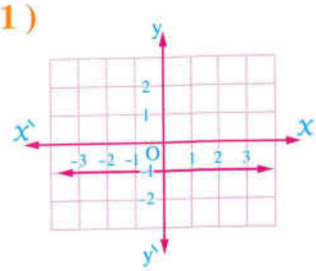
(10) $g(x) = \frac{x-1}{x^2-1}$

5 📖 The curve of f where $f(x) = |x|$ is graphed , then translated in the directions of the coordinate axes as in the opposite figure. Write the rule for each of the functions : g , h , z

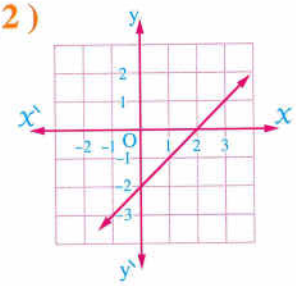


6 **Write the rule of the function f that is represented graphically by each of the following figures :**

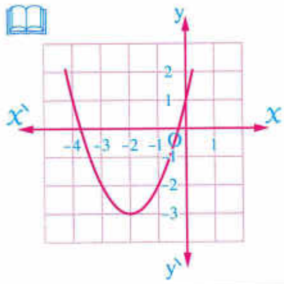
(1)



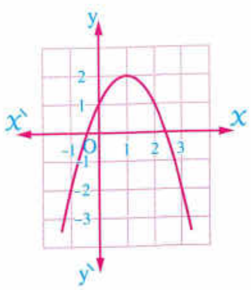
(2)



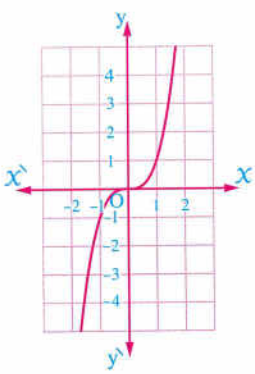
(3) 📖



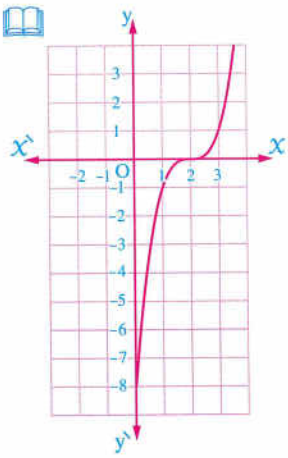
(4)



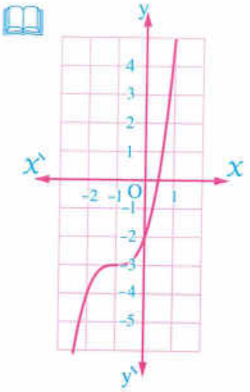
(5)



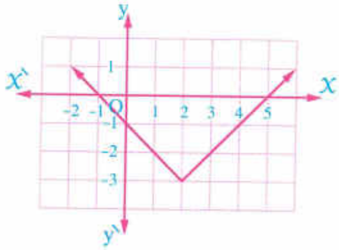
(6) 📖



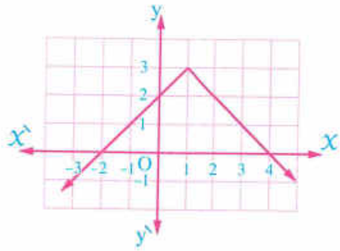
(7) 📖



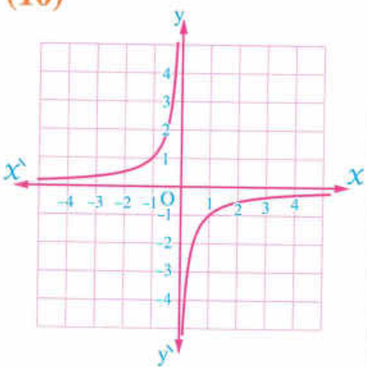
(8) 📖



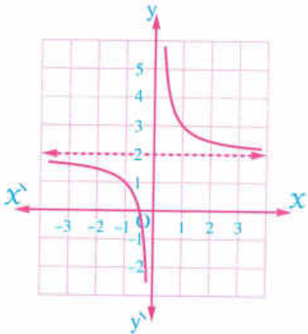
(9)



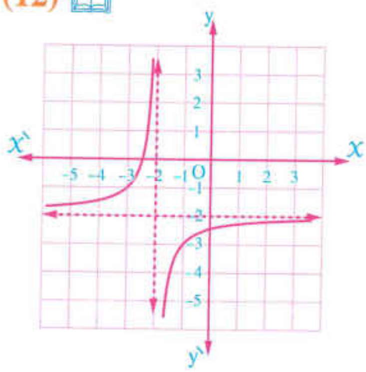
(10)



(11) 📖



(12) 📖



7 **If f , g , k , n are real functions where $f(x) = x^2$, $g(x) = x^3$, $k(x) = |x|$, $n(x) = \frac{1}{x}$, then represent each of the functions that are defined by the following rules showing its domain and range :**

(1) $f_1(x) = f(x + 1)$ (2) $f_2(x) = f(x) - 1$ (3) $f_3(x) = 2 - f(x - 1)$

(4) $g_1(x) = g(x - 1)$ (5) $g_2(x) = g(x) - \frac{1}{2}$ (6) $g_3(x) = g(x - 1) + 2$

(7) $k_1(x) = 2\,k(x)$ (8) $k_2(x) = \frac{1}{2}\,k(x) - 3$ (9) $k_3(x) = 2\,k(x - 1)$

(10) $n_1(x) = n(x - 2)$ (11) $n_2(x) = n(x) - 1$ (12) $n_3(x) = 2 - n(x + 1)$

8 **Draw the curve of the function f in each of the following and determine its range and discuss its monotonicity :**

(1) $f(x) = \begin{cases} x^2 + 1 , & x > 0 \\ -x^2 - 1 , & x < 0 \end{cases}$

(2) 📖 $f(x) = \begin{cases} x^2 + 1 , & -4 \le x < 0 \\ -x^2 - 1 , & 0 \le x \le 4 \end{cases}$

(3) $f(x) = \begin{cases} (x - 1)^3 , & x \ge 0 \\ -1 , & x < 0 \end{cases}$

(4) $f(x) = \begin{cases} (x + 1)^3 + 2 , & x \le -1 \\ \frac{1}{x + 1} + 1 , & x > -1 \end{cases}$

9 The following figures represent the curves of the functions f , g , h respectively. Write the rule of the function in each figure :

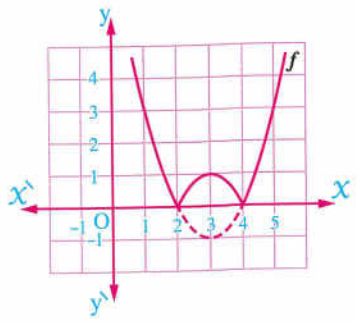


Fig. (1)

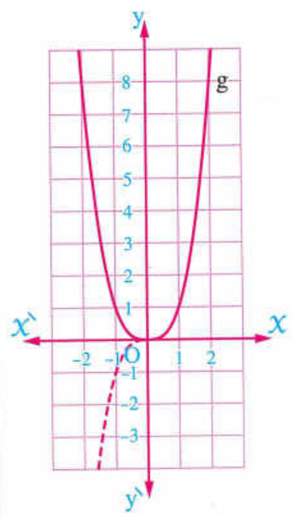


Fig. (2)

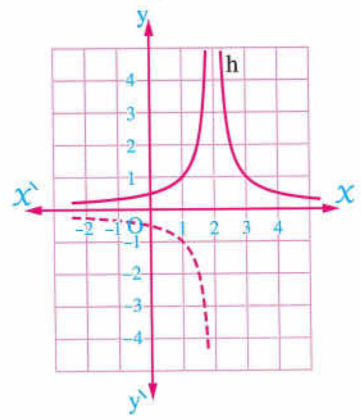


Fig. (3)

10 If $f(x) = \frac{1}{x}$, then draw the graph of the function g in each of the following :

(1) $g(x) = |f(x)|$

(2) $g(x) = -|f(x)|$

(3) $g(x) = 2 + |f(x)|$

(4) $g(x) = |f(x-2)|$

(5) $g(x) = |f(x+1)| - 2$

11 Draw the curve of the function f in each of the following :

(1) $f(x) = x|x|$

(2) $f(x) = x|x| - 1$

(3) $f(x) = |x^3|$

(4) $f(x) = -|x^3|$

(5) $f(x) = \frac{x^4}{|x|}$

(6) $f(x) = |x^3| + 2$

(7) $f(x) = x^2|x| + 2$

(8) $f(x) = \frac{x^3}{|x|} - 3$

(9) $f(x) = (x-2)|x|$

12 Draw the curve of the function f and determine its range and monotonicity if :

(1) $f(x) = |4 - x^2|$

(2) $f(x) = |x^2 - 2x - 3|$, $x \in [-1, 4]$

Third Higher skills

1 Choose the correct answer from those given :

(1) If f is a polynomial function and $f(X) = 0$ at $X \in \{-3, 1, 0\}$, then the function $g : g(X) = f(X - 3)$ cuts the X-axis at $X \in$ ....

(a) $\{-3, 1, 0\}$ (b) $\{3, 0, -2\}$ (c) $\{0, 3, 4\}$ (d) $\{-6, 2, 0\}$

(2) If $f : f(X) = (X - a + 1)^2 + b - 2$ is a quadratic function whose range is $[1, \infty[$ and the curve of f passes through $(3, 2)$, then a = ...

(a) ± 4 (b) 3 or 5 (c) 3 or -5 (d) -3 or 5

(3) If g (X) is decreasing on $]-\infty, 0[$ and is increasing on $]0, \infty[$, then $f(X) = g(X + 1)$ is increasing on ....

(a) $]-1, \infty[$ (b) $]-\infty, -1[$ (c) $]0, \infty[$ (d) $]-\infty, 0[$

(4) The curve $y = 3(X - 5)^2 + 7$ by translation 3 units in the positive direction of X-axis and one unit in the negative direction of y-axis is ....

(a) $y = 3(X + 8)^2 + 6$ (b) $y = 3(X - 8)^2 - 6$

(c) $y = 3(X - 8)^2 + 6$ (d) $y = 3(X + 8)^2 - 6$

(5) If $f(X) = |X| + 2$, then the range of the function $(f \circ f)$ = ...

(a) $]-\infty, 4[$ (b) $[4, \infty[$ (c) $[-4, 4]$ (d) $[0, 4]$

(6) If $g(X) = \frac{1}{2} X^2 - 8$, $f(X) = |X|$, then the range of the function $(f \circ g)$ = ...

(a) $[-8, \infty[$ (b) $[0, \infty[$ (c) $[8, \infty[$ (d) $\mathbb{R}$

(7) If f is an odd function, then $|f(X)|$ is ....

(a) odd. (b) even.

(c) both odd and even. (d) neither odd nor even.

(8) If $f : f(X) = \begin{cases} X^3 + 2 & , \quad X > 0 \\ g(X) & , \quad X < 0 \end{cases}$ is symmetric about the origin

, then g is ....

(a) decreasing. (b) increasing.

(c) not one - to - one. (d) even.

(9) If $f : f(x) = \begin{cases} x^3 + 2 & , \quad x \geq 0 \\ g(x) & , \quad x < 0 \end{cases}$ is symmetric about y-axis

, then g (x) = ...

(a) $x^3 - 2$ (b) $x^3 + 2$ (c) $-x^3 + 2$ (d) $-x^3 - 2$

2 **Graph the curve of the function f in each of the following and from the graph determine the domain and the range and discuss the monotony and show if the function is even , odd or otherwise :**

(1) $f(x) = x^3 + x^2 |x| + 2$

(2) $f(x) = (|x| - 1)^3$

(3) $f(x) = \frac{|x + 1|}{x^2 + 3x + 2}$

(4) $f(x) = \begin{cases} |x + 2| & , \quad x \leq -1 \\ 1 & , \quad -1 < x < 1 \\ |x - 2| & , \quad x \geq 1 \end{cases}$

(5) $f(x) = |x + 1| - |x - 1|$



Exercise 6 — Solving absolute value equations and inequalities

Test yourself

From the school book | Understand | Apply | Higher Order Thinking Skills

First — Multiple choice questions

Choose the correct answer from the given ones:

(1) The solution set of the equation $|X - 2| = 3$ is ....

(a) $\{2 , 3\}$ (b) $\{-1 , 5\}$ (c) $[-1 , 5]$ (d) $\{5 , -5\}$

(2) The solution set of the equation $|5X - 1| + 4 = 1$ in $\mathbb{R}$ is ....

(a) $\{\frac{-4}{5}\}$ (b) $\{\frac{-4}{5} , \frac{6}{5}\}$ (c) $\varnothing$ (d) $\{\frac{6}{5}\}$

(3) The solution set of the equation $|X - 3| = |3 - X|$ in $\mathbb{R}$ is ....

(a) $\{3\}$ (b) $\{3 , -3\}$ (c) $\mathbb{R}$ (d) $\varnothing$

(4) The solution set of the equation : $|2X - 4| = |X + 1|$ is ....

(a) $\{1 , 5\}$ (b) $\{5 , -1\}$ (c) $\{1 , -5\}$ (d) $\{-5 , -1\}$

(5) The solution set of the equation : $|2X - 1| = X + 2$ is ....

(a) $\{3\}$ (b) $\{3 , \frac{1}{3}\}$ (c) $\{3 , -\frac{1}{3}\}$ (d) $\varnothing$

(6) The solution set of the equation : $|X - 2| = X - 2$ is ....

(a) $\varnothing$ (b) $\{2\}$ (c) $[2 , \infty[$ (d) $\mathbb{R}$

(7) The solution set of the equation : $\frac{1}{|X - 3|} = \frac{1}{2}$ is .... where $X \neq 3$

(a) $\{5\}$ (b) $\{1\}$ (c) $\{5 , 1\}$ (d) $\varnothing$

(8) The solution set of the equation :
$|X| + X = 0$ is ....

(a) $\{0\}$ (b) $]-\infty , 0]$ (c) $]-\infty , 0[$ (d) $\varnothing$

(9) The solution set of the equation : $\frac{|x|}{x} = 1$ in $\mathbb{R}$ is ....

(a) $\{0\}$ (b) $\mathbb{R}^+$ (c) $\mathbb{R}^-$ (d) $[0, \infty[$

(10) The solution set of the equation $\sqrt{x^2} - 3x = 8$ is ....

(a) $\varnothing$ (b) $\{-2, -4\}$ (c) $\{-4\}$ (d) $\{-2\}$

(11) The solution set of the equation $x^2 = 2 - |x|$ is ....

(a) $\{1, -1\}$ (b) $\{2, -2\}$ (c) $\{1, -2\}$ (d) $\{-1, 2\}$

(12) If : $f(x) = \sqrt{x^2} + |x|$, then the solution set of the equation $f(x) = 2$ in $\mathbb{R}$ is ....

(a) $\{2, -2\}$ (b) $\{1, -1\}$ (c) $\{2, 0\}$ (d) $\{-1, 0\}$

(13) If $f(x) = |x - 2| + 4$, then the solution set of the equation $f(x + 2) = 3$ is ....

(a) $\{1, 3\}$ (b) $\mathbb{R}$ (c) $\varnothing$ (d) $\{-1, -3\}$

(14) If $f(x) = |x - 2| + 4$, then the solution set of the equation $f(x + 2) = 6$ is ....

(a) $\{0, 4\}$ (b) $\{2, -2\}$ (c) $\{2, 4\}$ (d) $\{-2, -4\}$

(15) The solution set of the equation $\sqrt{4x^2 - 12x + 9} = 5$ is ....

(a) $\{4\}$ (b) $\{-1\}$ (c) $\{4, -1\}$ (d) $\mathbb{R}$

(16) Which of the following does not belong to the solution set of the equation : $|x + 2| + |x - 1| = 3$?

(a) -2 (b) zero (c) -3 (d) 1

(17) The domain of the function $f(x) = \frac{2}{|x| - 2}$ is ....

(a) $\mathbb{R} - \{2\}$ (b) $\mathbb{R} - \{-2\}$ (c) $\mathbb{R}$ (d) $\mathbb{R} - \{2, -2\}$

(18) The domain of the function $f : f(x) = \frac{1}{|x| + 3}$ is ....

(a) $\mathbb{R}$ (b) $\{3, -3\}$ (c) $\mathbb{R} - \{3, -3\}$ (d) $\mathbb{R} - \{3\}$

(19) If the domain of the function $f : f(x) = \frac{x}{|x| + a}$ is $\mathbb{R} - \{2, -2\}$, then a = ...

(a) 2 (b) -2 (c) ± 2 (d) zero

(20) The type of the function $f : f(x) = \frac{x \tan x}{|x|}$ is ....

(a) odd. (b) even.

(c) neither odd nor even. (d) one - to - one.

(21) The false statement in the following is ....

(a) $|xy| = |x||y|$ (b) $|x| = |-x|$

(c) $|x + y| = |x| + |y|$ (d) $\sqrt{x^2} = |x|$

(22) If $a > 0$, $b < 0$, then which of the following is always negative ?

(a) $|a\,b|$ (b) $a\,|b|$ (c) $|a|\,b$ (d) $a + |b|$

(23) If $a - b > 0$, then which of the following could be equal to $|a - b|$?

(1) $a - b$ **(2)** $b - a$ **(3)** $|b - a|$

(a) only (3) (b) both (1) , (3) (c) both (2) , (3) (d) (1) , (2) and (3)

(24) If : $\frac{a}{|a|} = -1$, $\frac{b}{|b|} = 1$, then all the following statements are false except ....

(a) $a \times b > 0$ (b) $a - b > 0$ (c) $a - b = 0$ (d) $a^2 + b > 0$

(25) If $a < 0 < b$, then $|a| + |b| + |b - a| - |a - b| =$ ....

(a) 2 a (b) 2 b (c) a – b (d) b – a

(26) The number of intersecting points of the two curves $y = |X + 2|$, $y = 2 - |X|$ is ....

(a) zero. (b) 1 (c) 2 (d) an infinite number.

(27) The solution set of the inequality $|X| > -1$ is ....

(a) $[0 , \infty[$ (b) $\mathbb{R}$ (c) $\varnothing$ (d) $\mathbb{R} - \{0\}$

(28) The solution set of the inequality $\frac{1}{|X|} \geq 1$ is ....

(a) $[-1 , 1]$ (b) $]-1 , 1[$ (c) $[-1 , 1] - \{0\}$ (d) $]-1 , 1[- \{0\}$

(29) The solution set of the inequality : $|3 - X| > 0$ is ....

(a) $]-3 , 3[$ (b) $\mathbb{R} - [-3 , 3]$ (c) $\mathbb{R} - \{3\}$ (d) $\mathbb{R}$

(30) The solution set of the inequality $\frac{1}{|X - 2|} \geq \frac{1}{2}$ is ....

(a) $]0 , 4[- \{2\}$ (b) $[0 , 4] - \{2\}$ (c) $[0 , 4]$ (d) $]0 , 4]$

(31) The solution set of the inequality $|X + 3| \leq 0$ is ....

(a) $\varnothing$ (b) $]-\infty , -3]$ (c) $]-3 , \infty[$ (d) $\{-3\}$

(32) If : $|X| < a$, $a \in \mathbb{R}^+$, then $X \in$ ....

(a) $\mathbb{R} - [-a , a]$ (b) $[-a , a]$ (c) $\mathbb{R} -]-a , a[$ (d) $]-a , a[$

(33) The solution set of the inequality $|X - 2| \leq -4$ in $\mathbb{R}$ is ....

(a) $]-2 , 6[$ (b) $[-2 , 6]$ (c) $\mathbb{R}$ (d) $\varnothing$

(34) The solution set of the inequality $|2X - 5| \leq 9$ is ....

(a) $]-\infty , 7[$ (b) $[-2 , 7]$

(c) $\mathbb{R} -]-2 , 7[$ (d) $\mathbb{R} - [-2 , 7]$

(35) The solution set of the inequality $|x+2| > -1$ in $\mathbb{R}$ is ....

(a) $]-3, -1[$ (b) $\mathbb{R}$ (c) $\mathbb{R} - [-3, -1]$ (d) $\varnothing$

(36) The solution set of the inequality $|4-6x| \geq 14$ is ....

(a) $]\frac{-5}{3}, 3[$ (b) $[\frac{-5}{3}, 3]$ (c) $\mathbb{R} -]\frac{-5}{3}, 3[$ (d) $\mathbb{R} - [\frac{-5}{3}, 3]$

(37) If $x \in [-1, 4]$, then $|2x-3| \leq$ ....

(a) 3 (b) 4 (c) 5 (d) -5

(38) The solution set of the inequality $\sqrt{x^2-4x+4} > 0$ in $\mathbb{R}$ is ....

(a) $\mathbb{R} - \{2\}$ (b) $\mathbb{R} - \{-2\}$ (c) $\mathbb{R}$ (d) $\varnothing$

(39) The solution set of the inequality : $\sqrt{x^2-2x+1} \geq 4$ is ....

(a) $[-3, 5]$ (b) $\mathbb{R} -]-3, 5[$ (c) $]-3, 5[$ (d) $\mathbb{R} - [-3, 5]$

(40) The solution set of the inequality $\sqrt{4x^2-12x+9} \leq 9$ is ....

(a) $[-6, 12]$ (b) $[-3, 6]$ (c) $\mathbb{R} - [-3, 6]$ (d) $\mathbb{R} -]-3, 6[$

(41) The solution set of the inequality : $|2x-3| + |6-4x| \leq 21$ is ....

(a) $[-2, 5]$ (b) $\mathbb{R} -]-2, 5[$ (c) $]-2, 5[$ (d) $\{-2, 5\}$

(42) The solution set of the inequality $\sqrt{(x+2)^2} + |2x+4| > 6$ is ....

(a) $[-4, 0]$ (b) $\mathbb{R} - [-4, 0]$ (c) $]-4, 0[$ (d) $\mathbb{R} -]-4, 0[$

(43) If $1 < x < 2$, then : $\sqrt{x^2-2x+1} + \sqrt{x^2-4x+4} =$ ....

(a) $2x-3$ (b) $2x-1$ (c) 1 (d) 3

(44) The absolute value inequality that represents that the score of a student in one test (x) is including between 70 to 90 is ....

(a) $|x| \leq 90$ (b) $|x| \geq 70$ (c) $|x-80| \leq 10$ (d) $|x-70| \leq 90$

(45) Number of the integer solutions to the inequality : $|x-2| \leq 5$ is ....

(a) zero. (b) 7 (c) 9 (d) 11

(46) The domain of the function $f : f(x) = \sqrt{\sqrt{x^2} - 1}$ is ....

(a) $[-1, 1]$ (b) $]-1, 1[$ (c) $\mathbb{R} - [-1, 1]$ (d) $\mathbb{R} -]-1, 1[$

Second Essay questions

Exercises on solving absolute value equations :

1 Find algebraically in $\mathbb{R}$ the solution set of each of the following equations :

(1) $4|x|-20=0$ «$\{5, -5\}$»

(2) $|3-2x|=7$ «$\{5, -2\}$»

(3) $3-|x+2|=2$ «$\{-1, -3\}$»

(4) $2|x|=3-|x|$ «$\{1, -1\}$»

(5) $|x-2|=3x-4$ «$\left\{\frac{3}{2}\right\}$»

(6) $|x+2|-x+1=0$ «$\varnothing$»

(7) $|x+2|+x-2=0$ «$\{0\}$»

(8) $5x-2|x|=21$ «$\{7\}$»

(9) $|x+5|=|x-3|$ «$\{-1\}$»

(10) $|2x-6|=|x-3|$ «$\{3\}$»

(11) $|x-1|-2|2-x|=0$ «$\left\{3, \frac{5}{3}\right\}$»

(12) $x^2-5|x|+6=0$ «$\{2, 3, -2, -3\}$»

(13) $\sqrt{x^2}-2x=6$ «$\{-2\}$»

(14) $\sqrt{x^2-4x+4}=4$ «$\{6, -2\}$»

(15) $|x|+x=0$ «$]-\infty, 0]$»

(16) $|x+3|+2x=0$ «$\{-1\}$»

(17) $\sqrt{x^2-6x+9}+2x=9$ «$\{4\}$»

(18) $|x-3|^2-|x-3|=0$ «$\{3, 2, 4\}$»

(19) $5|3-x|-2\sqrt{x^2-6x+9}=12$ «$\{7, -1\}$»

(20) $|x^2-1|=|x-1|$ «$\{1, 0, -2\}$»

(21) $|x+1||x-1|=26$ «$\left\{3\sqrt{3}, -3\sqrt{3}\right\}$»

(22) $|x+1|^2-3|x+1|-10=0$ «$\{4, -6\}$»

(23) $(x-5)^2=|2x-10|$ «$\{5, 7, 3\}$»

(24) $x|x|-x=0$ «$\{0, 1, -1\}$»

(25) $x|5-x|=6$ «$\{2, 3, 6\}$»

(26) $|x^2+x-10|=10$ «$\{0, -1, -5, 4\}$»

(27) $\frac{4x-8}{|x-2|}=x$, $x\neq 2$ «$\{4, -4\}$»

(28) $x^3|x|=8x$ «$\{-2, 0, 2\}$»

2 Find graphically in $\mathbb{R}$ the solution set of each of the following equations and verify the results algebraically :

(1) $|x|-4=0$ «$\{4, -4\}$»

(2) $|x|+2=0$ «$\varnothing$»

(3) $2-|x+2|=0$ «$\{0, -4\}$»

(4) $\sqrt{x^2-4x+4}=3$ «$\{5, -1\}$»

(5) $|x+3|+x=5$ «$\{1\}$»

(6) $|x-2|=3x-4$ «$\{1\frac{1}{2}\}$»

(7) $|2x+4|=1-x$ «$\{-1, -5\}$»

(8) $|2x+5|=x-4$ «$\varnothing$»

(9) $|x+2|=|x-3|$ «$\left\{\frac{1}{2}\right\}$»

(10) $|x-2|+|x-1|=0$ «$\varnothing$»

(11) $|x-3|=|2x+1|$ « $\left\{-4, \frac{2}{3}\right\}$ »

(12) $|x-4|=4-x$ « $]-\infty, 4]$ »

(13) $|x+2|=x+2$ « $[-2, \infty[$ »

(14) $|x-3|-|x+2|=5$ « $]-\infty, -2]$ »

(15) $|x-3|=4-|x+1|$ « $[-1, 3]$ »

(16) $|x+2|=x^2+2$ « $\{0, 1\}$ »

3 Prove that the function $f : f(x) = \frac{12}{|x|+2}$ is even , then find the solution set of the equation $f(x) = 2$ algebraically. « $\{-4, 4\}$ »

4 Graph the function $f : f(x) = |2x+5|-3$, determine the range of the function and study its monotonicity.
From the graph , deduce the solution set of the equation : $|2x+5|-3=0$, then verify the solution algebraically. « $\{-1, -4\}$ »

5 Graph the function $f : f(x) = 1-|2x|$ and from the graph deduce its range and its monotonicity.
Prove also that f is even. From the graph or by any other method , find the solution set of the equation : $1-|2x|=-3$ « $\{-2, 2\}$ »

6 Prove that the function $f : f(x) = \frac{1-|x|}{|x|}$ is even , and graph its curve , then find graphically or algebraically the solution set of the equation $f(x) = 0$ « $\{-1, 1\}$ »

7 Draw the graph of the function $f : f(x) = |x-1|+x+1$, then deduce the range and monotony and find the solution set of the equation : $f(x) = 3$ « $\left\{\frac{3}{2}\right\}$ »

8 Draw in the same figure the two functions f , g where $f(x) = x^2|x|$, $g(x) = 2-|x|$, then from the graph find the solution set of the equation $f(x) = g(x)$ « $\{-1, 1\}$ »

Exercises on solving absolute value inequalities :

9 **Find algebraically in $\mathbb{R}$ the solution set of each of the following inequalities :**

(1) $|x-3| \le 5$ « $[-2, 8]$ »

(2) $|x-3| \ge 5$ « $\mathbb{R}-]-2, 8[$ »

(3) $|2x+5|>3$ « $\mathbb{R}-[-4, -1]$ »

(4) $|3-x|<7$ « $]-4, 10[$ »

(5) $\left|\frac{x-3}{4}\right| \le 1$ « $[-1, 7]$ »

(6) $|3x+2|+5<4$ « $\varnothing$ »

(7) $\frac{1}{|2x-5|} \ge 3$ « $[\frac{7}{3}, \frac{8}{3}]-\left\{\frac{5}{2}\right\}$ »

(8) $\frac{1}{|3x|} \ge 5$ « $[-\frac{1}{15}, \frac{1}{15}]-\{0\}$ »

(9) $\frac{1}{|2x-3|} > 2$ « $]\frac{5}{4}, \frac{7}{4}[-\left\{\frac{3}{2}\right\}$ »

(10) $|2x-3|+3>7$ « $\mathbb{R}-[-\frac{1}{2}, \frac{7}{2}]$ »

(11) 📖 $\sqrt{x^2-2x+1} \geq 4$ «$\mathbb{R}-]-3 , 5[$»

(12) 📖 $\sqrt{4x^2-12x+9} \leq 9$ «$[-3 , 6]$»

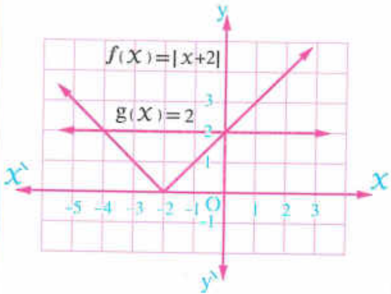
(13) $|x-2|+|2-x|<6$ «$]-1 , 5[$»

(14) $|3x-2|+2|6x-4| \geq 20$ «$\mathbb{R}-]-\frac{2}{3} , 2[$»

(15) $\sqrt{(x+2)^2}+|2x+4| \geq 6$ «$\mathbb{R}-]-4 , 0[$»

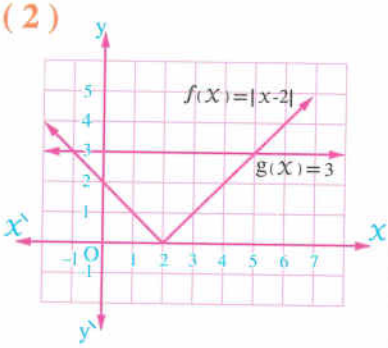
10 **Using the following figures :**

(1) 📖



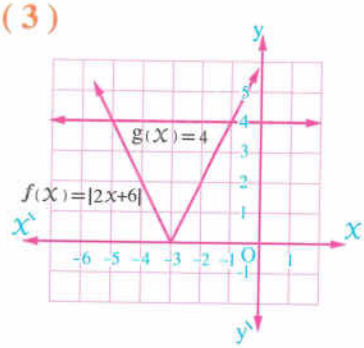
Find : The S.S. of the inequality $f(x) < g(x)$ in $\mathbb{R}$

(2)



Find : The S.S. of the inequality $f(x) > g(x)$ in $\mathbb{R}$

(3)



Find : The S.S. of the inequality $f(x) \leq g(x)$ in $\mathbb{R}$

11 **Find graphically in $\mathbb{R}$ the solution set of each of the following inequalities , then verify the result algebraically :**

(1) 📖 $|x-1|<2$

(2) 📖 $|2x-5| \geq 2$

(3) 📖 $\sqrt{4x^2-12x+9}>5$

(4) $|x+3|<-x$

(5) $|x-2|<\frac{-1}{3}x+6$

(6) $|x-3|+|x-4|>5$

12 📖 **Write in the form of an absolute value inequality each of the following :**

(1) $-4 \leq x \leq 4$

(2) $0<x<6$

(3) $x \geq 2$ or $x \leq -2$

(4) $x \in \mathbb{R}-[-2 , 6]$

13 📖 **Write the absolute value inequality which expresses :**

(1) The student's mark in an exam ranges between 60 and 100

(2) The temperature measured by a thermometer ranges between 35° , 42°C

(3) The Green algae found in Ocean reaches 30 meters deep.

Miscellaneous exercises :

14 **Find the domain of each of the functions defined by the following rules :**

(1) $f(x) = \frac{2}{|x| - 1}$

(2) $f(x) = \frac{3}{|x| + 1}$

(3) $f(x) = \frac{3x}{|x - 2| - 5}$

(4) $f(x) = \sqrt{5 - |x|}$

(5) $f(x) = \sqrt{|x| - 5}$

(6) $f(x) = \frac{1}{\sqrt{5 - |x|}}$

15 **Show whether each of the functions defined by the following rules is even , odd or otherwise :**

(1) $f(x) = x|x|$

(2) 📖 $f(x) = x^2 |x| - 1$

(3) $f(x) = \frac{|1 + x| + |1 - x|}{|1 + x| - |1 - x|}$

(4) $f(x) = \frac{x^2 \cos 2x}{5 + |2x|}$

(5) $f(x) = 2|x| \tan x + 2x|\tan x|$

(6) $f(x) = \sqrt{|x| + x}$

16 **Find in square units the area between the curves of the functions f and g where :**

(1) $f(x) = |x + 3| + 2$, $g(x) = 4$

(2) $f(x) = -|x - 2| + 3$, $g(x) = 0$

(3) 📖 $f(x) = |x - 2| - 1$, $g(x) = 5 - |x - 2|$

17 Prove that the function f where $f(x) = \frac{|x| + 1}{|x|}$ is even and graph the curve of f , then find graphically and algebraically the solution set of the equation $f(x) = 2x - 2$ and check your solution.

Third Higher skills

1 **Choose the correct answer from those given :**

(1) The solution set of the equation $|x + 1|^2 + |2x + 3| = 0$ in $\mathbb{R}$ is ....

(a) $\left\{-1 , \frac{-3}{2}\right\}$ (b) $\mathbb{R}$ (c) $\left\{1 , \frac{3}{2}\right\}$ (d) $\varnothing$

(2) The solution set of the equation $|x^2 - 4x + 3| = |x - 3|$ in $\mathbb{R}$ is ....

(a) $\{0 , 2\}$ (b) $\{2 , 3\}$ (c) $\{0 , 3\}$ (d) $\{0 , 2 , 3\}$

(3) The smallest value of the expression $\frac{|x|+|y|}{|x+y|}$ is ....

(a) -1 (b) zero (c) 1 (d) 2

(4) If $2^x = 61$, then $|x-6|+|x-5| =$ ....

(a) -11 (b) -1 (c) 1 (d) 11

(5) If $a^2 b > 0$, $\frac{a}{b} < 0$, then $\sqrt{a^2}+\sqrt{b^2}-(b-a) =$ ....

(a) $2a$ (b) $-2b$ (c) $-2a+2b$ (d) zero

(6) If $-4 \le x \le 6$ and $a \le |4x-8| \le b$, then $a+b =$ ....

(a) 24 (b) 20 (c) 18 (d) 16

(7) The product of the roots of the equation : $x^2 - 3|x| - 10 = 0$ equals ....

(a) -25 (b) -10 (c) 10 (d) 25

(8) The number of integers that satisfies $3 < |x| < 8$ equals ....

(a) 4 (b) 6 (c) 8 (d) 10

(9) If $f(x) = |2x-5|$, $g(x) = |x+1|$, then the solution set of the equation $(g \circ f)(x) = 3$ in $\mathbb{R}$ is ....

(a) $\{2, -4\}$ (b) $\left\{\frac{3}{2}, \frac{7}{2}\right\}$ (c) $\{3, 7\}$ (d) $\left\{\frac{3}{2}, \frac{9}{2}\right\}$

2 **Find algebraically in $\mathbb{R}^*$ the solution set of each of the following equations :**

(1) $(x+1)(|x|-1)+\frac{1}{2} =$ zero « $\left\{\frac{1}{\sqrt{2}}, -1+\frac{1}{\sqrt{2}}, -1-\frac{1}{\sqrt{2}}\right\}$ »

(2) $||2x+3|-8| = 5$ « $\{-8, 5, 0, -3\}$ »

(3) $\frac{2}{|x|}+|x| = 3$ « $\{2, -2, 1, -1\}$ »

(4) $\sqrt{\left(x-\frac{1}{x}\right)^2+4} = 2$ « $\{1, -1\}$ »

(5) $|x^2-3|+\frac{2}{|x^2-3|} = 3$ « $\{2, -2, \sqrt{2}, -\sqrt{2}, 1, -1, \sqrt{5}, -\sqrt{5}\}$ »

(6) $|4-x|+|x-2| = 10$ « $\{8, -2\}$ »

(7) $(|2x-3|-5)(|2x-3|+5) > 11$ « $\mathbb{R}-\left[\frac{-3}{2}, \frac{9}{2}\right]$ »

Life Applications on Unit One

From the school book

1 **Mechanics :** If the velocity v (t) of a motorcycle is given by

$$v(t) = \begin{cases} 8t & , \quad 0 \le t \le 10 \\ 80 & , \quad 10 < t < 200 \\ -4t + 880 & , \quad 200 \le t \le 220 \end{cases}$$

where t is the time in second , v in cm./sec.

, find :

(1) v (10) (2) v (150) (3) v (210)

---

2 **Geometry :**

If P is the perimeter of a square of side length ℓ , write P as a function of ℓ [P (ℓ)]

, then find :

(1) P (3) (2) $P\left(\frac{15}{4}\right)$

---

3 **Geometry :**

If A is the area of a circle of radius length r , write A as a function of r [A (r)]

, then find :

(1) $A\left(\frac{1}{2}\right)$ (2) A (5)

---

4 **Industry :**

Said works in a factory producing energy - saving lamps. If his salary was 8 pounds every working hour in addition to 0.3 pound for each lamp produced daily :

(1) Find the function f which expresses Said's salary if he works for 7 hours daily.

(2) Is the function f one - to - one ? Explain your answer.

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5 **Trade :**

A grains merchant pays 50 L.E for each ton getting in or out of his warehouse for loading or unloading the goods. Write down the function representing the cost of loading or unloading , then represent it graphically.

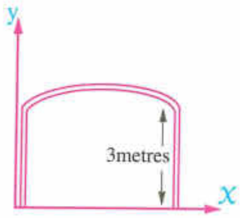
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6 **Mechanics :**

A body covered d meters in 3 minutes with a uniform velocity 30 m./min.
Show that the velocity (v) varies inversely with the time (t) for covering this distance and write the function which represents the velocity and time , then represent it graphically , then find the time taken to cover this distance if the body moves with a velocity of 45 m./min.

7 **Industry :**

An iron gate whose two sides are 3 metres high and its arc is in the form of a part of the curve of the function $f : f(X) = a(X-2)^2 + 4$ has been designed as shown in the opposite figure , **find :**



(1) The value of a

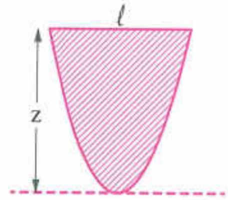
(2) The maximum height of the gate.

(3) The width of the gate.

« $-\frac{1}{4}$, 4 m. , 4 m. »

8 **Geometry :**

If you know that the area included between the curve of a quadratic function and a horizontal line segment joining between any two points lying on it (as shown in the figure) is calculated by the relation $a = \frac{2}{3} \ell z$



(1) Find the area of the figure included between the X-axis and the curve of the quadratic function $f : f(X) = X^2 - 6X + 5$ in square units.

(2) On the same lattice represent the curves of two functions f and g where $g(X) = |X-3| - 2$, then find the area of the part included between them in square units.

« $\frac{32}{3}$, $\frac{20}{3}$ square unit »

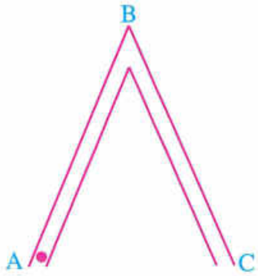
9 **Planning cities :** A piece of land is included between the curves of the two functions f , g where $f(X) = |X-3| - 2$ and $g(X) = 3$, calculate its area in square units. If the length unit is 8 metres , find the area of this land in square meters. « 1600 m^2 »

10 **Roads nets :** Two roads , the first is represented by the curve of the function $f : f(X) = |X-5|$, and the other is represented by the curve of the function $g : g(X) = 5 - \frac{2}{3}X$ If the two roads intersects at the points A and B , find the distance between A and B to the nearest km. if the length unit represents a distance of 5 km. « 36 km. »

11 **Life application on solving inequality :** One of the natural gas companies allows employing a counter reader if his length ranges between 178 cm. and 192 cm. Express all possible lengths for the persons applying to join this job using the absolute value inequality.

« $|X - 185| \leq 7$ »

12 **Mechanics :** A body moves with a uniform velocity of magnitude 8 cm./sec. from the position A to position C passing through the position B without stopping. If the distance between the body and the position B is given by $d(t) = 8 \mid 5 - t \mid$ where t is the time in seconds and d is the distance in cm.



(1) Calculate the distance between the body and the position B after 2 seconds and after 8 seconds. What do you notice ? Explain your answer.

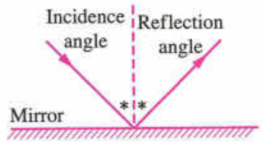
(2) When does the body become at a distance 16 cm. from the position B ? Explain your answer.

(3) When does the body become at a distance less than 8 cm. from the position B ?

« 24 cm. , 24 cm. , 3 , 7 ,]4 , 6[»

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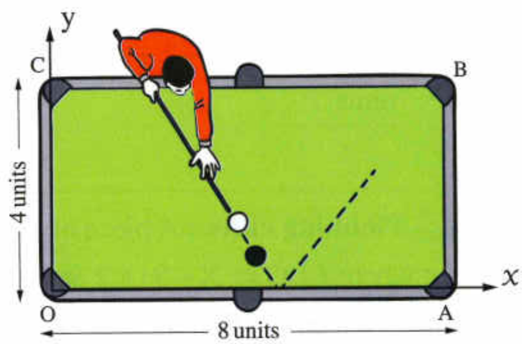
13 If a light ray falls on a reflective surface whose pathway is subjected to the modulus function , the measurement of the incidence angle equals the measurement of the reflection angle. In addition , the pathway of the billiard ball before and after colliding it against the table edge.



* The opposite figure shows a billiard player shooting at the black ball. Considering $\overrightarrow{Ox}$ and $\overrightarrow{Oy}$ the perpendicular coordinates axes , and the path of the ball follows the curve of the function f where $f(x) = \frac{4}{3} \mid x - 5 \mid$

Does the black ball fall in the pocket B ?

Explain your answer mathematically.



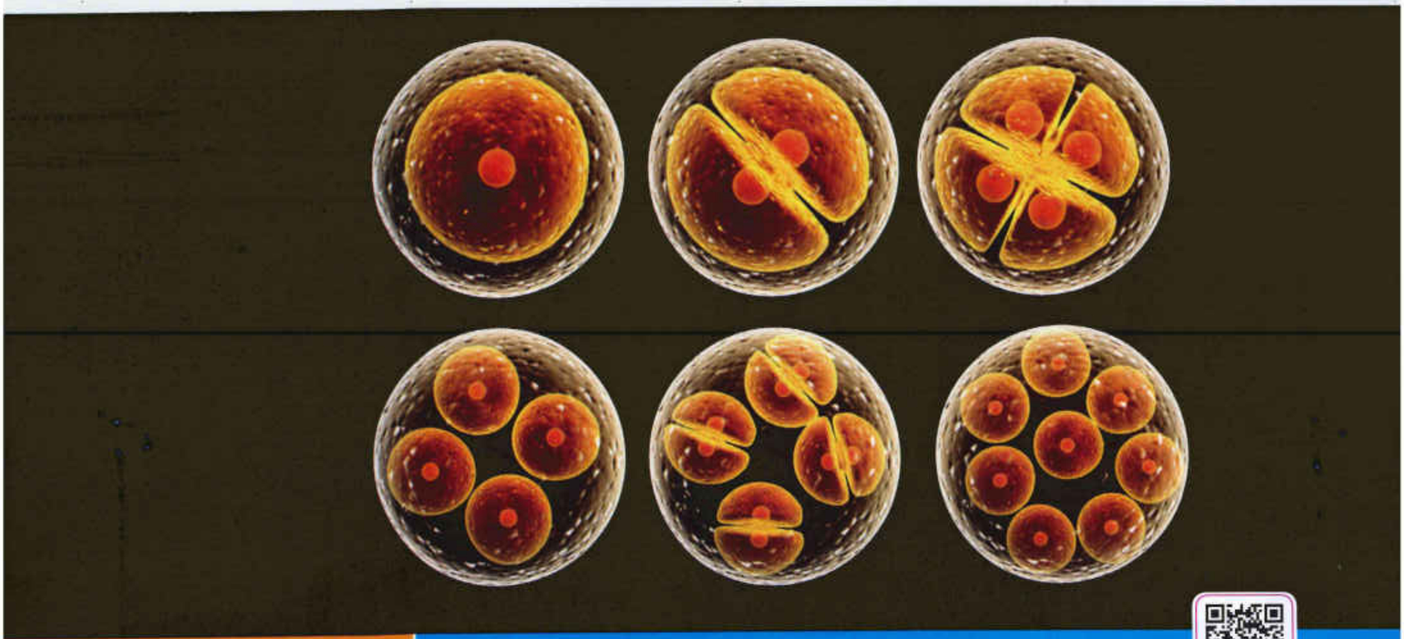
UNIT 2

Exponents, logarithms and their applications

Unit Exercises

- **At the end of the unit :**

Life applications on unit two.



Exercise 7 Rational exponents and exponential equations

Test yourself

From the school book ● Understand ● Apply ● Higher Order Thinking Skills

First Multiple choice questions

Choose the correct answer from those given :

(1) $a^m \times a^m = \dots\dots$

(a) a^{m^2} (b) a^{2m} (c) $2a^m$ (d) ma^2

(2) $\sqrt[5]{a^3} \times \sqrt{a^3} = \dots\dots$

(a) $\sqrt[7]{a^3}$ (b) $\sqrt[7]{a^6}$ (c) $\sqrt[7]{a^{14}}$ (d) $a^2 \sqrt[10]{a}$

(3) If $2^{x+1} = 8$, then $x = \dots\dots$

(a) 1 (b) 2 (c) 3 (d) 4

(4) If $3^{x+5} = \frac{1}{27}$, then $x = \dots\dots$

(a) – 3 (b) 8 (c) – 8 (d) 3

(5) If $5^{x-1} = 4^{x-1}$, then $x = \dots\dots$

(a) 5 (b) 1 (c) –1 (d) zero

(6) The solution set of the equation : $5^{x^2-4} = 7^{x^2-4}$ is $\dots\dots$

(a) $\{2\}$ (b) $\{-2\}$ (c) $\{2 , -2\}$ (d) $\{\text{zero}\}$

(7) If $7^{x+1} = 3^{2x+2}$, then : $5^{x+1} = \dots\dots$

(a) zero (b) 1 (c) 2 (d) 5

(8) 📖 If $\left(\frac{1}{2}\right)^{a^2 - a - 2} = 1$ where a > zero , then : a = ...

(a) 1 (b) –3 (c) 2 (d) 3

(9) The solution set of the equation : $7^{x^2} = 49^{x+4}$ is ....

(a) $\{-2\}$ (b) $\{-2 , 4\}$ (c) $\{-2 , 3\}$ (d) $\{2 , -4\}$

(10) If $3^x = 2$, $2^y = 9$, then $x\,y$ = ...

(a) 2 (b) 3 (c) 8 (d) 18

(11) 📖 If $5^x = 2$, then $(25)^x$ = ...

(a) 10 (b) 625 (c) 4 (d) 2

(12) If $2^x = 5$, then 2^{x+2} = ...

(a) 15 (b) 4 (c) 10 (d) 20

(13) 📖 If $x^{\frac{3}{2}} = 64$, then x = ...

(a) 512 (b) 16 (c) 4 (d) 2

(14) If $x^{\frac{2}{5}} = 4$, then x = ...

(a) 4 (b) 16 (c) ± 4 (d) ± 32

(15) 📖 If $4\,x^5 = 128$, then x = ...

(a) 4 (b) ± 2 (c) 2 (d) – 2

(16) $(128)^{-\frac{2}{7}}$ = ...

(a) 2 (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) 4

(17) 📖 $\sqrt[4]{(16)^{-3}}$ = ...

(a) 8 (b) –8 (c) $\frac{1}{8}$ (d) $-\frac{1}{8}$

(18) If x , y $\in \mathbb{R}$, then $\sqrt{x^2 y^6}$ = ...

(a) $x\,y^2$ (b) $|x\,y^3|$ (c) $\frac{1}{2}\,x^2 y^6$ (d) $\pm\,x\,y^3$

(19) 📖 $\sqrt[4]{16\,x^4 y^8}$ = ...

(a) $2\,x\,y^2$ (b) $2\,|x|\,y^2$ (c) $2\,x\,|y|^2$ (d) $2\,x\,|y^2|$

(20) If $2^{x-1} = 44$, then 2^{x-2} = ...

(a) 18 (b) 22 (c) 10 (d) 16

(21) If $x^{\frac{5}{3}} = 2\,y^{\frac{4}{3}} = 32$, then $x + y$ = ...

(a) 16 (b) zero. (c) 16 , – 16 (d) zero , 16

(22) The solution set of the equation : $3^{x+1} + 3^x = 12$ in $\mathbb{R}$ is ....

(a) $\{0\}$ (b) $\{3\}$ (c) $\{1\}$ (d) $\{1 , 0\}$

(23) The solution set of the equation : $3^{x} + 3^{3-x} = 12$ is ....

(a) $\{1, 2\}$ (b) $\{0, 3\}$ (c) $\{3, 4\}$ (d) $\{-1, -2\}$

(24) The solution set of the equation : $\sqrt[3]{x^2} - 3\sqrt[3]{x} + 2 = 0$ is ....

(a) $\{1, 8\}$ (b) $\{9, 3\}$ (c) $\{8\}$ (d) $\{1\}$

(25) The solution set of the equation : $9^{x} - 30 \times 3^{x-1} + 9 = 0$ is ....

(a) $\{0, 1\}$ (b) $\{1, 2\}$ (c) $\{0, 2\}$ (d) $\{0, 3\}$

(26) The number of real roots of the equation : $x^n = a$ where n is an odd number is ....

(a) 1 (b) 2 (c) 3 (d) n

(27) The number of real roots of the equation : $x^6 = a$ where $a > 0$, is ....

(a) 1 (b) 2 (c) 3 (d) 6

(28) The number of roots of the equation : $x^3 = 4$ is ....

(a) 1 (b) 2 (c) 3 (d) 4

(29) The number of real roots of the equation : $x^4 = -16$ is ....

(a) zero (b) 1 (c) 2 (d) 4

(30) The set of the real roots of the equation : $(x - 2)^4 = 16$ equals ....

(a) $\{0\}$ (b) $\{4\}$ (c) $\{8\}$ (d) $\{0, 4\}$

(31) The solution set of the equation : $(x - 3)^{\frac{5}{3}} = 32$ in $\mathbb{R}$ is ....

(a) $\{2\}$ (b) $\{11\}$ (c) $\{11, -5\}$ (d) $\{-11, 11\}$

(32) If $x \in \mathbb{R}^*$, n is an even integer, which of the following is true ?

(a) $x^n > 0$ (b) $x^n < 0$ (c) $x^n \le 0$ (d) $x^n = 0$

(33) If $x \in \mathbb{R}^-$, n is an odd integer, which of the following is true ?

(a) $x^n > 0$ (b) $x^n < 0$ (c) $x^n \le 0$ (d) $x^n = 0$

(34) Which of the following is not equal to $\sqrt[5]{x^4}$?

(a) $\left(\sqrt[5]{x}\right)^4$ (b) $\sqrt[4]{x^5}$ (c) $x^{\frac{4}{5}}$ (d) $\left(x^{\frac{1}{5}}\right)^4$

(35) If $a < b < 0 < c$, then $\dfrac{\sqrt[4]{b^4 c^4} + b\sqrt{(a-c)^2}}{\sqrt{a^2 b^2}} =$ ....

(a) 1 (b) -1 (c) $\dfrac{a}{b}$ (d) $\dfrac{-c}{2}$

(36) If $a < 0 < b < c$, then which of the following does not belong to $\mathbb{R}$?

(a) $\sqrt[3]{ab}$ (b) $\sqrt[4]{bc}$ (c) $\sqrt[5]{ab + c}$ (d) $\sqrt[6]{ac}$

(37) If $\sqrt{2} \times \sqrt[3]{3} = \sqrt[6]{x}$, then $x =$ ....

(a) 27 (b) 48 (c) 72 (d) 108

(38) $\sqrt{9^{16x^2}} =$ ....

(a) 3^{4x} (b) 3^{8x^2} (c) 9^{4x} (d) 9^{8x^2}

(39) If $3^{x-2} = \sqrt[4]{27}$, then $x =$ ....

(a) $\frac{11}{4}$ (b) $\frac{4}{3}$ (c) $\frac{3}{4}$ (d) 6

(40) If $3^{x+2} = 6^{x-1}$, then $2^{x} =$ ....

(a) 54 (b) 27 (c) $\frac{1}{9}$ (d) $\frac{1}{36}$

(41) If $2^{x} = 20$, $n < x < n + 1$, n is an integer , then n = ...

(a) 4 (b) 5 (c) 6 (d) 7

(42) If $3^{x} < 0$, then ....

(a) $0 < x < 1$ (b) $-1 < x < 0$

(c) $x < -1$ (d) there are no values for x satisfy this inequality.

(43) 📖 The number $(2^{24} + 2^{23} + 2^{22})$ is divisible by ....

(a) 3 (b) 5 (c) 7 (d) 9

(44) If $3^{a} = 4^{b}$, then $9^{\frac{a}{b}} + 16^{\frac{b}{a}} =$ ....

(a) 7 (b) 12 (c) 20 (d) 25

(45) If $2^{a} = 3$, $3^{b} = 7$, $7^{c} = 11$, then $2^{abc} =$ ....

(a) 11 (b) 27 (c) 21 (d) 231

(46) If $2^{x} = a$, $3^{x} = b$, $5^{x} = c$, then $(90)^{x} =$ ....

(a) abc (b) a^2bc (c) ab^2c (d) a + 2 b + c

Second Essay questions

1 Simplify to the simplest form :

(1) 📖 $\frac{(27)^{-3} \times (12)^2}{16 \times (81)^{-2}}$ « 3 »

(2) $\frac{4^{2n+1} \times 2^{1-n}}{8^{n+2}}$ « $\frac{1}{8}$ »

(3) $\frac{9^{4n+1} \times 4^{2-2n}}{3^{9n+1} \times 48^{1-n}}$ « 1 »

(4) $\frac{125 \times (15)^{n-3} \times (25)^{m+n}}{(75)^{n} \times (5)^{n+2m}}$ « $\frac{1}{27}$ »

(5) 📖 $(18)^{\frac{1}{2}} \times (12)^{\frac{3}{2}} \times \frac{1}{(24)^{\frac{1}{2}}}$ « 36 »

(6) 📖 $\frac{3^{\frac{1}{2}} \times (147)^{\frac{1}{6}}}{(63)^{\frac{1}{3}}}$ « 1 »

2 Prove that :

(1) $\dfrac{(27)^{y-\frac{1}{3}} \times \sqrt[y]{7^{y^2+3y}}}{(81)^{y-1} \times (21)^{5-y} \times (49)^{y-1}} = \dfrac{1}{9}$

(2) $\dfrac{(343)^{2x-\frac{1}{3}} \times (4)^{3x+1}}{(196)^{3x} \times 4} = \dfrac{1}{7}$

(3) $\dfrac{125 \times \sqrt[8]{4^3} \times 10^{-\frac{1}{4}}}{4^{\frac{5}{8}} \times \sqrt[4]{6^{-3}} \times (15)^{\frac{3}{4}}} = 25$

(4) $\dfrac{3 \times 2^{x+2} + \sqrt{4^{x+2}}}{2^{x+3} - 6 \times 2^x} = 8$

3 Find in ℝ the solution set of each of the following equations :

(1) $x^{\frac{7}{2}} = 128$

(2) $x^{\frac{5}{2}} = \frac{1}{32}$

(3) $x^{-\frac{5}{3}} = \sqrt[3]{32}$

(4) $\sqrt[3]{x^4} = 81$

(5) $\sqrt[3]{(x-1)^5} = 32$

(6) $(2x+3)^{\frac{4}{3}} = 81$

(7) $(x+1)^{-\frac{5}{2}} = (32)^{-\frac{1}{2}}$

(8) $(x^2 - 5x + 9)^{\frac{5}{2}} = 243$

(9) $\left(\sqrt{x} + 2\right)^{\frac{1}{2}} = 3$

(10) $x^{\frac{4}{5}} - 5x^{\frac{2}{5}} + 4 = 0$

(11) $x^{\frac{4}{3}} - 10x^{\frac{2}{3}} + 9 = 0$

(12) $x - 3x^{\frac{1}{2}} + 2 = 0$

(13) $x + 15 = 8\sqrt{x}$

(14) $\sqrt[3]{x^2} - 25\sqrt[3]{x} - 54 = 0$

(15) $\sqrt[3]{x^5} - \sqrt[6]{x^5} = 6$

(16) $\sqrt[5]{x^4} - 3\sqrt[5]{x^2} = 4$

(17) $\sqrt[5]{x} - 3 = -\dfrac{2}{\sqrt[5]{x}}$

4 Find in ℝ the solution set of each of the following equations :

(1) $5^{2x-1} = \frac{1}{125}$ « {−1} »

(2) $5^{x+2} = x^{x+2}$ « {5 , −2} »

(3) $2^{x^2-9} = 1$ « {3 , −3} »

(4) $x^{x^2-25} = 3^{x^2-25}$ « {3 , −3 , 5 , −5} »

(5) $\left(3\sqrt{3}\right)^{|x|} = 27$ « {2 , −2} »

(6) $3^{|3x-4|} = 9^{2x-2}$ « $\{\frac{8}{7}\}$ »

(7) $2 \times 4^{x+3} = \frac{1}{32}$ « {−6} »

(8) $\left(\frac{3}{5}\right)^{2x-1} = \frac{27}{125}$ « {2} »

(9) $5^{x-1} \times 7^{1-x} = \frac{25}{49}$ « {3} »

(10) $\dfrac{9^{x+1} \times 4^{x-1}}{36^x} = \left(\frac{3}{2}\right)^x$ « {2} »

(11) $\dfrac{12^{3x-2} \times 9^{x+1}}{18^{2x} \times 4^{2x-2}} = 9$ « {2} »

(12) $\frac{1}{27}\left(\sqrt{3}\right)^{x+2} = 1$ « {4} »

(13) $(\sqrt{3})^{x^2-5x} = 1$ « {0 , 5} »

(14) $5^{x^2-5x} = 0.0016$ « {1 , 4} »

(15) $5^{x^2} = 25^{x+4}$ « {–2 , 4} »

(16) 📖 $3^{x^2-42} = \left(\frac{1}{3}\right)^x$ « {6 , –7} »

(17) $(\sqrt{7})^{|x+2|} = 49$ « {2 , –6} »

(18) $3^{2x-3} \times 7^{6-4x} = 1$ « $\{\frac{3}{2}\}$ »

(19) $\sqrt{9^x - 2 \times 3^{x+1} + 9} = 24$ « {3} »

(20) $\sqrt{5^{2x-2} - \frac{2}{5} \times 5^x + 1} = 124$ « {4} »

5 **Find in $\mathbb{R}$ the solution set of each of the following equations :**

(1) $5^{x+1} + 5^{x-1} = 26$ « {1} »

(2) 📖 $3^{x+3} - 3^{x+2} = 162$ « {2} »

(3) 📖 $7^{2-x} + 7^{-x} = 50$ « {0} »

(4) 📖 $5^{2x} + 25 = 26 \times 5^x$ « {0 , 2} »

(5) 📖 $2^x + 2^{5-x} = 12$ « {2 , 3} »

(6) 📖 $\left(\frac{1}{2}\right)^{x+1} + \left(\frac{1}{2}\right)^{x+3} + \left(\frac{1}{2}\right)^{x+5} = 84$ « {–7} »

(7) $4^x + 2^{x+1} = 8$ « {1} »

(8) $2^{2x+1} - 33 \times 2^x + 16 = 0$ « {–1 , 4} »

(9) $5^{2x-2} - 6 \times 5^{x-1} + 5 = 0$ « {1 , 2} »

(10) $9^{x^2-1} - 36 \times 3^{x^2-3} + 3 = 0$ « $\{1 , -1 , \sqrt{2} , -\sqrt{2}\}$ »

(11) $10^x - 5^{x-1} \times 2^{x-2} = 950$ « {3} »

6 (1) If $y^{\frac{3}{4}} = 2x^{\frac{5}{3}} = 64$, then find the value of : $5x + 2y$ « 552 »

(2) 📖 If $x^{\frac{3}{2}} = 3y^{\frac{2}{3}} = 27$, then find the value of : $x + y$ « 36 or –18 »

(3) If $x^{\frac{4}{3}} = 9y^{-\frac{2}{3}} = 81$, then find the value of : $|2xy|$ « 2 »

7 📖 **Find the error :**

(1) $-9 = (-9)^{\frac{2}{2}} = \sqrt{(-9)^2} = \sqrt{81} = 9$

(2) If $x^4 = 81$, then $x = \sqrt[4]{81}$ $\therefore x = 3$

8 **Find in $\mathbb{R} \times \mathbb{R}$ the solution set of each of the following equations :**

(1) $4^{x+y} = 128$, $5^{x-2y-3} = 1$ « $\{(\frac{10}{3} , \frac{1}{6})\}$ »

(2) 📖 $3^x \times 5^y = 75$, $3^y \times 5^x = 45$ « {(1 , 2)} »

(3) $3^x \times 3^y = 27$, $3^x + 3^y = 12$ « {(1 , 2) , (2 , 1)} »

9 Creative thinking :

Find in $\mathbb{R}$ the solution set of the equation : $9^{x+1} - 3^{x+3} - 3^{x} + 3 = 0$ « $\{-2 , 1\}$ »

10 Creative thinking :

If $x^3 = y^2$ and $x^{n+1} = y^{n-1}$, find the value of n « 5 »

Third Higher skills

1 **Choose the correct answer from the given ones :**

(1) If $x < 0$, then $\sqrt{x^2} - \sqrt[3]{x^3} - \sqrt{x^2 - 2x + 1} + 1 =$ ....

(a) x (b) $-x$ (c) 0 (d) -1

(2) If $a = \sqrt[4]{\dfrac{\sqrt[3]{2}}{\sqrt{7}}}$, then which of the following numbers is rational ?

(a) a^{12} (b) a^{16} (c) a^{18} (d) a^{24}

(3) Which of the following is not always true ?

(a) $\left(\sqrt[5]{\sqrt[6]{2}}\right)^{30} = 2$ (b) $\sqrt[5]{128} = 2\sqrt[5]{4}$

(c) $\sqrt{x^2 y^6} = x y^3$ (d) $\sqrt[6]{x^4}\ \sqrt[6]{x^4} = x^{\frac{4}{3}}$

(4) The solution set of the equation $x^{\frac{2}{3}} = x^{\frac{3}{2}}$ in $\mathbb{R}$ is ....

(a) $\{1\}$ (b) $\{-1 , 1\}$ (c) $\{0 , 1\}$ (d) $\{0 , 1 , -1\}$

(5) $\sqrt[5]{\dfrac{2}{27}} = \dfrac{\cdots\cdots\cdots}{3}$

(a) $\sqrt[5]{2}$ (b) $\sqrt[5]{4}$ (c) $\sqrt[5]{18}$ (d) $3\sqrt{5}$

(6) The relation $\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$ is true for all values of ....

(a) $a \in \mathbb{R}$ (b) $n \in \mathbb{Z}^+ - \{1\}$, n is even

(c) $\sqrt[n]{a} \in \mathbb{R}$, $n \in \mathbb{Z}^+ - \{1\}$ (d) nothing of the previous

(7) The equation $x^{\frac{2}{3}} = a$ has a solution in $\mathbb{R}$ if ....

(a) $a \in \mathbb{R}$ (b) $a \in \mathbb{R}^+$ (c) $a \in \mathbb{R}^-$ (d) $a \in \mathbb{R}^+ \cup \{0\}$

(8) If $\sqrt[3]{2\sqrt{3\sqrt{2}}} = \sqrt[6]{m\sqrt{2}}$, then m = ...

(a) 6 (b) 8 (c) 10 (d) 12

(9) $\sqrt[n]{a} \times \sqrt[m]{a} =$ ....

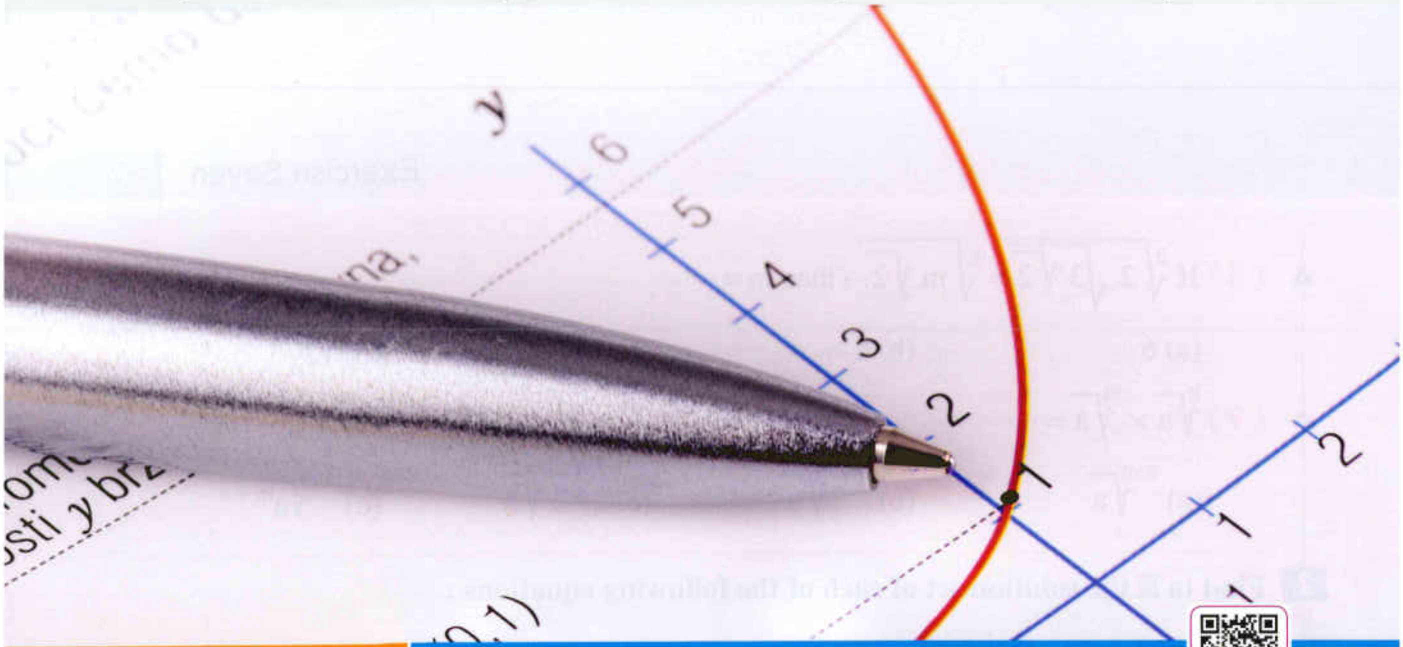
(a) $\sqrt[nm]{a}$ (b) $\sqrt[n+m]{a}$ (c) $\sqrt[\frac{1}{n}+\frac{1}{m}]{a}$ (d) $\sqrt[nm]{a^{m+n}}$

2 Find in ℝ the solution set of each of the following equations :

(1) $5^{x-1} \times x^{2-x} = 5$ « {5 , 2} »

(2) $(x-3)^{x-5} = 1$ « {5 , 4} »

(3) $(x-3)^{x-6} = 1$ « {6 , 4 , 2} »



Exercise 8 Exponential function and its applications

Test yourself

From the school book ● Understand ● Apply ● Higher Order Thinking Skills

First Multiple choice questions

Choose the correct answer from the given ones :

(1) If $f : f(x) = a^{x}$ is an exponential function, then a ∈ ....

(a) $\mathbb{R}$ (b) $\mathbb{R}^{+}$ (c) $\mathbb{R}^{-}$ (d) $\mathbb{R}^{+} - \{1\}$

(2) If $f : f(x) = 3^{x+2}$, then $f(-2) =$ ....

(a) 3 (b) zero (c) – 1 (d) 1

(3) If : $f(x) = 4^{x-1}$, then $f(x+1) =$ ....

(a) 4^{x} (b) 4^{x+1} (c) 4^{x+2} (d) 2^{x}

(4) If : $f(x) = 2^{x}$, then $f(-x) =$ ....

(a) -2^{x} (b) $\left(\frac{1}{2}\right)^{x}$ (c) 2^{x+1} (d) $\left(\frac{1}{2}\right)^{-x}$

(5) If : $f(x) = (5)^{-x}$, then $\frac{f(x-1)}{f(x+1)} =$ ....

(a) 5 (b) $\frac{1}{5}$ (c) 25 (d) $\frac{1}{25}$

(6) If : $f(x-1) = 2^{x+1}$, then $f(x) =$ ....

(a) 2^{x} (b) 2^{x-1} (c) 2^{x+2} (d) 2^{x-2}

(7) If : $f(x) = a^{x}$, then $f(x+1) \times f(x-1) = f($...............$)$

(a) $2x + 1$ (b) a^{2x} (c) $2x$ (d) 2

(8) If : $f(x+1) = 2^{x}$ and $f(a) = 8$, then a = ...

(a) 3 (b) 2 (c) 4 (d) 5

(9) If : $f(X) = 3^{X-2}$, then the solution set of the equation : $f(X-1) = 81$ is ....

(a) $\{7\}$ (b) $\{5\}$ (c) $\{4\}$ (d) $\{3\}$

(10) If $f(X) = 2^{X}$, then the solution set of the equation $f(2X) - f(X+1) = \text{zero}$ is ....

(a) $\{0\}$ (b) $\{0 , 1\}$ (c) $\{1\}$ (d) $\{-1\}$

(11) If $f(X) = 3^{X}$, then the value of X which satisfy the equation : $f(X+1) - f(X-1) = 24$ is ....

(a) 2 (b) 3 (c) 8 (d) zero.

(12) If $f(X) = 3^{X}$, then the value of X which satisfy the relation :
$f(2X) - 24 f(X-1) - f(2) = 0$ is ....

(a) $2 , \frac{1}{3}$ (b) 2 , zero (c) 2 (d) $2 , -1$

(13) The exponential function of base a is increasing if ....

(a) $a > 0$ (b) $a > 1$ (c) $0 < a < 1$ (d) $a = 1$

(14) The exponential function of base a is decreasing if ....

(a) $a > 0$ (b) $a < 0$ (c) $0 < a < 1$ (d) $-1 < a < 0$

(15) The range of the function $f : f(X) = \left(\frac{1}{2}\right)^{X}$ is ....

(a) $]-\infty , \infty[$ (b) $]-\infty , 0[$ (c) $]0 , \infty[$ (d) $]1 , \infty[$

(16) If $f(X) = 2^{-X}$, then $f(X)$ is decreasing when $X \in$ ....

(a) $\mathbb{R}$ (b) $\mathbb{R}^{+}$ (c) $\mathbb{R}^{-}$ (d) $\varnothing$

(17) Which of the following functions is increasing on its domain ?

(a) $f(X) = \left(\frac{1}{2}\right)^{X}$ (b) $f(X) = 3^{-X}$ (c) $f(X) = \left(\frac{2}{3}\right)^{X}$ (d) $f(X) = 5^{X}$

(18) If $n(X) = \left(-\frac{1}{2}\right)^{X+1}$, then it represents ....

(a) an exponential function with base $\left(-\frac{1}{2}\right)$

(b) an exponential function with exponent $(X+1)$

(c) not an exponential function because the base < 0

(d) both (a) and (b)

(19) If $f(X) = 2^{X+1}$ and the point $\left(a , \frac{1}{2}\right) \in$ to the curve of the function f , then a = ...

(a) $\frac{1}{2}$ (b) -1 (c) 2 (d) -2

(20) If $y_1 = 2^{X}$, $y_2 = 3^{X}$, $y_3 = 4^{X}$ where $X > 0$, then ....

(a) $y_1 > y_2 > y_3$ (b) $y_1 > y_3 > y_2$ (c) $y_3 > y_2 > y_1$ (d) $y_3 > y_1 > y_2$

(21) If $f(X) = a^X$, then ....

(a) $f(X + y) = f(X) + f(y)$
(b) $f(X - y) = f(X) - f(y)$
(c) $f(X + y) = f(X) \cdot f(y)$
(d) $f(X y) = f(X) \cdot f(y)$

(22) If the curve of the function $f : f(X) = 2^X$ is shifted one unit to the left , then the new function is $g : g(X) =$ ....

(a) 2^{X+1}
(b) 2^{X-1}
(c) -2^X
(d) -2^{1-X}

(23) The curve of the function $f : f(X) = 3^X$ is the image of the curve of the function $g : g(X) = -3^X$ by reflection in ....

(a) $y = 0$
(b) $X = 0$
(c) $y = X$
(d) $y = -X$

(24) The equation of the symmetry axis of the two functions f , g where $f(X) = 3^X$ and $g(X) = \left(\frac{1}{3}\right)^X$ is ....

(a) $y = 0$
(b) $X = 0$
(c) $y = X$
(d) $y = -X$

(25) The curve of the function $f : f(X) = 5^X$ intersects the y-axis at the point ....

(a) (1 , 0)
(b) (0 , 1)
(c) (5 , 1)
(d) (1 , 5)

(26) The curve of the function $f : f(X) = 2^{X+2}$ intersects the y-axis at the point ....

(a) (0 , 1)
(b) (0 , 2)
(c) (0 , 4)
(d) (0 , 8)

(27) The straight line y = 9 cuts the curve of the function $f : f(X) = 3^X$ at the point ....

(a) (0 , 1)
(b) (2 , 0)
(c) (2 , 9)
(d) (1 , 9)

(28) If the point (a , b) where $a \neq 0$ lies on the curve of the function $y = 2^X$, which of the following points lies on the curve of the function $y = \left(\frac{1}{2}\right)^X$?

(a) (a , b)
(b) (– a , b)
(c) (a , – b)
(d) $\left(a , \frac{1}{2} b\right)$

(29) If the point (a , b) lies on the curve of the function $y = 2^X$, then which of the following points lies on the curve of the function $y = 2^{X+3}$

(a) (a , b)
(b) (a + 3 , b)
(c) (a , b + 3)
(d) (a , 8 b)

(30) The exponential function f where $f(X) = a^X$, a > 1 , its curve approaches ....

(a) the X-axis (positive direction)
(b) the X-axis (negative direction)
(c) the y-axis (positive direction)
(d) the y-axis (negative direction)

(31) In the exponential function $f : f(X) = a^X$, a > 1 , then $f(X) > 1$ when $X \in$ ....

(a) $\mathbb{R}$
(b) $\mathbb{R}^+$
(c) $\mathbb{R}^-$
(d) $\mathbb{Z}$

(32) Which of the functions that are defined by the following rules represents an exponential growth function ?

(a) $f(X) = 2^{-X}$ (b) $f(X) = \left(\frac{1}{3}\right)^{X}$ (c) $f(X) = 3^{X}$ (d) $f(X) = \left(\frac{2}{3}\right)^{X}$

(33) Which of the functions that are defined by the following rules represents an exponential decay function ?

(a) $f(X) = 2^{X}$ (b) $f(X) = \left(\frac{1}{3}\right)^{-X}$ (c) $f(X) = 3^{X}$ (d) $f(X) = \left(\frac{2}{3}\right)^{X}$

(34) The opposite figure represents the curve of the function $y = a^{X}$

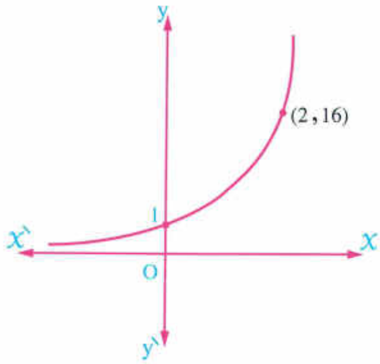
, then a = ...

(a) 2

(b) 3

(c) 4

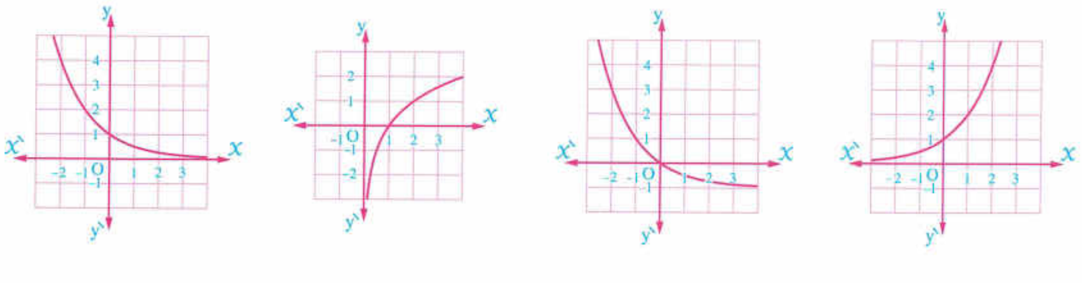
(d) 9



(35) In the exponential function g where $g(X) = a^{X}$, $0 < a < 1$, then $0 < a^{X} < 1$ when $X \in$ ....

(a) $]0 , \infty[$ (b) $]-\infty , 0]$ (c) $]1 , \infty[$ (d) $]-\infty , 1]$

(36) The function f where $f(X) = 2^{X}$ is represented by the figure ....



(a) (b) (c) (d)

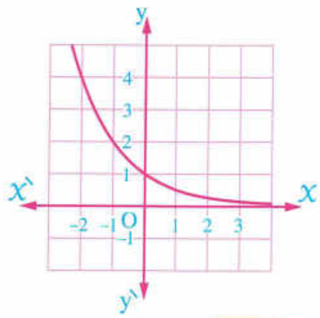
(37) The opposite figure shows the function f where ....

(a) $f(X) = 2^{X+1}$

(b) $f(X) = 2^{-X}$

(c) $f(X) = 3^{-X}$

(d) $f(X) = 2^{X}$



(38) An amount of 5000 pounds is deposited in a bank gives a yearly compound interest 5% for 7 years ≃ .... pounds.

(a) 6750 (b) 7035.5 (c) 5350 (d) 8500

(39) Galal bought a car for 200000 pounds , if the car price depreciated by 0.4 % yearly. Which of the following functions express the car price after n years ?

(a) $y = 200000 \times (0.4)^n$ (b) $y = 200000\ (0.996)^n$

(c) $y = 200000 \times (1.4)^n$ (d) $y = 200000\ (0.2)^n$

Second Essay questions

1 **Show which of the functions defined by the following rules is an exponential function , then determine the base and the power of each :**

(1) $f(X) = 2X^3$ (2) $f(X) = \frac{2}{3}(5)^X$ (3) $f(X) = \frac{1}{X-1}$

(4) $f(X) = 3X^2 - 1$ (5) $f(X) = \left(\frac{2}{3}\right)^{X-1}$ (6) $f(X) = (-7)^X$

2 **If $f(X) = 5^X$, then find the value of :** $\frac{f(X+4) - f(X+3)}{f(X+5) - f(X+4)}$ «$\frac{1}{5}$»

3 **If $f(X) = 7^{X+1}$, then find the value of X that satisfies :**

$f(2X-1) + f(X-2) = 50$ « 1 »

4 **If $f_1(X) = 3^X$ and $f_2(X) = 9^X$, then find the value of X that satisfies :**

$f_1(2X-1) + f_2(X+1) = 756$ « 2 »

5 **If $f : \mathbb{R} \longrightarrow \mathbb{R}^+$ where $f(X) = a^X$, then prove that :** $\frac{f(X+1) + f(X+2)}{f(X+1) + f(X)} = a$

6 **If $f(X) = 3^X$, then prove that :** $\frac{f(2X+2) + f(2X-1)}{5f(2X) - 7f(2X-1)} = \frac{7}{2}$

7 **If $f(X) = 3^{3X-1}$, then prove that :** $\frac{f(X+1) \times f(X+2)}{f(X+3)} = f(X)$

8 **If $f(X) = 2^X$, then prove that :** $\frac{f(X+1)}{f(X-1)} + \frac{f(X-1)}{f(X+1)} = \frac{17}{4}$

9 **If $f(X) = 7^X$, then find the value of X satisfying :**

$f(2X-1) + f(2X+1) = \frac{50}{49}$ «$\frac{-1}{2}$»

10 **If $f(X) = 2^{X}$ and $\frac{f(X+1) - f(X)}{f(X) - f(X-1)} = f(X-2)$, then find X** « 3 »

11 **If $f(X) = 3^{-X}$, then find the S.S. of the equation : $\frac{f(2X+1)}{f(X-1)} = 729$** « {– 8} »

12 **If $f(X) = 3^{X-1}$, then find the value of X satisfying : $f(X+2) + f(4-X) = 30$** « 0 or 2 »

13 **If $f(X) = 2^{X}$, then find the S.S. of the equation : $f(2X) - 6f(X) + f(3) = 0$** « {1 , 2} »

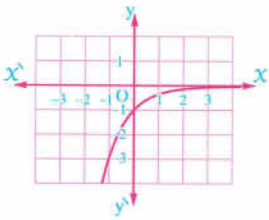
14 **If $f(X) = 3^{X}$, then find the S.S. of the equation : $f(X) + f\left(\frac{X}{2}\right) = 12$** « {2} »

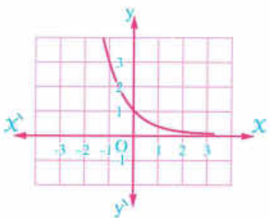
15 **If $f(X-2) = 3^{X}$, then find the value of X that satisfies : $f(3X+1) - f(3X-1) = 24$** « 0 »

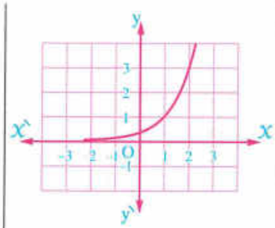
16 **Write the rule of each function under its suitable graph :**

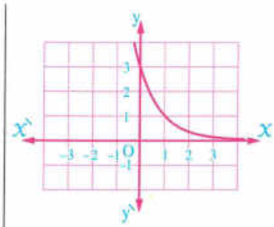
(1) $y = 3^{X}$ (2) $y = 3^{-X}$ (3) $y = -3^{X}$ (4) $y = -3^{-X}$

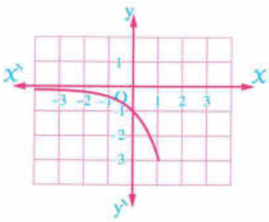
(5) $y = 3^{X} - 1$ (6) $y = 3^{X-1}$ (7) $y = 3^{1-X}$ (8) $y = 1 - 3^{X}$

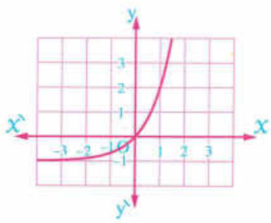

Fig. (1)

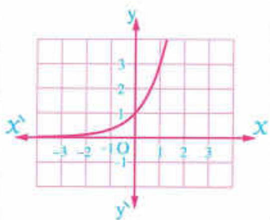

Fig. (2)

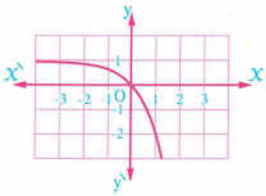

Fig. (3)


Fig. (4)


Fig. (5)


Fig. (6)


Fig. (7)


Fig. (8)

17 **Represent graphically each of the following functions , then find the domain and the range of each , also determine which is increasing and which is decreasing :**

(1) $f(X) = 3^{X}$

(2) $f(X) = \left(\frac{1}{2}\right)^{X}$

(3) $f(X) = -3(2)^{X}$

(4) $f(X) = 2^{X+1} + 1$

(5) $f(X) = \left(\frac{1}{2}\right)^{X+2} - 2$

(6) $f(X) = 2\left(\frac{2}{3}\right)^{X-1} + 1$

(7) $f(X) = -\left(\frac{1}{2}\right)^{2X} + \frac{3}{4}$

18 **Find graphically in $\mathbb{R}$ the solution set of each of the following equations :**

(1) $2^{x+3} = 4$
(2) $2^{x} = 8$
(3) $3^{x} = 4 - x$
(4) $3^{x-2} = 3 - x$
(5) $2^{1-x} = x + 5$
(6) $4^{x+1} = 4 - x$
(7) $2^{x} = 2x$
(8) $2^{x} = 3^{x+1}$
(9) $2^{x+1} = 3^{1-x}$

19 **If $f : \mathbb{R} \longrightarrow \mathbb{R}^+$ where $f(x) = 3^{x-1}$, graph the function where $x \in [-2, 3]$, from the graph find :**

(1) $f\left(\frac{3}{2}\right)$
(2) The value of x when : $3^{x-1} = 7\frac{1}{2}$

« 1.7 , 2.8 »

20 **Graph the function $f : f(x) = \begin{cases} 1 - x^2 & , \quad x \le 0 \\ \left(\frac{1}{2}\right)^x & , \quad x > 0 \end{cases}$**

and from the graph , deduce the domain and the range of the function and its monotonicity.

21 Draw the function $f : f(x) = 2^{|x|}$, then from the graph , deduce the range of the function and its monotonicity and show whether it is odd , even or otherwise.

22 Draw the function $f : f(x) = \left(\frac{1}{2}\right)^{|x|}$, then from the graph , deduce the range of the function and its monotonicity and show whether it is odd , even or otherwise.

23 **If $f(x) = 5^x$ and $f(1 + \sin x) + f(-1 + \sin x) = 26$, find the real values of x**

« $x = \frac{\pi}{2} + 2\pi n$ where $n = \mathbb{Z}$ »

24 **Creative thinking :**

If $f(x) = 2^x$, then prove that : $\frac{1}{f(x) + 1} + \frac{1}{f(-x) + 1}$ has a constant value whatever the value of x

Applications on the exponential growth and decay

25 **Saving :**

Ziad deposit L.E. 80000 in a bank which gives an annual interest of 10.5% ,

Find the total amount of money after 10 years.

given that the total amount is given by $C = a(1 + r)^t$ where t is the number of years , a is the starting amount , r is the annual interest.

« L.E. 217126 »

26 **In-habitation :**

If the population of a country at the end of the year 2000 is 43.3 million and the rate of population increasing is 1.5% yearly.

(1) Find a form represents the population of this country after n years from the year 2000

(2) Use this form to find the expected population of this country at the year 2020

« 58.3 millions »

---

27 **Sport :**

The number of spectators of a football team decreases at the rate of 4 % each match as a result of recurrent loss in a championship. If the number of spectators in the first match was 36400 , write the exponential function which represents the number of spectators (y) in the match (t) , then estimate the number of fans in the tenth match. « 24200 fans »

---

28 **Investment :**

The number of cows in a cattle farm is 80 cows and the reproduction rate of these cows is 18 % annually. Find the number of cows after 4 years. « 155 cows »

---

29 **Population :**

The number of population in a city of A.R.E. reached 4.6 million people with an average increase 4 % annually.

(1) Write the exponential growth function after t years.

(2) Estimate the number of population after 5 years. « 5.6 millions »

---

30 **Industry :**

The production of gold mine is decreasing with the rate 5% yearly , if the production of the mine in the first year was about 254 kg. evaluate the production of the mine in the 9th year. « 160 kg. »

---

31 **Biology :**

If the amount of the bacterium at an instant 2000 bacterium and it is increasing with the rate 7% per hour.

Find the amount of the bacterium after 11 hours. « 4210 bacterium »

---

32 A man deposited a capital of L.E. 5000 in one of the banks with annual compound interest 8%. Find the sum of the capital after 10 year in each of the following :

(1) The interest compounded annually. « L.E. 10794.62 »

(2) The interest compounded quarter annually. « L.E. 11040.2 »

(3) The interest compounded monthly. « L.E. 11098.2 »

Third Higher skills

Choose the correct answer from the given ones :

(1) The function $f : f(X) = (2a)^X$ is decreasing when a ∈ ....

(a)]0 , 1[(b)]1 , ∞[(c)]0 , 2[(d) $]0 , \frac{1}{2}[$

(2) If the function $f : f(X) = \left(\frac{a}{3}\right)^X$ is an increasing exponential function , then ....

(a) $a > 0$ (b) $a > 1$ (c) $a > 3$ (d) $a < 3$

(3) If $f : f(X) = (a-2)^X$ is an exponential function , then ....

(a) $a \in \mathbb{R}^+ - \{2\}$ (b) $a > 2$ (c) $a < 2$ (d) $a \in]2 , \infty[- \{3\}$

(4) Which of the following curves intersects X-axis ?

(a) $f(X) = \left(\frac{1}{3}\right)^X$ (b) $f(X) = 2^X + 3$ (c) $f(X) = 3^X - 1$ (d) $f(X) = 3^{X-1}$

(5) If the straight line $y = 8$ cuts the two curves $y = 2^X$, $y = \left(\frac{1}{2}\right)^X$ at the two points A , B respectively , then the length of $\overline{AB}$ = ... length unit.

(a) 2 (b) 3 (c) 4 (d) 6

(6) When the curve of the function $f : f(X) = 3^X$ is reflected about y-axis , then shifted 5 units upwards , then the new function is $g : g(X) =$ ....

(a) $5 + 3^{-X}$ (b) $-3^X + 5$

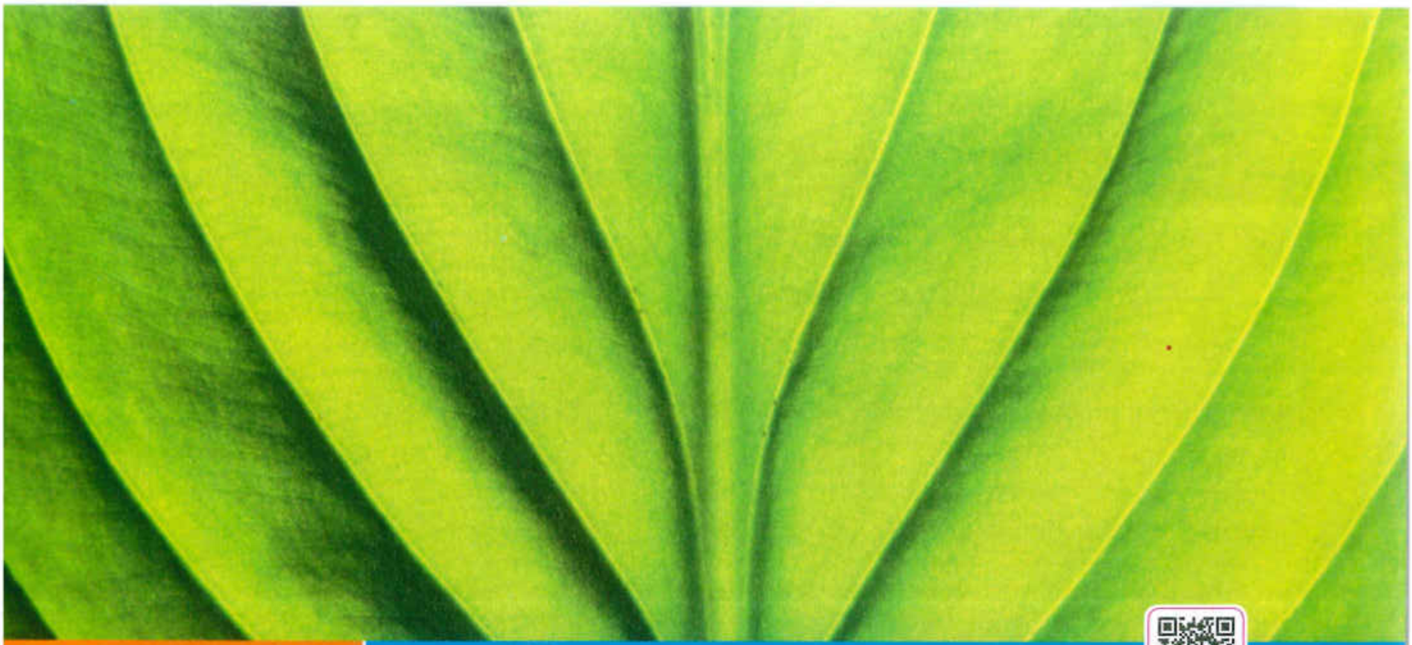
(c) $3^X - 5$ (d) $3^{-X} - 5$

(7) If $f(X) = \frac{9^X}{9^X + 3}$, then $f(X) + f(1 - X) =$ ....

(a) $\frac{2}{9^X + 3}$ (b) $\frac{9^X + 3}{2}$ (c) $\frac{1}{3}$ (d) 1

(8) If $f(X) = \frac{4^X}{4^X + 2}$, then $f\left(\frac{1}{11}\right) + f\left(\frac{2}{11}\right) + f\left(\frac{3}{11}\right) + \cdots + f\left(\frac{10}{11}\right) =$ ....

(a) 5 (b) 6 (c) $\frac{6}{11}$ (d) 10



Exercise 9 — The inverse function

Test yourself

From the school book | ● Understand | ● Apply | Higher Order Thinking Skills

First: Multiple choice questions

Choose the correct answer from those given :

(1) If f is an odd function and g is a function where the curve of g is the image of the curve of f by reflection in the straight line $y = x$, then ....

(a) $g(x) = f(x)$ (b) $g(x) = \frac{1}{f(x)}$ (c) $g(x) = f^{-1}(x)$ (d) $g(x) = (f \circ f)(x)$

(2) Which of the following does not have an inverse function ?

(a) $y = \sqrt{x}$ (b) $y = 3x$ (c) $y = x^3$ (d) $y = x^2$

(3) The function f, f^{-1} are image of each other by reflection in the straight line ....

(a) $y = 0$ (b) $x = 0$ (c) $y = -x$ (d) $y = x$

(4) If $(a , b) \in$ the curve of the function f, then .... $\in$ the curve of the function f^{-1}

(a) (a , b) (b) $(a , -b)$ (c) (b , a) (d) $(b , -a)$

(5) If $f(7) = 3$, then $f^{-1}(3) =$ ....

(a) 3 (b) 4 (c) 7 (d) 10

(6) If the function $f = \{(1 , 4) , (2 , -3) , (3 , 1) , (4 , 0)\}$, then $f^{-1}(1) + f(2) =$ ....

(a) -1 (b) zero (c) 1 (d) 3

(7) If the function f^{-1} where $f^{-1} = \{(2 , 3) , (5 , b)\}$ is the inverse function of the function f where $f = \{(4 , 5) , (a , 2)\}$, then $a - b =$ ....

(a) zero (b) 1 (c) -1 (d) 2

(8) If the straight line $y = x$ intersects the one - to - one function f at the point $(2, 2)$, then it intersects the function f^{-1} at the point ....

(a) $(-2, 2)$ (b) $(2, 2)$ (c) $(-2, -2)$ (d) $(2, -2)$

(9) If f is a function where $f(x) = x + 2$, then $f^{-1}(x) =$ ....

(a) $x + 2$ (b) $-x + 2$ (c) $x - 2$ (d) $\frac{x}{2}$

(10) If f is a function where $f(x) = 7x$, then $f^{-1}(x) =$ ....

(a) $7x$ (b) $\frac{x}{7}$ (c) $\frac{7}{x}$ (d) $7 - x$

(11) The image of the point $(3, -1)$ by reflection in the straight line $y = x$ is ....

(a) $(3, -1)$ (b) $(-1, -3)$ (c) $(-1, 3)$ (d) $(3, 1)$

(12) If the curve of the function f intersects the curve of the function f^{-1} at the point $(k, 2k - 3)$, then $k =$ ....

(a) 2 (b) 3 (c) 4 (d) 5

(13) If the curve of the function f intersects the curve of the function f^{-1} at the point $\left(a, \frac{4}{a}\right)$, then $a =$ ....

(a) 2 (b) ± 2 (c) 4 (d) ± 4

(14) If f^{-1} is the inverse function of the function f, then ....

(a) domain of f^{-1} = domain of f (b) domain of f^{-1} = range of f

(c) range of f^{-1} = range of f (d) range of f^{-1} = domain of f^{-1}

(15) If $y = \sqrt[3]{x}$, then its inverse function is $y =$ ....

(a) $\frac{1}{3}x^3$ (b) x^3 (c) $x^3 - 1$ (d) $3x^3$

(16) If $f(x) = x^3 + 7$, then $f^{-1}(-1) =$ ....

(a) 1 (b) 2 (c) -2 (d) 8

(17) If the function $f : f(x) = \frac{1}{x - 2} + 2$, then $f^{-1}(x) =$ ....

(a) $\frac{1}{2 + x} - 2$ (b) $\frac{1}{2 - x} - 2$ (c) $f(x)$ (d) $f(-x)$

(18) If f is a function where $f(x) = 3 + \sqrt{x - 1}$, then the range of f^{-1} is ....

(a) $[3, \infty[$ (b) $]3, y[$ (c) $[1, \infty[$ (d) $]1, \infty[$

(19) If f, g are two functions where $f(x) = 4x - 12$, $g(x) = ax + 3$ and each one of them is the inverse function of the other, then $a =$ ....

(a) -4 (b) $\frac{1}{4}$ (c) 4 (d) 3

(20) If $f(\mathcal{X}) = 5\mathcal{X} + b$ is the inverse function of $g(\mathcal{X}) = c\mathcal{X} + 4$, then $b \times c =$ ....

(a) 4 (b) – 4 (c) 20 (d) – 20

(21) If $f^{-1}(\mathcal{X}) = 2\mathcal{X} + 1$, then $f(\mathcal{X}) =$ ....

(a) $-2\mathcal{X} - 1$ (b) $\frac{1}{2}\mathcal{X} + 1$ (c) $\frac{1}{2}\mathcal{X} - \frac{1}{2}$ (d) $\frac{1}{2\mathcal{X}} + 1$

(22) If $f : \mathcal{X} \longrightarrow y$, and $f(\mathcal{X}) = \frac{\mathcal{X} + k}{\mathcal{X} - 1}$ and $(5, 2) \in f^{-1}$, then k = ...

(a) zero (b) 1 (c) 2 (d) 3

(23) If $f(\mathcal{X}) = \frac{1}{\mathcal{X} - a} + b$, then $f^{-1}(a) + f(b) =$ ....

(a) undefined. (b) a – b (c) a + b (d) zero.

(24) If $3^{f(\mathcal{X})} = 2\mathcal{X} - 1$, then $f^{-1}(0) =$ ....

(a) 1 (b) – 1 (c) 2 (d) 5

(25) $(f^{-1} \circ f)(\mathcal{X}) =$ ....

(a) $\mathcal{X}$ (b) $\frac{1}{f(\mathcal{X})}$ (c) $f(\mathcal{X})$ (d) $f^{-1}(\mathcal{X})$

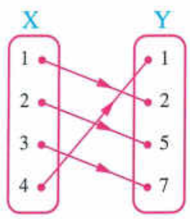
(26) If $f : \mathbb{R} - \{2\} \longrightarrow \mathbb{R} - \{1\}$ where $f(\mathcal{X}) = \frac{\mathcal{X} + 1}{\mathcal{X} - 2}$, then $f^{-1}(4) =$ ....

(a) 2 (b) 3 (c) 4 (d) 5

(27) If $f : f(\mathcal{X}) = \frac{2\mathcal{X} + 3}{3\mathcal{X} + 5}$, then the domain of the inverse function f^{-1} is ....

(a) $\mathbb{R}$ (b) $\mathbb{R} - \left\{\frac{-5}{3}\right\}$ (c) $\mathbb{R} - \left\{\frac{2}{3}\right\}$ (d) $\mathbb{R} - \left\{\frac{-5}{3}, \frac{2}{3}\right\}$

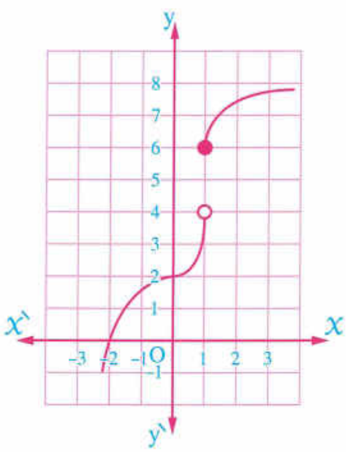
(28) The opposite figure represents the function $f : X \longrightarrow Y$, then $f^{-1}(2) =$ ....



(a) 1 (b) 5

(c) 4 (d) 7

(29) The opposite figure represents the curve of the function f , then $f^{-1}(0) + f^{-1}(6) =$ ....



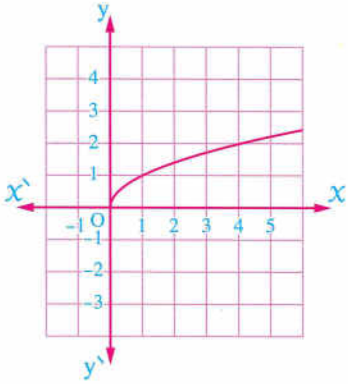
(a) – 1

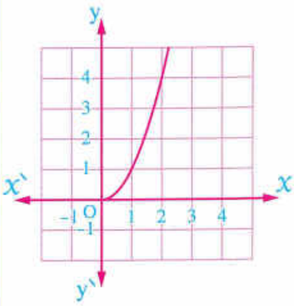
(b) 3

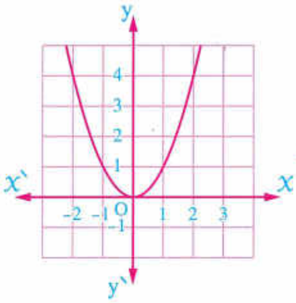
(c) 5

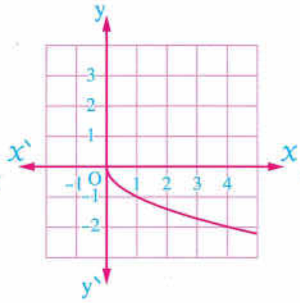
(d) 8

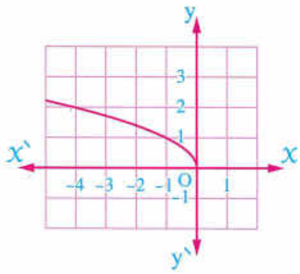
(30) If the opposite figure represent the function $f : f(X) = \sqrt{X}$, $X \geq 0$, then which of the following figures represent the function f^{-1} ?




(a)


(b)


(c)


(d)

Second Essay questions

1 **Find the inverse function for each of the following :**

(1)

X	−2	1	2	5
$f(X)$	7	4	1	−1

(2) $f = \{(1, 2), (2, 3), (3, 4)\}$

(3) $f(X) = 2X + 5$

(4) $f(X) = \frac{1}{2}X + 4$

(5) $f(X) = 5 + \frac{4}{X}$

(6) $f(X) = 8X^3 - 1$

(7) $f(X) = \sqrt[3]{X + 1}$

(8) $f(X) = \sqrt[3]{4 - X}$

(9) $f(X) = 2 + \sqrt{3 - X}$

(10) $f(X) = X^2$ where $X \geq 0$

(11) $f(X) = (X + 2)^2$ where $X \leq -2$

(12) $f(X) = (X - 1)^2 + 2$ where $X \geq 1$

(13) $f(X) = X^2 + 8X + 7$ where $X \geq -4$

(14) $f(X) = \sqrt{9 - X^2}$ where $-3 \leq X \leq 0$

(15) $f(X) = \sqrt{9 - X^2}$ where $0 \leq X \leq 3$

(16) $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ where $f(X) = \frac{1}{X^2 + 2}$

2 **Which of the following functions has an inverse function :**

(1) $f_1 = \{(1 , 2) , (2 , 4) , (3 , 5)\}$

(2) $f_2 = \{(-1 , 3) , (0 , 2) , (1 , 3)\}$

(3) $f_3(x) = x^2 - 1$ where $x \in \mathbb{R}$

(4) $f_4(x) = |x|$ where $x \in \mathbb{R}$

3 **Determine whether each of the two functions f and g is inverse function to the other or not in each of the following :**

(1) $f(x) = 2x - 3$, $g(x) = \frac{x+3}{2}$

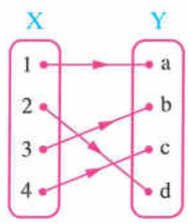
(2) $f(x) = x^2 + 4$ where $x \geq 0$, $g(x) = \sqrt{x-4}$

(3) $f(x) = \frac{-2}{x-5}$, $g(x) = \frac{5x-2}{x}$

(4) $f(x) = \sqrt[3]{4x}$, $g(x) = \frac{x^3}{4}$

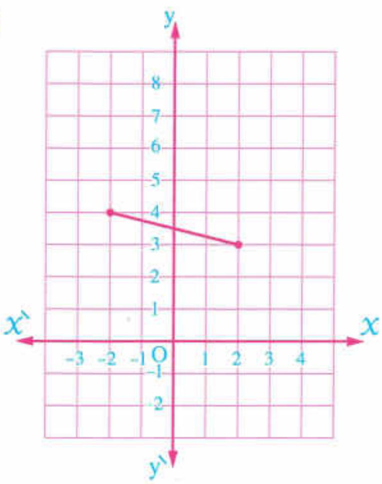
4 (1) If $f(x) = 5x$, find $f^{-1}(x)$ and represent it graphically.

(2) The opposite figure represents the function f from X to Y , then find the value of : $f^{-1}(b) + 2f^{-1}(c)$

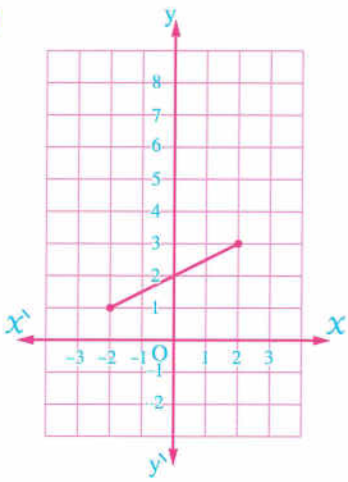


5 **In each of the following figures draw in the same figure the curve of the inverse function f^{-1} :**

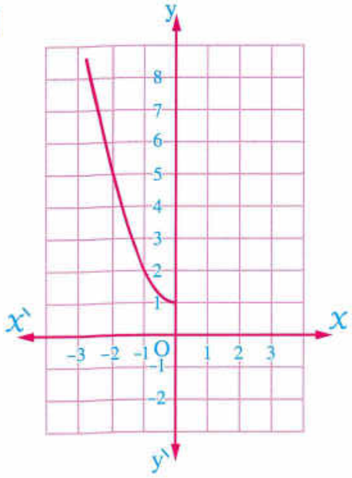
(1)



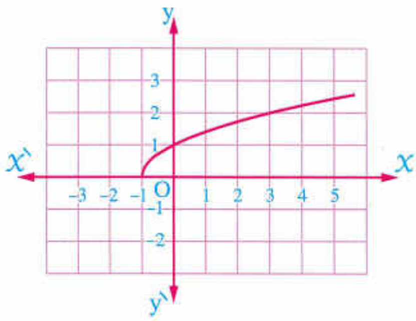
(2)



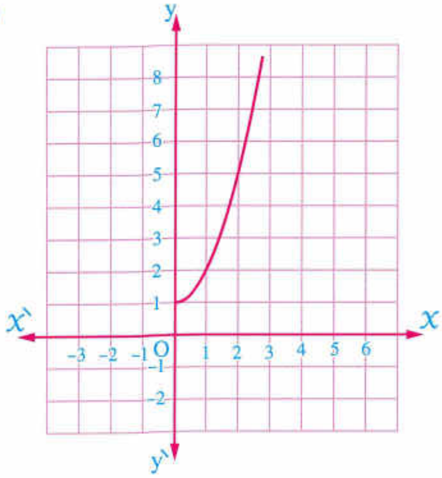
(3)



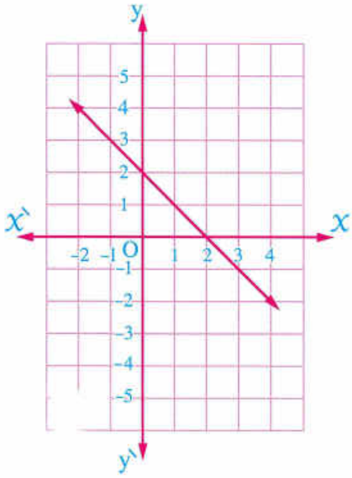
(4)



(5)



(6)



6 **Which of the functions defined by the following rules its inverse is the function itself :**

(1) $f(X) = 2X$

(2) $f(X) = -X$

(3) $f(X) = \frac{2}{X}$

(4) $f(X) = 7 - X$

(5) $f(X) = \frac{1}{X-3} + 5$

(6) $f(X) = \frac{1}{X-k} + k$ where $k \in \mathbb{R}$

7 Discover the error :

Wael and Rana tried to find the inverse function of the function $f : f(x) = \frac{x-5}{x}$

Wael's answer

$\because f(x) = \frac{x-5}{x}$

$, \because f^{-1}(x) = \frac{1}{f(x)}$

$\therefore f^{-1}(x) = 1 \div \frac{x-5}{x} = 1 \times \frac{x}{x-5}$

$\therefore f^{-1}(x) = \frac{x}{x-5}$

Rana's answer

$\because y = \frac{x-5}{x}$

$\therefore x = \frac{y-5}{y}$ «exchanging the variables»

$\therefore y\,x = y - 5$ «cross multiplication»

$\therefore y\,x - y = -5$ $\quad$ $\therefore y(x-1) = -5$

$\therefore f^{-1}(x) = \frac{-5}{x-1}$

Which answer is correct ? Why ?

8 In each of the following , determine the domain in which the function f has an inverse function :

(1) $f(x) = x^2$ $\quad$ (2) $f(x) = x^3$ $\quad$ (3) $f(x) = \frac{1}{2}x$

9 Find the inverse function of the function f where $f(x) = \frac{x-1}{x+5}$

10 If each of the two functions f , g where $f(x) = 2x + a$, $g(x) = bx + 3$ is an inverse function to the other , then find the value of each of a and b

«-6 , $\frac{1}{2}$»

Third Higher skills

Choose the correct answer from the given ones :

(1) If $f(x) = ax + b$, $f^{-1}(9) = 3$, $f^{-1}(5) = 2$, then $a \times b =$ ....

(a) -12 $\quad$ (b) -10 $\quad$ (c) -8 $\quad$ (d) -7

(2) If $f(x) = x^2$, $g(x) = x - 3$, then the solution set of the equation $g\left(f(x)\right) = g^{-1}(x)$ is ....

(a) $\{2 , -3\}$ $\quad$ (b) $\{3\}$ $\quad$ (c) $\{3 , -2\}$ $\quad$ (d) $\{2 , 3\}$

(3) If $f : \mathbb{R} - \{-1\} \longrightarrow \mathbb{R} - \{2\}$ where $x = \frac{f(x)+1}{2-f(x)}$, then $f^{-1}(3) =$ ....

(a) -4 $\quad$ (b) -3 $\quad$ (c) -2 $\quad$ (d) -1

(4) If $f : \mathbb{R} - \{1\} \longrightarrow \mathbb{R} - \{3\}$ where $f(x) = \frac{a\,x + 3}{x - b}$, then $f^{-1}(1) = \cdots\cdots$

(a) -2 (b) zero (c) 2 (d) undefined

(5) If $f : \mathbb{R} \longrightarrow \mathbb{R}$ where $f(x) = \sqrt[3]{x - 5}$, $g : \mathbb{R} \longrightarrow \mathbb{R}$ where $g(x) = 2x - 7$,

then $(g \circ f^{-1})(x) = \cdots\cdots$

(a) $2x^3 - 3$ (b) $2x^3 - 5$ (c) $2x^3 + 3$ (d) $2x^3 + 5$

(6) If $f : \mathbb{R} \longrightarrow \mathbb{R}$ where $f(x) = 3x - 4$, then $f^{-1}(x + 2) = \cdots\cdots$

(a) $\frac{x-2}{3}$ (b) $\frac{x+2}{3}$ (c) $\frac{x+4}{3}$ (d) $\frac{x+6}{3}$

(7) If $f(x) = x^3 + 3x^2 + 3x + 1$, then $f^{-1}(x) = \cdots\cdots$

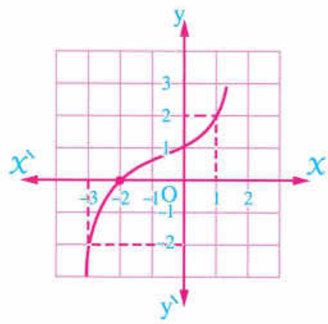
(a) $(x+1)^3$ (b) $\sqrt[3]{x} - 1$ (c) $\sqrt[3]{x} + 1$ (d) $x^3 - 1$

(8) The opposite figure represents the curve of the function $f^{-1}(x)$,

then $(f \circ f)(\text{zero}) = \cdots\cdots$

(a) -3 (b) -2

(c) zero (d) 1





Exercise 10 Logarithmic function and its graph

Test yourself

From the school book · Understand · Apply · Higher Order Thinking Skills

First Multiple choice questions

Choose the correct answer from those given :

(1) The form $\log_a \mathcal{X} = y$ is equivalent to ....

(a) $\log_a y = \mathcal{X}$ (b) $a^y = \mathcal{X}$ (c) $a^{\mathcal{X}} = y$ (d) $y = a\,\mathcal{X}$

(2) $\text{Log}_{\frac{5}{2}} \frac{16}{625} =$ ....

(a) – 2 (b) – 4 (c) 3 (d) 5

(3) $\text{Log}_3 \log_2 8 =$ ....

(a) – 2 (b) – 1 (c) 1 (d) 4

(4) If $\log 0.01 = 3\mathcal{X} + 1$, then $\mathcal{X} =$ ....

(a) – 3 (b) – 1 (c) 2 (d) 7

(5) If $\log_3 \mathcal{X} = 2$, then $\mathcal{X} =$ ....

(a) 3 (b) 5 (c) 8 (d) 9

(6) If $\log_{\frac{1}{3}} \mathcal{X} = -1$, then $\mathcal{X} =$ ....

(a) – 3 (b) – 1 (c) 1 (d) 3

(7) If $\log_2 \mathcal{X} = \log_3 9$, then $\mathcal{X} =$ ....

(a) 1 (b) 2 (c) 3 (d) 4

(8) If $\log_5 \mathcal{X} = 2$, then $\log (40\,\mathcal{X}) =$ ....

(a) 3 (b) 25 (c) 100 (d) 1000

(9) If $\text{Log}_5 \mathcal{X} = 3$, then $\log_5 \frac{\mathcal{X}}{5} =$ ....

(a) 2 (b) 3 (c) 25 (d) 125

(10) If $\log (\mathcal{X} + 11) = 2$, then $\mathcal{X} =$ ....

(a) -9 (b) 22 (c) 89 (d) 91

(11) If $\log_6 \sqrt{\mathcal{X} + 4} = \frac{1}{2}$, then $\mathcal{X} =$ ....

(a) 2 (b) 4 (c) 6 (d) 8

(12) The S.S. of the equation $\log_{\mathcal{X}} 81 = 4$ in $\mathbb{R}$ is ....

(a) $\{-3\}$ (b) $\{3\}$ (c) $\{3 , -3\}$ (d) $\{9\}$

(13) The solution set of the equation : $\log_{\mathcal{X}} 3 = -2$ in $\mathbb{R}$ is ....

(a) $\left\{\frac{1}{9}\right\}$ (b) $\{9\}$ (c) $\left\{\sqrt{3}\right\}$ (d) $\left\{\frac{1}{\sqrt{3}}\right\}$

(14) If $\text{Log}_{\mathcal{X}} 25 = 2$, then $\mathcal{X}^3 + \mathcal{X}^2 - \mathcal{X} =$ ....

(a) 95 (b) 105 (c) 145 (d) 155

(15) If $\log_3 (2\mathcal{X} + 3) = 2$, then $\mathcal{X} =$ ....

(a) 3 (b) 2 (c) 9 (d) 4

(16) The solution set of the equation $\log (\mathcal{X} - 1) =$ zero is ....

(a) $\left\{\frac{1}{10}\right\}$ (b) $\{1\}$ (c) $\{2\}$ (d) $\{-1\}$

(17) The S.S. of the equation $\log_{\mathcal{X}} (3\mathcal{X} - 2) = 2$ in $\mathbb{R}$ is ....

(a) $\{1 , 2\}$ (b) $\{1\}$ (c) $\{2\}$ (d) $\varnothing$

(18) If $\log_{\mathcal{X}} (\mathcal{X} + 6) = 2$, then $\mathcal{X} \in$ ....

(a) $\{3 , -2\}$ (b) $\{3\}$ (c) $\{3 , 1\}$ (d) $\{6 , 1\}$

(19) If the solution set of the equation : $\log_{\mathcal{X}} 64\mathcal{X} = 4$ in $\mathbb{R}$ is ....

(a) $\{2\}$ (b) $\{4\}$ (c) $\{6\}$ (d) $\{0 , 4\}$

(20) If $\log_{|\mathcal{X}+2|} 64 = 3$, then $\mathcal{X} \in$ ....

(a) $\{6 , -2\}$ (b) $\{2 , -6\}$ (c) $\{0 , -8\}$ (d) $\{4 , -8\}$

(21) If $\log_3 (4 + \log_2 \mathcal{X}) = 2$, then $\mathcal{X} =$ ....

(a) 16 (b) 32 (c) 64 (d) 128

(22) If $\log_2 \log_5 \log_2 \mathcal{X} =$ zero , then $\mathcal{X} =$ ....

(a) 4 (b) 8 (c) 16 (d) 32

(23) If $f(\mathcal{X}) = 2^{\mathcal{X}}$, then $\log_2 f(\mathcal{X}) =$ ....

(a) 1 (b) $\mathcal{X}$ (c) $f(\mathcal{X})$ (d) 2

(24) If $f(\mathcal{X}) = 3^{\mathcal{X}}$, then $f^{-1}(\mathcal{X}) =$ ....

(a) $\log_{\mathcal{X}} 3$ (b) $\log_3 \mathcal{X}$ (c) $\mathcal{X}^{-3}$ (d) $3^{\mathcal{X}}$

(25) $f(\mathcal{X}) = \log_3 (\mathcal{X} + 4)$, then $f^{-1}(2) =$ ....

(a) 6 (b) 5 (c) $\log_3 6$ (d) $\log_6 3$

(26) If $f(\mathcal{X}) = \log_2 (\mathcal{X} + k)$ and $f^{-1}(3) = 1$, then k = ...

(a) 4 (b) 5 (c) 6 (d) 7

(27) The value of $\log_6 33 \approx$ .... (by using calculator)

(a) 1.95 (b) 0.0512 (c) 2.297 (d) 0.74

(28) The value of $\mathcal{X}$ where $\log \mathcal{X} = 0.35$ is .... (to the nearest thousandths)

(a) 3.534 (b) 2.839 (c) 2.239 (d) ± 2.239

(29) The curve of the function $f : f(\mathcal{X}) = \log_2 (\mathcal{X} + 1)$ intersects the $\mathcal{X}$-axis at the point ....

(a) (0 , 0) (b) (1 , 0) (c) (2 , 0) (d) (1 , 1)

(30) The curve of the function $f : f(\mathcal{X}) = \log_2 (3 - \mathcal{X})$ intersects the $\mathcal{X}$-axis at the point ....

(a) (1 , 0) (b) (2 , 0) (c) (0 , 1) (d) (3 , 0)

(31) The range of the function $f : f(\mathcal{X}) = \log_2 \mathcal{X}$ is ....

(a) $\mathbb{R}^+$ (b) $\mathbb{R}^-$ (c) $\mathbb{R}$ (d) $\mathbb{R}^*$

(32) The function $f : f(\mathcal{X}) = \log_a \mathcal{X}$ is decreasing for every a ∈ ....

(a)]0 , ∞[(b)]− ∞ , 0[(c)]0 , 1[(d)]1 , ∞[

(33) If the function $f : f(\mathcal{X}) = \log_{\frac{1}{2}} \mathcal{X}$, then $f\left(\frac{1}{4}\right) + f(8) =$ ....

(a) − 3 (b) − 1 (c) 2 (d) 5

(34) If $\log_{\frac{1}{2}} f(\mathcal{X}) = \mathcal{X}$, then $8 f(2) + f(-3) + f(0) =$ ....

(a) $\frac{1}{16}$ (b) $\frac{1}{8}$ (c) 11 (d) 22

(35) 📖 If the curve of the function $y = \log_4 (1 - a\mathcal{X})$ passes through the point $\left(\frac{1}{4} , -\frac{1}{2}\right)$, then a = ...

(a) 2 (b) 3 (c) 4 (d) 8

(36) 📖 If the curve of the function f where $f(\mathcal{X}) = \log_a \mathcal{X}$ passes through the point (8 , 3) , then $f(4) =$ ....

(a) 1 (b) 2 (c) $\frac{1}{4}$ (d) −2

(37) The domain of the function f where $f(X) = \log_{(1-X)} 3$ is ....

(a) $]-\infty, 0[\cup]0, 1[$ (b) $]-\infty, 1[$ (c) $]1, \infty[$ (d) $]-1, 1[$

(38) The domain of the function $f : f(X) = \log_{(1-X)} X$ is ....

(a) $X > 0$ (b) $X < 1$ (c) $0 < X < 1$ (d) $0 \leq X \leq 1$

(39) The domain of the function $f : f(X) = \log_X (5 - X)$ is ....

(a) $]0, 5[- \{1\}$ (b) $[0, 5]$ (c) $]0, 5[$ (d) $]-\infty, 5[$

(40) If $\log_a X = \log_a y$, then ....

(a) $y = X$ (b) $y = -X$ (c) $y = a^X$ (d) $X = a^y$

(41) If $\log_a X = \log_a y$, then $X = y$ because $f(X) = \log_a X$ is .... function.

(a) an odd (b) an even (c) an increasing (d) one - to - one

(42) If $f(X) = \log_a X$, $a \in]0, 1[$, then all the following statements are true except ....

(a) if $\log_a X = \log_a y$, then $X = y$

(b) the function f is increasing on its domain.

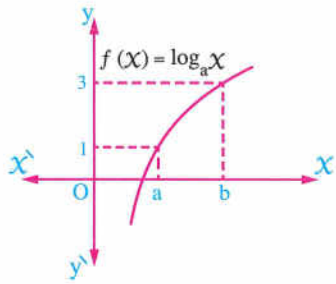
(c) if $\log_a X > \log_a y$, then $X < y$

(d) the function f is neither odd nor even.

(43) If $\log_a X < 0$ where $a > 1$, then $X \in$ ....

(a) $]0, 1[$ (b) $]1, \infty[$ (c) $]-\infty, 0[$ (d) $]0, \infty[$

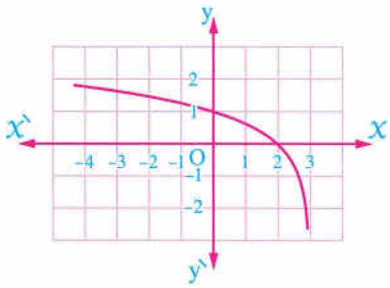
(44) The opposite figure shows the curve of the function $f : f(X) = \log_a X$, then $b =$ ....



(a) a^2 (b) $a + 3$

(c) a^3 (d) 3^a

(45) The opposite figure represents the function ....



(a) $y = 3^{X-1}$ (b) $y = 3^{X+1}$

(c) $y = \log_3 (2 - X)$ (d) $y = \log_3 (3 - X)$

Second Essay questions

1 Express each of the following in the equivalent exponential form :

(1) $\log_2 128 = 7$ (2) $\log_{\frac{2}{5}} \frac{4}{25} = 2$ (3) $\log_2 4\sqrt{2} = \frac{5}{2}$

2 Express each of the following in the equivalent logarithmic form :

(1) $5^0 = 1$ (2) $10^{-4} = 0.0001$ (3) $5^{-3} = \frac{1}{125}$

3 Find the value of each of the following :

(1) $\log_2 16$ « 4 »
(2) $\log_8 1$ « 0 »
(3) $\log 0.00001$ « –5 »
(4) $\log_{\frac{1}{2}} 128$ « –7 »
(5) $\log_2 \frac{1}{8}$ « –3 »
(6) $\log_{\sqrt{2}} 8\sqrt{2}$ « 7 »
(7) $\log_3 \sqrt[4]{27}$ « $\frac{3}{4}$ »
(8) $\log_2 \cos 45°$ « $-\frac{1}{2}$ »

4 Solve each of the following equations in $\mathbb{R}$:

(1) $\log_2 x = 7$ « 128 »
(2) $\log_5 x = -2$ « $\frac{1}{25}$ »
(3) $\log_3 x^2 = 4$ « ±9 »
(4) $\log_{81} x = \frac{3}{4}$ « 27 »
(5) $\log_2 x^{-1} = -5$ « 32 »
(6) $\log_2 (\log_3 x) = 1$ « 9 »
(7) $\log_3 (2x - 5) = 0$ « 3 »
(8) $\log_8 \sqrt[3]{x^2 + 48} = \frac{2}{3}$ « ±4 »
(9) $\log_3 \log_2 (x^2 - 2x) = 1$ « 4 or –2 »
(10) $\log_5 |2x + 1| = 1$ « 2 or – 3 »
(11) $\log_2 x(x + 6) = 4$ « 2 or – 8 »
(12) $\log_3 (x^2 - 2x) = 1$ « 3 or – 1 »
(13) $(\log_3 x)^2 - 9 \log_3 x + 20 = 0$ « 81 or 243 »
(14) $\log_2 (3^x - 3^{x-2}) = 3$ « 2 »
(15) $3^{\log_4 (x + 1.25)} = \frac{1}{3}$ « – 1 »
(16) $|\log_{10} x - 2| = 2$ « 10^4 or 1 »

5 Find in $\mathbb{R}$ the S.S. of each of the following equations :

(1) $\log_x 125 = 3$ «{5}»
(2) $\log_x 2 = 5$ «$\{\sqrt[5]{2}\}$»
(3) $\log_x 3 = -2$ «$\{\frac{1}{\sqrt{3}}\}$»
(4) $\log_x 0.001 = \frac{-3}{4}$ «{10000}»
(5) $\log_{-x} 81 = 4$ «{–3}»
(6) $\log_{x-1} 27 = 3$ «{4}»
(7) $\log_{x-1} (7 - x) = 2$ «{3}»
(8) $\log_{\frac{2}{x}} 64 = -3$ «{8}»
(9) $\log_x 5x = 2$ «{5}»
(10) $\log_x x^6 = 5x$ «$\{\frac{6}{5}\}$»
(11) $\log_9 \log_3 \log_x 27 = 0$ «{3}»
(12) $\log_x (x^2 - 12) = 1$ «{4}»
(13) $\log_x \left(\sqrt{x - 2} + 2\right) = 1$ «{2 , 3}»
(14) $(\log_3 x)^2 + 15 = 8 \log_3 x$ «{243 , 27}»
(15) $\log_{27} (x - 2) + \log_3 (x - 2) = 4$ «{29}»

6 **Find the value of X in each of the following :**

(1) $\log_4 8\sqrt{2} = X$ «$\frac{7}{4}$»

(2) $\log_{\sqrt{5}} 625\sqrt{5} = X^2$ «± 3»

(3) $\log_{0.3} 0.09 = X^{-2}$ «$\pm \frac{1}{\sqrt{2}}$»

(4) 📖 $\log_2 (2^X - 4) + X - 5 = 0$ «3»

(5) 📖 $\log_4 [13 + \log_2 (X - 1)] = 2$ «9»

(6) 📖 $\log_3 \frac{X^2}{2X - 3} = 1$ «3»

---

7 **Using the calculator , find the value of each of the following approximating to the nearest 4 decimals :**

(1) 📖 $\log 3.15$

(2) $\log_2 27$

(3) $4 \log 7 - 5 \log 13$

---

8 **Using the calculator , find the value of X in each of the following approximating to the nearest 4 decimals :**

(1) $\log X = 0.2345$

(2) $\log X = 1.412$

(3) $\log X = -0.3$

---

9 **Find in $\mathbb{R} \times \mathbb{R}$ the solution set for each pair of the following equations :**

(1) $X^y = 5X - 4$, $\log_X 16 = y$ «$\{(4 , 2)\}$»

(2) $\log_X \log_2 \log_X y = 0$, $\log_y 9 = 1$ «$\{(3 , 9)\}$»

---

10 If $\log_{16} 49 = a$, $\log_7 2.5 = b$, prove that : $4^{ab + 1} = 10$

---

11 **Determine the domain of each of the functions that are defined by the following rules :**

(1) 📖 $f(X) = \log_3 (2X + 1)$

(2) 📖 $f(X) = 2 \log X$

(3) $f(X) = \log_X X$

(4) $f(X) = \log_{X - 2} X$

(5) $f(X) = \log_{2 - X} X$

(6) 📖 $f(X) = \log_{(5 - X)} (X - 3)$

(7) $f(X) = \log_{X - 3} X^2$

(8) $f(X) = \log_{X + 2} (3X + 1)$

(9) $f(X) = \log_4 (3 - X)(X + 1)$

(10) $f(X) = \log_X (3 - X)(X + 1)$

---

12 **If the curve of the function $f : f(X) = \log_a X$ is passing through the point (81 , 4) , find the value of a , then graph the function f taking $X \in [\frac{1}{9} , 9]$, from the graph :**

(1) Deduce domain and range and monotonicity and the point of intersection with X-axis.

(2) Find approximated value to the number $\log_3 5$

13 If the curve of the function $f : f(X) = \log_a X$ is passing through $\left(\frac{1}{8}, 3\right)$, find the value of a, then graph the curve of the function f, taking $X \in [\frac{1}{4}, 4]$, from the graph deduce the range, monotonicity and the intersection point of the curve with the X-axis, then find approximated value to the number $\log_{\frac{1}{2}} 3.5$

14 **Use the curve of the function $f : f(X) = \log_{\frac{1}{2}} X$ to represent each of the functions that are defined by the following rules, from the graph determine domain, range and monotonicity of each function :**

(1) $g(X) = \log_{\frac{1}{2}} X + 2$ (2) $\ell(X) = \log_{\frac{1}{2}} (X + 1)$ (3) $h(X) = -\log_{\frac{1}{2}} X$

15 **Use the curve of the function $f : f(X) = \log_2 X$ to represent each of the functions that are defined by the following rules, from the graph determine the domain, range and monotonicity of each function :**

(1) $\ell(X) = \log_2 (X - 2)$

(2) $t(X) = \log_2 X + 1$

(3) $h(X) = -\log_2 X$

(4) $g(X) = \log_2 (-X)$

16 Graph in the same lattice each of the two functions g, f where $g(X) = \log_2 X$, $f(X) = 6 - X$, then use that to find the solution set of the equation $\log_2 X = 6 - X$ «{4}»

17 Graph in the same lattice each of the two functions g, f where $g(X) = \log_3 X$, $f(X) = 4 - X$, then use that to find the solution set of the equation $\log_3 X = 4 - X$ «{3}»

Third Higher skills

Choose the correct answer from the given ones :

(1) If $f(X) = \log_a (2X + 4)$ and $f^{-1}(5) = 14$, then a = ...

(a) 1 (b) 2 (c) 3 (d) 4

(2) If $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$, $f(X) = \log_4 X$ and $f^{-1}(a + 3) = 32$, then a = ...

(a) -4 (b) -2 (c) -1 (d) $-\frac{1}{2}$

(3) If $f(X) = \log_2 (X + 1)$, $g(X) = 5 + \log_3 (X - 1)$, then $(f \circ g)(10) =$ ....

(a) 3 (b) 4 (c) 5 (d) 6

(4) If $\log (X - 5) >$ zero, then ....

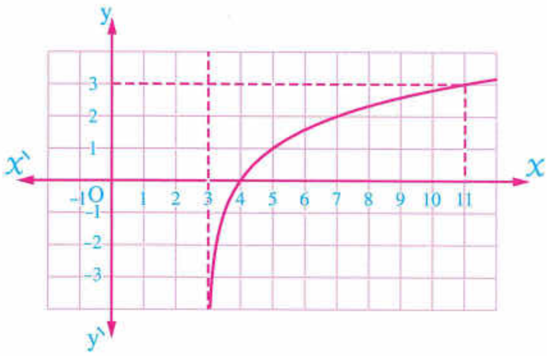
(a) $X > 5$ (b) $X > 6$ (c) $X > 1$ (d) $X < 5$

(5) The solution set of the equation : $\log_3 \log_2 \left| |x - 1| + 5 \right| = 1$ in $\mathbb{R}$ is ....

(a) $\{2 , 4\}$ (b) $\{2 , -2 , 4 , -4\}$

(c) $\{-2 , 4\}$ (d) $\{4\}$

(6) The opposite figure represents the curve of the function $f : f(x) = \log_a (x + b)$, then $f(7) + f^{-1}(2) =$ ....



(a) 9

(b) 8

(c) 7

(d) 6

(7) The domain of the function $f : f(x) = \log |x^2 - 9|$ is ....

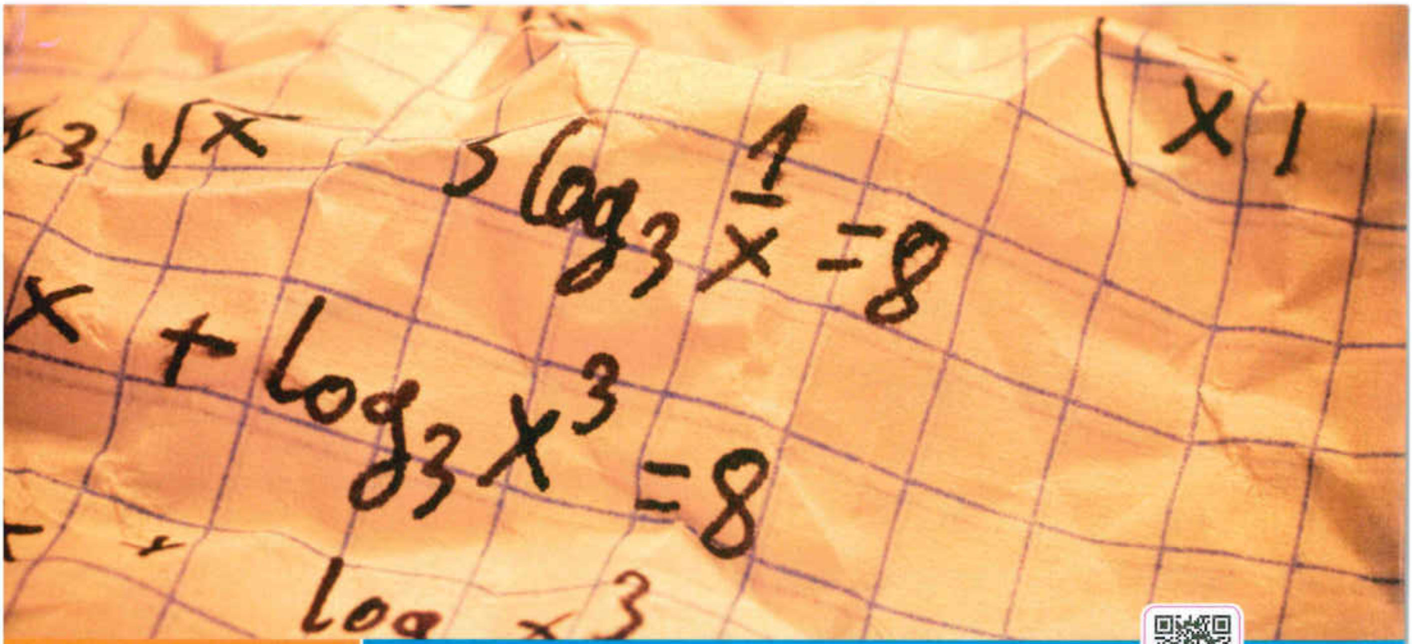
(a) $\mathbb{R}^*$ (b) $\mathbb{R} - \{3 , -3\}$

(c) $\mathbb{R} - [-3 , 3]$ (d) $]-3 , 3[$

(8) The domain of the function $f : f(x) = \dfrac{\log_2 (x + 3)}{x^2 + 3x + 2}$ is ....

(a) $\mathbb{R} - \{-2 , -1\}$ (b) $]-2 , \infty[$

(c) $\mathbb{R} - \{-2 , -1 , -3\}$ (d) $]-3 , \infty[- \{-1 , -2\}$



Exercise 11 Some properties of logarithms

Test yourself

From the school book • Understand • Apply • Higher Order Thinking Skills

First Multiple choice questions

Choose the correct answer from those given :

(1) $\log_2 5 \times \log_5 2 =$ ....

(a) 1 (b) 10 (c) $\log_2 10$ (d) $\log_5 10$

(2) $1 + \log 2 =$ ....

(a) log 5 (b) log 2 (c) log 20 (d) – log 5

(3) The value of the expression : $2 \log 25 + \log \frac{8}{15} + 2 \log 3 - \log 30 =$ ....

(a) 6 (b) 2 (c) 3 (d) – 1

(4) Which of the following statements is true ?

(a) $\log 2 - \log \sqrt{2} = \log \sqrt{2}$

(b) $\log_1 1 =$ zero

(c) $\log \frac{7}{5} = \frac{\log 7}{\log 5}$

(d) $\log 7 \div \log 2 = \log 5$

(5) If $\log X - \log 2 = \log 4$, then $X =$ ....

(a) 4 (b) 6 (c) 8 (d) 16

(6) If $\log X + \log 5 = 2$, then $X =$ ....

(a) 3 (b) 8 (c) 17 (d) 20

(7) $2 \log_5 3 + 3 \log_5 2 =$ ....

(a) $\log_5 6$ (b) $6 \log_5 6$ (c) $\log_5 72$ (d) $\log_5 36$

(8) $\log_b a \times \log_c b \times \log_d c \times \log_a d =$ ....

(a) zero (b) 1 (c) abcd (d) ad

(9) $\sqrt{3^{\log 4} \times 3^{\log 25} + 7^{\log_7 16}} =$ ....

(a) 9 (b) 25 (c) 5 (d) 16

(10) $\frac{1}{\log_2 30} + \frac{1}{\log_3 30} + \frac{1}{\log_5 30} =$ ....

(a) 1 (b) 2 (c) 5 (d) 30

(11) $2 \log_a X + \log_a y - \log_a (X y) =$ ....

(a) $\log X$ (b) $\log_a X$ (c) $\log_a X y$ (d) $\log_a X^2$

(12) The simplest form of the expression : $\log_b a^2 \times \log_c b^3 \times \log_a c =$ ....

(a) 2 (b) 3 (c) 6 (d) 1

(13) $\log \left(\frac{a^2}{bc}\right) =$ .... where a , b , c $\in \mathbb{R}^+$

(a) $2 \log a + \log b + \log c$ (b) $2 \log a - \log b + \log c$

(c) $2 \log a - \log b - \log c$ (d) $2 (\log a - \log b - \log c)$

(14) If $X - 2 = \log_2 3$, then $X =$ ....

(a) $\log_2 6$ (b) $\log_2 9$ (c) $\log_2 12$ (d) $\log_2 18$

(15) $\log_{ab} \frac{1}{a} + \log_{ab} \frac{1}{b} =$ ....

(a) $\frac{a}{b}$ (b) $\frac{b}{a}$ (c) -1 (d) 1

(16) The expression $\frac{3 \log 2}{\log 4 + \log 3}$ is equivalent to the expression ....

(a) $\log_3 2$ (b) $\log_7 2$ (c) $\log_{12} 8$ (d) $\log_7 8$

(17) If $3^X = 5$, then $X =$ ....

(a) 3 (b) $\log_3 5$ (c) $\log_5 3$ (d) $\frac{5}{3}$

(18) The solution set of the equation : $\log_2 (2^{X-4}) = 5 - X$ is ....

(a) $\{4\}$ (b) $\{4.5\}$ (c) $\{5\}$ (d) $\{5.4\}$

(19) The solution set of the equation : $\log_{\sqrt{2}} X + \log_{\sqrt{2}} (X + 1) = 2$ is ....

(a) $\{1 , 2\}$ (b) $\{-2\}$ (c) $\{1 , -2\}$ (d) $\{1\}$

(20) The solution set of the equation : $2 \log 2 - \log X = \log (X + 3) - \log 7$ is ....

(a) $\{7\}$ (b) $\{4\}$ (c) $\{7 , 4\}$ (d) $\varnothing$

(21) If $\log_b X + \log_b 3 = \log_b 27 - 1$, then the value of $X =$ .... in terms of b

(a) 9 b (b) $\frac{1}{9}$ b (c) $\frac{9}{b}$ (d) $\frac{1}{9 b}$

(22) If $\log_2 X = \log_4 9$, then $X =$ ....

(a) 3 (b) 4 (c) 9 (d) 12

(23) The solution set of the equation : $\log_2 X + \log_4 X = 3$ is ....

(a) $\{2\}$ (b) $\{4\}$ (c) $\{2 , 4\}$ (d) $\{0\}$

(24) If $\log_2 X + \log_2 X^2 = 6$, then $X =$ ....

(a) 2 (b) 4 (c) 6 (d) 216

(25) The solution set of the equation : $\log X^2 - (\log X)^2 = 0$ is ....

(a) $\{1\}$ (b) $\{1 , 10\}$ (c) $\{1 , 100\}$ (d) $\{100\}$

(26) The solution set of the equation : $(\log_3 X)^2 - \log_3 X^3 + 2 = 0$ is ....

(a) $\{3\}$ (b) $\{3 , 9\}$ (c) $\{9\}$ (d) $\{1 , 2\}$

(27) The solution set of the equation : $\log_3 X - 2 \log_X 3 = 1$ in $\mathbb{R}$ is ....

(a) $\{8 , 3\}$ (b) $\left\{8 , \frac{1}{3}\right\}$ (c) $\left\{9 , \frac{1}{3}\right\}$ (d) $\{8\}$

(28) If $X^{\log_X X^2} = \log_2 16$, then $X =$ ....

(a) 2 (b) -2 (c) ± 2 (d) 4

(29) Which of the following statements is true ?

(a) $\log 3 + \log 3 = \log 6$ (b) $1 - \log 5 = 2$

(c) $\log 2 \times \log 2 = \log 4$ (d) $\log (1 + 2 + 3) = \log 1 + \log 2 + \log 3$

(30) If $X = \sqrt{y\,z}$ where X , y , z are positive numbers , then log y = ...

(a) $\frac{\log X^2}{\log z}$ (b) $\frac{2 \log X}{\log z}$ (c) $2 \log X - \log z$ (d) $2 (\log X - \log z)$

(31) If log 23 = a , then log 2300 = ...

(a) a + 2 (b) a − 2 (c) 100 a (d) a^2

(32) If $\log_3 5 = a$, then $\log_{15} 5 =$ ....

(a) a^2 (b) 3 a (c) $\frac{1}{a + 3}$ (d) $\frac{a}{a + 1}$

(33) If $\log 3 = X$, log 4 = y , then log 12 = ...

(a) $X + y$ (b) $X y$ (c) $X - y$ (d) $\log X + \log y$

(34) ABC is a right-angled triangle at A in which AB = $(\log_4 3)$ cm. , AC = $(\log_3 64)$ cm. , then its area = ... cm^2

(a) 1.5 (b) 3 (c) log 16 (d) $\log_3 16$

(35) If $\frac{\log x}{\log 5} = \frac{\log 36}{\log 6} = \frac{\log 64}{\log y}$, then $x + y =$ ....

(a) 25 (b) 8 (c) 17 (d) 33

(36) If L and M are the two roots of the equation : $3x^2 - 16x + 12 = 0$

, then $\log_2 L + \log_2 M =$ ....

(a) 2 (b) 4 (c) 12 (d) 16

(37) If L and M are the roots of the equation : $x^2 + x \log_3 a = 9$ and $\frac{L + M}{LM} = \frac{1}{3}$,

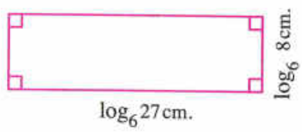
then a = ...

(a) 12 (b) 24 (c) 27 (d) 729

(38) The solution set of the equation : $\log_b (x + 5) = \log_b x + \log_b 5$ in $\mathbb{R}$ is ....

(a) $\{5\}$ (b) $\{4\}$ (c) $\left\{\frac{4}{5}\right\}$ (d) $\left\{\frac{5}{4}\right\}$

(39) **In the opposite figure :**



The perimeter of the figure = ... cm.

(a) $2 \log_6 35$ (b) $\log_6 70$

(c) 3 (d) 6

(40) 📖 $\log (\cos \theta) + \log (\sec \theta) =$ .... , $\theta \in \left[0 , \frac{\pi}{2}\right[$

(a) 1 (b) 0 (c) 2 (d) – 1

Second Essay questions

1 **Without using the calculator , find the value of each of the following :**

(1) 📖 $\log 2 + \log 5$ « 1 »

(2) 📖 $\log_5 15 - \log_5 3$ « 1 »

(3) 📖 $\log_2 5 \times \log_5 2$ « 1 »

(4) $\log_3 81 \times \log_9 3$ « 2 »

(5) $\log_2 15 + \log_2 14 - \log_2 105$ « 1 »

(6) 📖 $\log 54 - 3 \log 3 - \log 2$ « 0 »

(7) $\log_4 \log_2 \log_2 4$ « 0 »

(8) 📖 $1 + \log 3 - \log 2 - \log 15$ « 0 »

(9) 📖 $\log 0.009 - \log \frac{27}{16} + \log 15 \frac{5}{8} - \log \frac{1}{12}$ « 0 »

(10) $\log_{100} \frac{7}{170} - \log_{100} \frac{35}{18} + \log_{100} \frac{17}{36}$ « –1 »

(11) 📖 $2 \log 25 + \log \left(\frac{1}{3} + \frac{1}{5}\right) + 2 \log 3 - \log 30$ « 2 »

(12) $\log 25 + \frac{\log 8 \times \log 16}{\log 64}$ « 2 »

(13) $\frac{\log_2 25 + \log_2 4}{\log_2 30 - \log_2 3}$ « 2 »

(14) $\frac{(\log 3)^2 - \log 3^2}{\log 3 - \log 100}$ « log 3 »

(15) $\log_5 \frac{1 - \log_3 2}{\log_3 \left(\frac{3}{2}\right)^5}$ « – 1 »

(16) $\log_3 (\log x^{18}) - \log_3 (\log x^2)$ « 2 »

(17) $\log_{abc} a + \log_{abc} b + \log_{abc} c$ « 1 »

(18) $\frac{1}{2} \log_3 a + \frac{1}{2} \log_3 b + 2 \log_3 c - \log_3 \sqrt{ab} - \log_3 3c^2$ « – 1 »

(19) $\frac{1}{\log_2 12} + \frac{1}{\log_8 12} + \frac{1}{\log_9 12}$ « 2 »

2 **Without using the calculator , prove each of the following :**

(1) $2 \log_3 15 + \log_3 \frac{7}{3} - \log_3 175 = 2 \log_3 \sqrt{3}$

(2) $\log_2 \frac{3}{11} + \log_2 \frac{297}{98} - \left(2 \log_2 2 \times \log_2 \frac{9}{7}\right) = -1$

(3) $(1 - \log 5)(2 - \log 25) = 2 (\log 2)^2$

(4) $\frac{\log 729 - \log 64}{\log 9 - \log 4} = 3$

(5) $\frac{(\log 5)^2 - \log 5^2}{\log 5 - \log 100} = 1 - \log 2$

(6) $\frac{\log_2 \log_5 \log_3 243}{\log_7 \log_3 \log_2 512} = 0$

3 **Using the calculator , find the value of x in each of the following , approximating to two decimals :**

(1) $5^x = 13$ « 1.59 »

(2) $3^{x+2} = 6$ « – 0.37 »

(3) $3^{7-2x} = 13.4$ « 2.32 »

(4) $7^{x+1} = 3^{x-2}$ « – 4.89 »

(5) $3^{2x-3} = 11^{1-x}$ « 1.24 »

(6) $x^{1.6} = 94.5$ « ± 17.17 »

(7) $\frac{5}{(10)^{2x}} = 7$ « – 0.07 »

(8) $7^{x+1} + 7^{x-1} = 300$ « 1.92 »

(9) $2^x \times 3 = 2 \times 3^{x+2}$ « – 4.42 »

(10) $3 \times 2^{x-11} = 8 \times 6^{5x-1}$ « – 0.82 »

(11) $25^x - 27 \times 5^x + 50 = 0$ « 2 or 0.43 »

(12) $8^{3-5x} \div 7^{4+x} = 2$ « – 0.18 »

(13) $\log_5 (4^{x+1}) = x - 2$ « 20.64 »

4 **If $\log_3 5 \simeq 1.465$, then find without using the calculator :**

(1) $\log_3 15$

(2) $\log_3 135$

(3) $\log_3 \frac{5}{9}$

5 **If $\log 2 = x$, $\log 3 = y$, then find in terms of x , y each of :**

(1) $\log 6$

(2) $\log_{18} 12$

6 **Discuss the true of each of the following and correct the false one [where x , $y \in \mathbb{R}^+$, a , b $\in \mathbb{R}^+ - \{1\}$ in the questions (3) to (7)]**

(1) $\log_x x^x = x$ for every $x \in \mathbb{R}$

(2) $\log x^n = n \log x$ for every $x \in \mathbb{R}$

(3) $\log_a (x + y) = \log_a x + \log_a y$

(4) $\log_a (x + y) = \log_a x \times \log_a y$

(5) $\log_a (x\,y) = \log_a x + \log_a y$

(6) $\log_a \frac{x}{y} = \log_a x + \log_a y^{-1}$

(7) $\frac{\log_a x}{\log_a y} = \frac{\log_b x}{\log_b y}$

(8) If $x < 0$, then $\log_a x^4 = 4 \log_a x$, $a \in \mathbb{R}^+ - \{1\}$

7 **Find in $\mathbb{R}$ the solution set of each of the following equations :**

(1) $\log_3 (x + 6) = 2 \log_3 x$ « $\{3\}$ »

(2) $\log_4 x + \log_4 (x + 6) = 3$ « $\{-3 + \sqrt{73}\}$ »

(3) $\log_3 x + \log_3 x^2 = 3$ « $\{3\}$ »

(4) $\log_3 (x - 1) + \log_3 (x + 1) = \log_3 8$ « $\{3\}$ »

(5) $\log (x + 8) - \log (x - 1) = 1$ « $\{2\}$ »

(6) $\log_4 x = 1 - \log_4 (x - 3)$ « $\{4\}$ »

(7) $\log_5 x^2 + \log_5 2 = \log_5 18$ « $\{3 , -3\}$ »

(8) $\log_3 (7\,x^2 - 4) = 2 \log_3 x + \frac{1}{2} \log_3 9$ « $\{1\}$ »

(9) $5 \log_3 (x + 2) - \log_3 (x - 1)^5 = \log_3 32$ « $\{4\}$ »

(10) $\log (8 - x) + 2 \log \sqrt{x - 6} = 0$ « $\{7\}$ »

(11) $\log (x + 2) + \log (x - 2) = 1 - \log 2$ « $\{3\}$ »

(12) $\log 7 \times \log 729 = \log 49 \times \log x^3$ « $\{3\}$ »

(13) $\log_2 (x^2 + 6\,x + 9) - \log_2 (x - 1) = \log_5 625$ « $\{5\}$ »

(14) $\log x = \frac{(\log 3)^2 - \log 27}{\log 0.003}$ « $\{3\}$ »

(15) $(\log x)^2 - \log x^2 = 3$ « $\{0.1 , 1000\}$ »

(16) $(\log x)^2 + \log x^2 + 1 = (\log 5)^2$ « $\{\frac{1}{2} , \frac{1}{50}\}$ »

(17) $2 \log_x 2 + \log_x 14 - 3 \log_x 3 = 3 + \log_x 7$ « $\{\frac{2}{3}\}$ »

(18) $\log \sqrt[3]{3\,x - 1} + \log \sqrt[3]{x - 2} = \log 20 - 1$ « $\{3\}$ »

8 **Find in $\mathbb{R}$ the solution set of each of the following equations :**

(1) $\log_2 x = \log_4 9$ « $\{3\}$ »

(2) $3^{\log x} = 2^{\log 3}$ « $\{2\}$ »

(3) $x^{\log x} = 10$ « $\{10, 0.1\}$ »

(4) $\log_3 x = \log_x 3$ « $\{3, \frac{1}{3}\}$ »

(5) $x^{\log_x x^2} = \log 10^9$ « $\{3\}$ »

(6) $2^{2\log x} \times 5^{\log x} = 8000$ « $\{1000\}$ »

(7) $2^{(\log x)^2} \times 64 = 2^{\log x^5}$ « $\{100, 1000\}$ »

(8) $\log x - \frac{1}{\log x} = \frac{3}{2}$ « $\{\frac{1}{\sqrt{10}}, 100\}$ »

(9) $\log_2 x + \log_x 2 = 2$ « $\{2\}$ »

(10) $\log x - \log_x 100 = 1$ « $\{\frac{1}{10}, 100\}$ »

(11) $\log \frac{x}{2} \times \log \frac{2}{x} = -1$ « $\{20, 0.2\}$ »

(12) $(\log x)^3 = \log x^9$ « $\{1, 1000, 0.001\}$ »

(13) $x^{\log x} = 100\,x$ « $\{100, \frac{1}{10}\}$ »

(14) $(\log x + 2)\left(\log \frac{x}{100}\right) = 5$ « $\{1000, 0.001\}$ »

(15) $\log_3 x + \log_9 x^2 + 3 = 0$ « $\{\frac{1}{3\sqrt{3}}\}$ »

(16) $\log_2 x = \log_4 (x + 6)$ « $\{3\}$ »

(17) $\log_2 x + \log_4 x = \frac{-3}{2}$ « $\{\frac{1}{2}\}$ »

(18) $\sqrt{\log_2 x} = \log_2 \sqrt{x}$ « $\{1, 16\}$ »

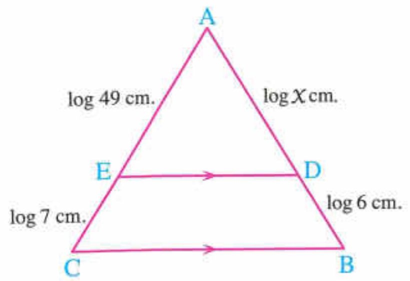
(19) $\sqrt{x^{\log \sqrt{x}}} = 10$ « $\{100, \frac{1}{100}\}$ »

(20) $\sqrt{\log\left(91 + 3^{\sqrt{\frac{x}{2}}}\right)} = \log 10^{\sqrt{2}}$ « $\{8\}$ »

(21) $\log \frac{1}{2^x + x - 1} = x(\log 5 - 1)$ « $\{1\}$ »

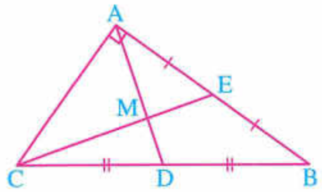
9 **In each of the following figures, find the required below the figure in the simplest form :**

(1)



If $\overline{DE} \mathbin{/\mkern-6mu/} \overline{BC}$, find the value of X

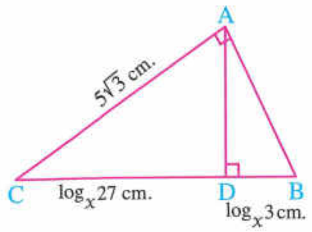
(2)



If $MD = \log_{81} 243$ cm.

, find the length of $\overline{BC}$

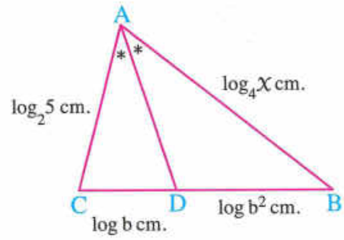
(3)



If $\overline{AB} \perp \overline{AC}$, $\overline{AD} \perp \overline{BC}$

, find the value of X

(4)



If $\overrightarrow{AD}$ bisects $\angle A$

, find the value of X

10 **Find the S.S. of each pair of the following equations in $\mathbb{R} \times \mathbb{R}$:**

(1) $\log X y = 1 - \log 5$, $\log y^3 = 3 - \log 125$ « $\{(1, 2)\}$ »

(2) $\log_2 X + \log_2 y + \log_2 4 = 2 + \log_2 9$, $X + y = 10$ « $\{(1, 9), (9, 1)\}$ »

(3) $\log X^3 + \log y^2 = 11$, $\log X^2 - \log y^3 = 3$ « $\{(1000, 10)\}$ »

11 **Answer the following :**

(1) If $X y = 9\sqrt{3}$, then prove that : $5 \log_3 X + 4 \log_3 y - \log_3 X^3 y^2 = 5$

(2) If $\frac{1}{3} \log a = \frac{1}{5} \log b = \log c$, then prove that : $c^8 = a\, b$

(3) If $\log \frac{X + y}{3} = \frac{1}{2} (\log X + \log y)$, then prove that : $\frac{X}{y} + \frac{y}{X} = 7$

(4) If $3^{\log_a X} = a$, then prove that : $\log_a X$ is the multiplicative inverse of the number $\log_a 3$, then find the value of : $\log_9 X$ « 2 »

(5) Prove that : $\log_{a^2} b^2 = \log_a b$ and prove that : $\log_a b^2 + \log_{a^2} b^4 = 4 \log_a b$

(6) If $3 \log X + 4 \log y - \log X y^2 = 2 (\log 2 + \log 3)$, then prove that : $X = \frac{6}{y}$

(7) 📖 If $X^2 + y^2 = 8 X y$, then prove that : $2 \log (X + y) = 1 + \log X + \log y$

(8) 📖 If $\log (X + y) = \frac{1}{2} (\log X + \log y) + \log 2$, then prove that : $X = y$

(9) If $\log x y^3 = 1$, $\log x^2 y = 1$, then find the value of $\log x y$ « $\frac{3}{5}$ »

(10) Prove that : $\log_y x = \frac{1}{\log_x y}$, then find the solution set of the equation :

$\log_3 x + 3 \log_x 3 = 4$ « $\{3 , 27\}$ »

(11) If x , y , z are positive integers $\frac{4}{\log_x y} + \frac{7}{\log_z y} = 3 \log_y z$, find the value of x z « 1 »

(12) If $y = a^{\log_a x}$, then prove that : $x = y$ and hence find the value of : $3^{\frac{1}{2}\log_3 49}$ « 7 »

(13) Prove that : $\log_{a^y} a^x = \frac{x}{y}$ and hence find the value of : $\log_{\sqrt[3]{5}} 125 \times \log_8 \sqrt{2}$ « $\frac{3}{2}$ »

(14) Prove that : $\log_b a = \frac{\log a}{\log b}$ and from this , prove that : $(\log_a ab)^{-1} + (\log_b ab)^{-1} = 1$

(15) If $\log_c a + \log_c b - 2 \log_c \frac{a+b}{2} = 0$, prove that : $a - b = 0$

(16) If $\left(x^{\log x}\right)^2 \times x^3 = 10^5$, find the value of : x « $\frac{1}{100\sqrt{10}}$ or 10 »

(17) Putting $a^x = y$ in the equation : $a^x - a^{-x} = 1$, evaluate y and

show that : $x = \log_a \frac{1+\sqrt{5}}{2}$ « $\frac{1 \pm \sqrt{5}}{2}$ »

(18) If $f(x) = \log x^2$, find the solution set of the equation :

$f(x+1) + f(x-1) = f(3)$ « $\{2 , -2\}$ »

(19) Discuss whether the function $f : f(x) = \log\left(\sqrt{x^2+1} - x\right)$ is even , odd or otherwise.

(20) Use the calculator to find the number of digits of the number 4^{47} « 29 »

Third Higher skills

1 Choose the correct answer from the given ones :

(1) If $\log x = z + \log y$, then $x =$ ....

(a) $y \times 10^z$ (b) $\frac{z}{y}$ (c) $z - (10)^y$ (d) $\frac{1}{y} \times (10)^z$

(2) $\frac{1}{1 + \log_b a + \log_b c} + \frac{1}{1 + \log_c a + \log_c b} + \frac{1}{1 + \log_a b + \log_a c} =$ ....

(a) $\log_a bc$ (b) $\log_b ac$ (c) $\log_c ab$ (d) 1

(3) The sum of the roots of the equation : $(25)^x - 12 \times (5)^x + 27 = 0$ equals ....

(a) $\log_5 12$ (b) 12 (c) $\log_5 27$ (d) 27

(4) $3^{\log x} =$ ....

(a) $3 \log x$ (b) $x \log 3$ (c) $\log 3 x$ (d) $x^{\log 3}$

(5) If $a > b > c > 1$, then $\log_c \log_b \log_a a^{b^c} =$ ....

(a) zero (b) 1 (c) 2 (d) abc

(6) If $b^{x} - 2\,b^{-x} = 1$ where $b > 1$, then $x =$ ....

(a) 2 (b) log 2 (c) $\log_b 2$ (d) $\log_2 b$

(7) If $\frac{1}{\log_2 x} + \frac{1}{\log_4 x} + \frac{1}{\log_8 x} + \frac{1}{\log_{16} x} = 5$, then $x =$ ....

(a) 1 (b) 2 (c) 4 (d) 8

(8) If $\log a \in]0 , 1[$, then $a \in$ ....

(a) $]0 , 1[$ (b) $]1 , 2[$ (c) $]1 , 10[$ (d) $]1 , \infty[$

(9) If $a \in]0 , 9]$, then $\log_3 a \in$ ....

(a) $]-\infty , 2]$ (b) $]2 , 81]$ (c) $[2 , \infty[$ (d) $]-\infty , 0]$

(10) The solution set of the equation : $2^{\log_2 (x+4)} + 3^{\log_3 (x+5)} = 25^{\log_5 7}$ is ....

(a) $\{2 , 3\}$ (b) $\{-4 , -5\}$ (c) $\{2 , -2\}$ (d) $\{20\}$

(11) If $x , y \in \mathbb{R}^+ - \{1\}$ and $\log_y x = \log_x y$, then ....

(a) $x = y$ (b) $x = \frac{1}{y}$ (c) $x = \pm 1$ (d) a , b together

(12) If $3^a = 5^b = 7^c$, then $\frac{a}{b} + \frac{a}{c} =$ ....

(a) 35 (b) 7 (c) $\log_5 7$ (d) $\log_3 35$

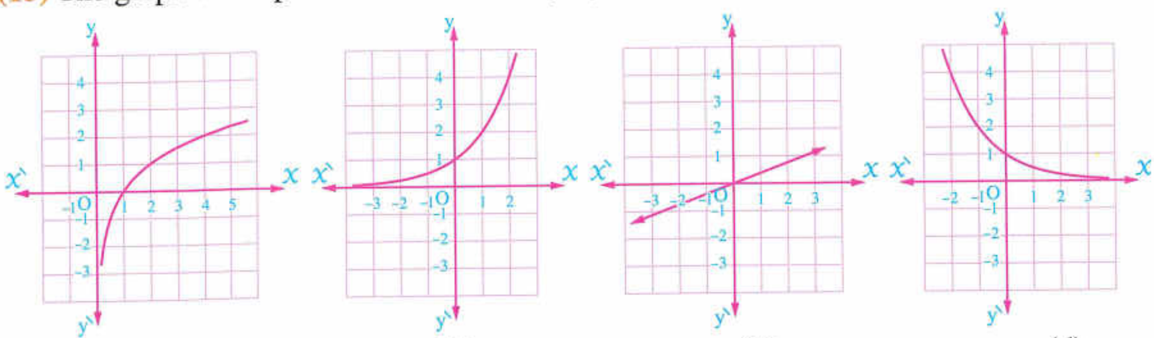
(13) If $\log_2 x + \log_4 x + \log_8 x = 11$, then $x =$ ....

(a) 32 (b) 36 (c) 64 (d) 121

(14) If $\log_2 3 \times \log_3 4 \times \log_4 5 \times \cdots \times \log_n(n+1) = 10$, then $n =$ ....

(a) 9 (b) 10 (c) 1023 (d) 1024

(15) The graph that represents the function $f : f(x) = \log 2^{x}$ is ....



(a) (b) (c) (d)

2 Without using the calculator , find the value of each of the following :

(1) $\log_3 \frac{1}{2} + \log_3 \frac{2}{3} + \log_3 \frac{3}{4} + \cdots + \log_3 \frac{80}{81}$ « −4 »

(2) log tan 1° + log tan 2° + log tan 3° + ··· + log tan 89° « 0 »

(3) log tan 1° × log tan 2° × log tan 3° × ··· × log tan 73° « 0 »

3 Prove that : $\log_b x + \log_{b^2} x^2 + \log_{b^3} x^3 + \cdots + \log_{b^n} x^n = \log_b x^n$

Life Applications on Unit Two

📖 From the school book

1 📖 **Geometry:**

If the radius length of a sphere r is given by the relation $r = \left(\frac{3\,V}{4\,\pi}\right)^{\frac{1}{3}}$ where V is the volume of the sphere , find the increase in the radius length when the volume changes from $\frac{32}{3}\pi$ to $36\,\pi$ cube unit.

« 1 length unit »

---

2 📖 **Numbers :**

If the sum of $2 + 4 + 8 + 16 + \cdots + 2^n$ is given by the relation $S_n = 2\,(2^n - 1)$

(1) Find the sum of the first ten numbers in this pattern.

(2) Find the number of terms of this pattern starting from the first term to give the sum 131070

« 2046 , 16 »

---

3 📖 **Education :**

If the relation between retention of materials of a student in the first secondary form and the number of months (t) starting from the end of study of the class is :
$f\,(t) = 70 - 4 \log_2 (t + 1)$, find the score of the student :

(1) At the end of the study of the class (t = 0) « 70 marks »

(2) After 7 months from the end of the study of the class. « 58 marks »

---

4 📖 In a study to measure the students retain what has been studied in a certain subject they re-examed from time to time in the same subject. If the student score follows the relation $f\,(t) = 85 - 25 \log (t + 1)$ where t is the period after studying in months , $f\,(t)$ is the student score in percentage **, find :**

(1) The score of the student in the first exam for this subject. « 85% »

(2) The score of the student after 3 months from studying this subject. « 69.95% »

(3) The score of the student after one year from studying this subject. « 57.15% »

---

5 **A country use a taxes system such that the taxpayer pays yearly the decided taxes according to the following function :**

$$f\,(X) = \begin{cases} 10\,\%\,X & , \quad \text{where } X \le 5000 \\ 10\,\%\,X + 100 \log (X - 4999) & \text{where } X > 5000 \end{cases}$$

Where X is the yearly net profit , find :

(1) The decided taxes on a taxpayer whose yearly net profit is 3600 pounds.

(2) The decided taxes on a taxpayer whose yearly net profit 8000 pounds.

« 360 pounds , 1147.7266 pounds »

6 **Chemistry : The (PH) level of a solution is known as negative the logarithm of the concentration of the hedrogen (H^+) in the solution i.e. PH = – log (H^+) :**

(1) What is the PH of a solution for which the concentration of hydrogen ions is 10^{-3} ?

(2) Find the concentration of hydrogen ions of a solution whose PH is 9 « 3 , 10^{-9} »

---

7 **Population : If the population of a city increases by yearly rate 7 % :**

(1) Find a formula of the population of the city after 1 year.

(2) After how many years the population is doubled assuming that it rises at the same rate ? « 10 years »

---

8 **If the population of a city starting from 2010 is given by $N = 10^5 (1.3)^{t-2010}$, where N is the number of population , t is the year.**

(1) Find the population of this city in 2015

(2) In which year the population of this city is 1.4 million people ? « 371293 people , 2020 »

---

9 **If the magnitude of the intensity M (I) of an earthquake on Rickter scale is given by $M(I) = \log\left(\frac{I}{I_0}\right)$ where I is the earthquake intensity , I_0 represents the smallest earth movement that can be recorded , called the reference intensity.**

(1) Find on Richter scale the magnitude of the earthquake of intensity 1.5×10^6 times the reference intensity.

(2) If the magnitude of the intensity of an earthquake is on Richter scale is 8 , find how many times does this earthquake intensity equal from the reference intensity. « 6.2 , 10^8 »

---

10 If the efficiency of a machine decreases yearly according to the relation $k = k_0 (0.9)^n$ where k is the machine efficiency , k_0 is the primary efficiency of the machine and n is the number of years the machine works.

If you know that the machine stops working if its efficiency is 40 % of its primary efficiency , how many years does the machine work before it stops working ? « 9 years »

---

11 **Industry :**

If the efficiency of a machine decreases yearly by a rate of 5 % , if you know that the machine stops working if its efficiency is 60 % of its primary efficiency , how many years does this machine work ? « 10 years »

Second

Calculus and Trigonometry



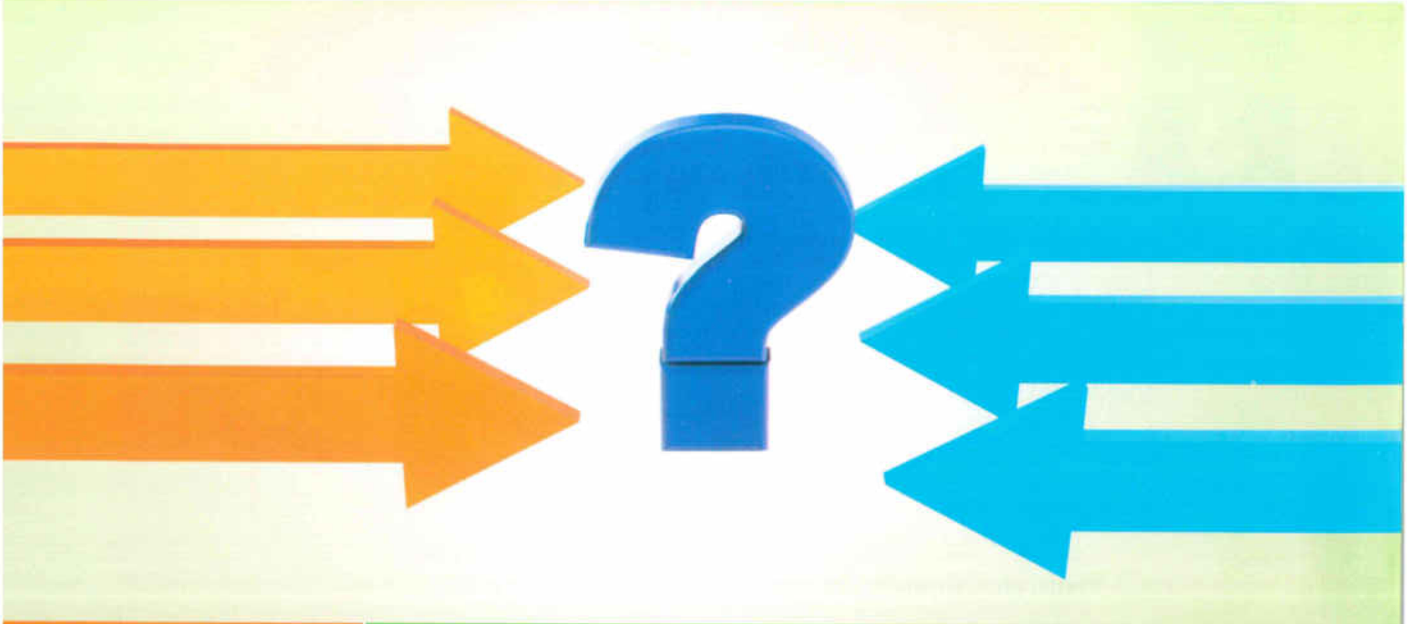
UNIT 3 Limits and continuity.

UNIT 4 Trigonometry.

UNIT 3

Limits and continuity

Unit Exercises



Exercise 12 Introduction to limits of functions "Evaluation of the limit numerically and graphically"

From the school book ● Understand ● Apply ● Higher Order Thinking Skills

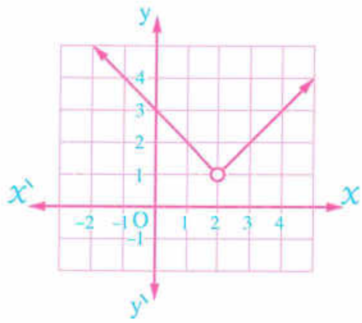
First Multiple choice questions

Choose the correct answer from those given :

(1) All the following are unspecified quantities except ....

(a) zero ÷ zero (b) $\infty - \infty$ (c) $\infty + \infty$ (d) $\infty \div \infty$

(2) In the opposite figure :



$\lim\limits_{x \to 2} f(x) =$ ....

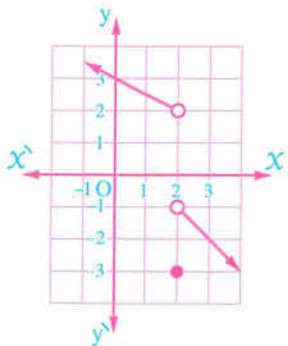
(a) 1

(b) – 1

(c) does not exist.

(d) 2

(3) In the opposite figure :



$\lim\limits_{x \to 2} f(x) =$ ....

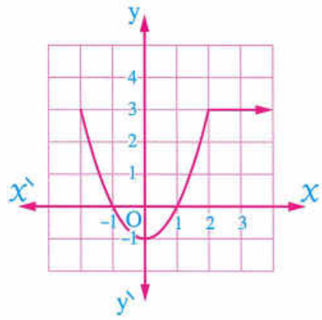
(a) – 3

(b) 2

(c) – 1

(d) does not exist.

(4) In the opposite figure :



$\lim\limits_{x \to 2} f(x) = \dots\dots$

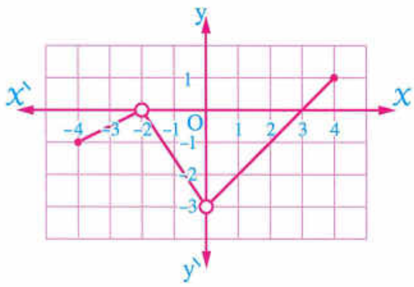
(a) zero

(b) 2

(c) 3

(d) does not exist.

(5) From the opposite figure :



First : $\lim\limits_{x \to -2} f(x) = \dots\dots$

(a) zero (b) – 3

(c) – 2 (d) does not exist.

Second : $\lim\limits_{x \to 0} f(x) = \dots\dots$

(a) zero (b) – 2

(c) – 3 (d) does not exist.

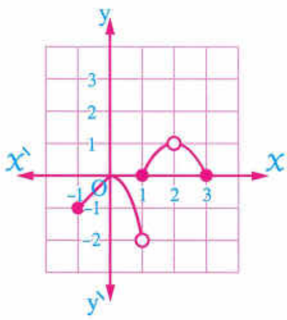
Third : $\lim\limits_{x \to -4} f(x) = \dots\dots$

(a) zero. (b) – 4 (c) – 1 (d) does not exist.

Fourth : $\lim\limits_{x \to 4} f(x) = \dots\dots$

(a) zero. (b) 4 (c) 1 (d) does not exist.

(6) Using the opposite figure :



First : $\lim\limits_{x \to 1^+} f(x) = \dots\dots$

(a) zero. (b) – 2

(c) 1 (d) does not exist.

Second : $\lim\limits_{x \to 1^-} f(x) = \dots\dots$

(a) zero (b) – 2

(c) 1 (d) does not exist.

Third : $\lim\limits_{x \to 1} f(x) = \dots\dots$

(a) zero (b) – 2 (c) 1 (d) does not exist.

Fourth : $\lim\limits_{x \to -1^+} f(x) = \dots\dots$

(a) zero (b) – 1 (c) – 2 (d) does not exist.

Fifth : $\lim\limits_{x \to -1} f(x) = \dots\dots$

(a) zero (b) – 1 (c) – 2 (d) does not exist.

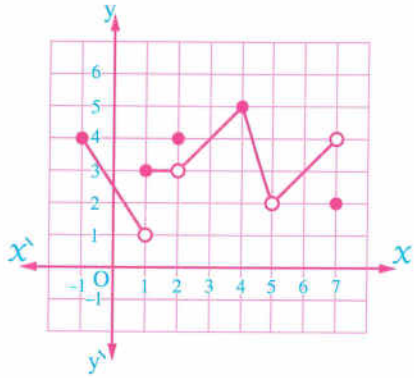
Sixth : $\lim_{x \to 2} f(x) = \dots\dots$

(a) zero. (b) 1 (c) 2 (d) does not exist.

Seventh : $\lim_{x \to 0} f(x) = \dots\dots$

(a) zero. (b) -1 (c) -2 (d) does not exist.

(7) Using the opposite figure :



First : $\lim_{x \to -1^+} f(x) = \dots\dots$

(a) zero. (b) -1

(c) 4 (d) does not exist.

Second : $\lim_{x \to -1^-} f(x) = \dots\dots$

(a) zero. (b) -1

(c) 4 (d) does not exist.

Third : $\lim_{x \to -1} f(x) = \dots\dots$

(a) zero. (b) -1 (c) 4 (d) does not exist.

Fourth : $f(2) = \dots\dots$

(a) zero. (b) 3 (c) 4 (d) undefined.

Fifth : $f(5) = \dots\dots$

(a) zero. (b) 2 (c) 5 (d) undefined.

Sixth : $\lim_{x \to 5^-} f(x) = \dots\dots$

(a) zero. (b) 2 (c) 3 (d) does not exist.

Seventh : $\lim_{x \to 5^+} f(x) = \dots\dots$

(a) zero. (b) 2 (c) 3 (d) does not exist.

Eighth : $\lim_{x \to 5} f(x) = \dots\dots$

(a) zero. (b) 2 (c) 3 (d) does not exist.

Ninth : $\lim_{x \to 7^-} f(x) = \dots\dots$

(a) zero. (b) 2 (c) 4 (d) does not exist.

Tenth : $\lim_{x \to 7} f(x) = \dots\dots$

(a) zero. (b) 2 (c) 4 (d) does not exist.

Second Essay questions

1 Complete the following table and deduce $\lim_{x \to 2} f(x)$ where $f(x) = 5x + 4$:

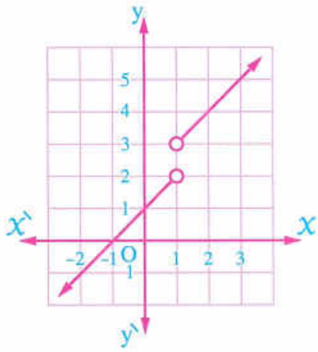
x	1.9	1.99	1.999	⟶	2	⟵	2.001	2.01	2.1
$f(x)$	....	....	....	⟶	?	⟵	....	....	....

2 Estimate each of the following limits graphically and numerically :

(1) $\lim_{x \to 2} (1 - 3x)$

(2) $\lim_{x \to 0} (x^2 - 2)$

3 From the opposite figure :



Find :

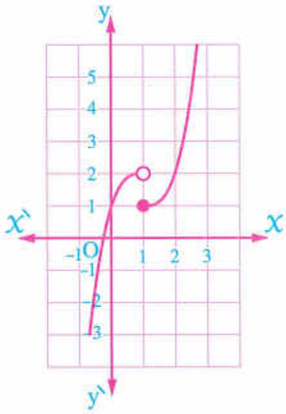
(1) $f(1)$

(2) $f(1^+)$

(3) $f(1^-)$

(4) $\lim_{x \to 1} f(x)$

4 Study the opposite figure , then find :



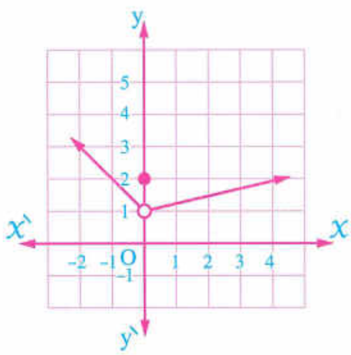
(1) $f(1)$

(2) $f(1^-)$

(3) $f(1^+)$

(4) $\lim_{x \to 1} f(x)$

5 Study the opposite figure , then find :



(1) $f(0)$

(2) $f(0^-)$

(3) $f(0^+)$

(4) $\lim_{x \to 0} f(x)$

(5) $f(2)$

(6) $\lim_{x \to 2} f(x)$

6 **From the opposite figure :**

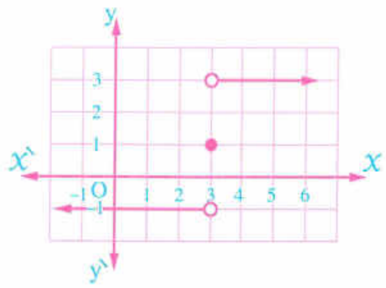
Find each of the following :

(1) 📖 $f(3)$

(2) $\lim\limits_{x \to 3^-} f(x)$

(3) $\lim\limits_{x \to 3^+} f(x)$

(4) 📖 $\lim\limits_{x \to 3} f(x)$



7 **From the opposite figure :**

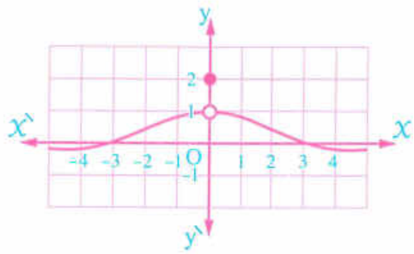
Find each of the following :

(1) $f(0)$

(2) $f(0^+)$

(3) $f(0^-)$

(4) $\lim\limits_{x \to 0} f(x)$



8 **From the opposite figure :**

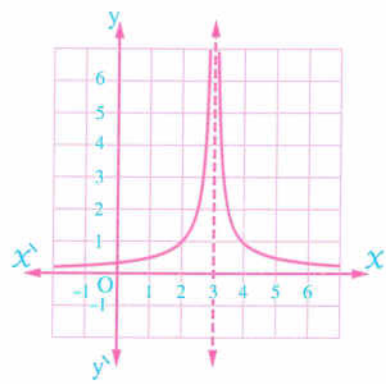
Find (if possible) each of the following :

(1) $f(3)$

(2) 📖 $\lim\limits_{x \to 3^-} f(x)$

(3) 📖 $\lim\limits_{x \to 3^+} f(x)$

(4) $\lim\limits_{x \to 3} f(x)$



9 **From the opposite figure :**

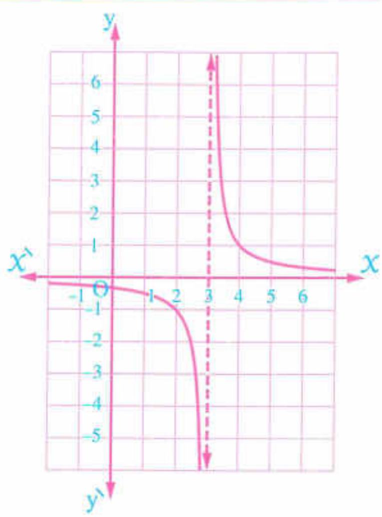
Find (if possible) each of the following :

(1) $f(3)$

(2) $f(3^+)$

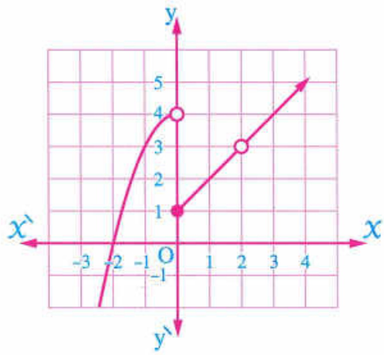
(3) $f(3^-)$

(4) $\lim\limits_{x \to 3} f(x)$



10 From the opposite figure :

Find :



(1) $f(0)$

(2) $f(0^+)$

(3) $f(0^-)$

(4) $\lim_{x \to 0} f(x)$

(5) $f(2)$

(6) $f(2^+)$

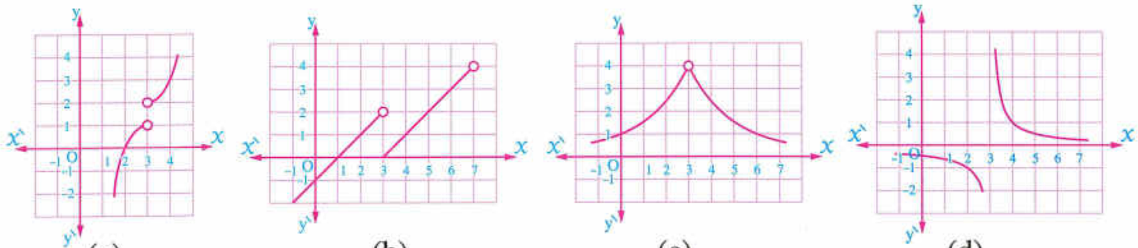
(7) $f(2^-)$

(8) $\lim_{x \to 2} f(x)$

Third Higher skills

Choose the correct answer from those given :

(1) Which of the functions represented by the following figures does have a limit at $x = 3$?



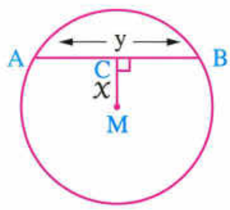
(a) (b) (c) (d)

(2) The opposite figure represents circle M

, its radius length is 5 cm. , $\overline{MC} \perp \overline{AB}$

, the length of $\overline{MC} = x$ and the length of $\overline{AB} = y$

When $x \longrightarrow 0$, then : $y \longrightarrow$ .... cm.

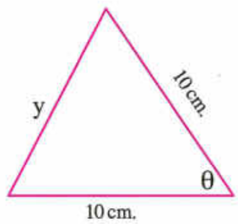


(a) 2.5 (b) 5 (c) 10 (d) 20

(3) **In the opposite figure :**

When $\theta \longrightarrow \frac{\pi}{2}$

, then : $y \longrightarrow$ .... cm.



(a) 0 (b) 5 (c) 10 (d) $10\sqrt{2}$

(4) If the curve of the polynomial function f intersects the x-axis at $x = 3$, then ....

(a) $\lim_{x \to 3} f(x) = 0$ (b) $\lim_{x \to 0} f(x) = 3$

(c) $\lim_{x \to 0} f(x) = 0$ (d) $\lim_{x \to 3} f(x) = 3$

(5) If the curve of the polynomial function f intersects the y-axis at y = 3 , then ....

(a) $\lim_{x \to 3} f(x) = \text{zero}$ (b) $\lim_{x \to 3} f(x) = 3$

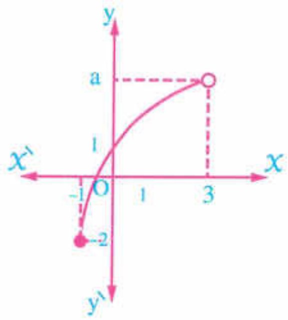
(c) $\lim_{x \to 0} f(x) = \text{zero}$ (d) $\lim_{x \to 0} f(x) = 3$

(6) The opposite figure represents the curve of the function

$y = f(x)$ and $\lim_{x \to 0} f(x) = \lim_{x \to -1^+} f(x) + \lim_{x \to 3^-} f(x)$

, then a = ...

(a) 3 (b) 4 (c) 5 (d) 6



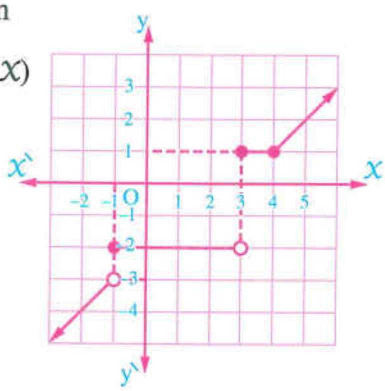
(7) The opposite figure represents the curve of the function

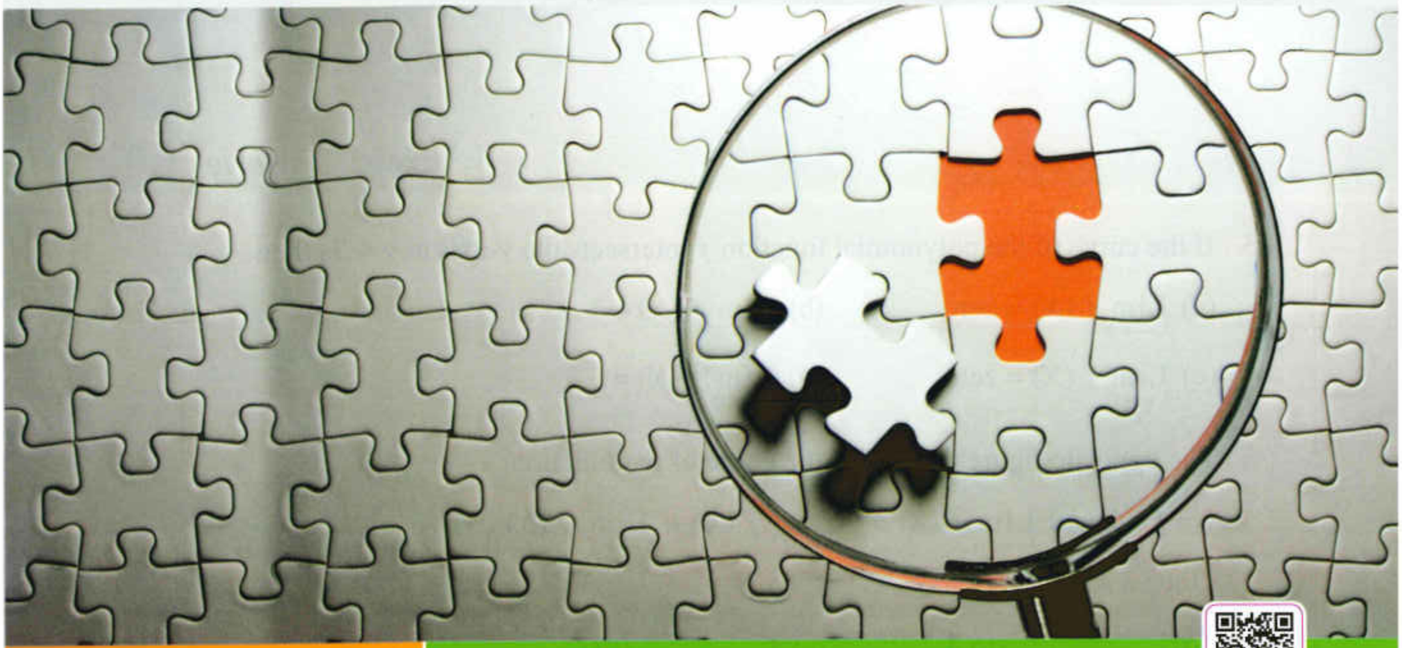
$y = f(x)$ and $\lim_{x \to a^-} f(x) = \lim_{x \to -1^-} f(x) + \lim_{x \to 3^+} f(x)$

, then the greatest possible value of a = ...

(a) 1 (b) 2

(c) 3 (d) 4





Exercise 13 Finding the limit of a function algebraically

Test yourself

From the school book ● Understand ● Apply ● Higher Order Thinking Skills

First Multiple choice questions

Choose the correct answer from the given ones :

(1) $\lim\limits_{x \to \frac{1}{2}} (10) = \ldots\ldots$

(a) 5 (b) 20 (c) 10 (d) $10\frac{1}{2}$

(2) $\lim\limits_{x \to 4} \left(3x - \sqrt{x}\right) = \ldots\ldots$

(a) 8 (b) 10 (c) 14 (d) 16

(3) $\lim\limits_{x \to 2} (3a^2) = \ldots\ldots$

(a) 3 (b) 6 (c) $3a^2$ (d) 12

(4) $\lim\limits_{x \to a} (2x^2 - ax - a^2) = \ldots\ldots$ where $a \in \mathbb{R}$

(a) zero (b) a (c) $4a^2$ (d) $a^2 - 2a$

(5) $\lim\limits_{x \to \frac{\pi}{2}} (2x - \sin x) = \ldots\ldots$

(a) π (b) $\pi + 1$ (c) $\pi - 1$ (d) $2\pi - 1$

(6) $\lim\limits_{x \to 2} \sqrt{\frac{3 + 2x}{4x - 1}} = \ldots\ldots$

(a) -3 (b) 1 (c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

(7) $\lim\limits_{x \to 3} \frac{2x - 6}{7x - 21} = \ldots\ldots$

(a) $\frac{2}{3}$ (b) $\frac{2}{7}$ (c) $\frac{3}{7}$ (d) 3

(8) $\lim\limits_{x \to 0} \dfrac{x^2 - x}{x} = \ldots\ldots\ldots\ldots$

(a) zero. (b) -1 (c) does not exist. (d) 1

(9) $\lim\limits_{x \to 3} \dfrac{x^2 - 7x + 12}{x - 3} = \ldots\ldots\ldots\ldots$

(a) 1 (b) -1 (c) 7 (d) -2

(10) $\lim\limits_{x \to -1} \dfrac{x^2 + x}{x^3 + 1} = \ldots\ldots\ldots\ldots$

(a) zero. (b) $-\frac{1}{3}$ (c) -1 (d) has no existence.

(11) $\lim\limits_{x \to 3} \dfrac{x^2 - x - 6}{x^2 + x - 12} = \ldots\ldots\ldots\ldots$

(a) $\frac{5}{7}$ (b) $\frac{1}{7}$ (c) -1 (d) -5

(12) $\lim\limits_{x \to -4} \dfrac{3x^2 - 48}{x + 4} = \ldots\ldots\ldots\ldots$

(a) 32 (b) -32 (c) 24 (d) -24

(13) $\lim\limits_{x \to 2} \dfrac{x^2 + x - 6}{x^2 - 4} = \ldots\ldots\ldots\ldots$

(a) $\frac{4}{5}$ (b) $\frac{5}{4}$ (c) $\frac{2}{5}$ (d) $\frac{-2}{5}$

(14) $\lim\limits_{x \to 0} \dfrac{3x + 2x^{-1}}{x + 4x^{-1}} = \ldots\ldots\ldots\ldots$

(a) 2 (b) 4 (c) $\frac{1}{2}$ (d) $\frac{1}{4}$

(15) $\lim\limits_{x \to \sqrt{5}} \dfrac{x^4 - x^2 - 20}{x - \sqrt{5}} = \ldots\ldots\ldots\ldots$

(a) 9 (b) $2\sqrt{5}$ (c) $9\sqrt{5}$ (d) $18\sqrt{5}$

(16) $\lim\limits_{x \to 4} \dfrac{(x - 3)^2 - 1}{x - 4} = \ldots\ldots\ldots\ldots$

(a) zero (b) 2 (c) 3 (d) 4

(17) $\lim\limits_{x \to 0} \dfrac{\sqrt{x + 1} - 1}{x} = \ldots\ldots\ldots\ldots$

(a) zero (b) $\sqrt{2}$ (c) $\frac{1}{2}$ (d) has no existence.

(18) $\lim\limits_{x \to 3} \dfrac{\sqrt{x + 1} - 2}{x - 3} = \ldots\ldots\ldots\ldots$

(a) 4 (b) $\frac{1}{4}$ (c) 6 (d) $\frac{1}{6}$

(19) $\lim\limits_{x \to 9} \dfrac{\sqrt{2}-\sqrt{x-7}}{x-9} = \dots\dots$

(a) $2\sqrt{2}$ (b) $\dfrac{\sqrt{2}}{4}$ (c) $\dfrac{-\sqrt{2}}{4}$ (d) $-2\sqrt{2}$

(20) $\lim\limits_{x \to 1} \dfrac{4-\sqrt{x+15}}{1-x^2} = \dots\dots$

(a) 4 (b) $\frac{1}{8}$ (c) $\frac{1}{16}$ (d) -4

(21) $\lim\limits_{x \to 4} \dfrac{\sqrt{x}-2}{5x^2-23x+12} = \dots\dots$

(a) $\frac{1}{92}$ (b) $\frac{1}{68}$ (c) 68 (d) 92

(22) $\lim\limits_{x \to 2} \dfrac{(x-3)^2-1}{\sqrt{x+2}-2} = \dots\dots$

(a) -6 (b) -8 (c) -2 (d) does not exist.

(23) $\lim\limits_{x \to 1} \dfrac{\sqrt{2x-1}-1}{\sqrt{3x+1}-2} = \dots\dots$

(a) 1 (b) $\frac{2}{3}$ (c) $\frac{1}{2}$ (d) $\frac{4}{3}$

(24) $\lim\limits_{x \to 1} \left(\dfrac{x^3}{x-1} - \dfrac{1}{x-1}\right) = \dots\dots$

(a) zero. (b) -3 (c) 3 (d) does not exist.

(25) $\lim\limits_{x \to 2} \dfrac{x^3-7x+6}{3x^2-8x+4} = \dots\dots$

(a) $\frac{5}{4}$ (b) $\frac{3}{2}$ (c) $\frac{4}{5}$ (d) $\frac{1}{3}$

(26) $\lim\limits_{x \to 2} \dfrac{3(x-2)^3+2x^2-8}{x-2} = \dots\dots$

(a) 0 (b) 2 (c) 4 (d) 8

(27) $\lim\limits_{x \to 0} \dfrac{7+2x}{\cos x} = \dots\dots$

(a) 7 (b) 8 (c) 9 (d) 1

(28) 📖 $\lim\limits_{x \to \frac{\pi}{4}} \dfrac{\tan x}{x} = \dots\dots$

(a) 0 (b) 1 (c) $\frac{4}{\pi}$ (d) does not exist.

(29) If $\lim\limits_{x \to 5} f(x) = 4$, then $\lim\limits_{x \to 5} [f(x) - 4] = \dots\dots$

(a) 4 (b) zero. (c) 5 (d) 8

(30) $\lim\limits_{x \to 2} \frac{x-3}{x-2} = \ldots\ldots$

(a) -1 (b) $\frac{-3}{2}$ (c) $\frac{3}{2}$ (d) does not exist.

(31) If $\lim\limits_{x \to m} \frac{2x^2-x-3}{4x^2-9} = \frac{5}{12}$, then m = ……

(a) $\frac{3}{2}$ (b) $\frac{-3}{2}$ (c) $\frac{2}{3}$ (d) $\frac{-2}{3}$

(32) If $\lim\limits_{x \to 1} \frac{x^2+3x-4}{x+a} = 5$, then a = ……

(a) -1 (b) 1 (c) 0 (d) 4

(33) If $\lim\limits_{x \to 2} \frac{x^2-4a}{x-2}$ exists, then a = ……

(a) -1 (b) 1 (c) 2 (d) 4

(34) If $\lim\limits_{x \to 3} \frac{x^2-2x+k}{x^2-9} = m$, where $m \in \mathbb{R}$, then $k \times m =$ ……

(a) $\frac{2}{3}$ (b) -3 (c) -2 (d) -1

(35) If $\lim\limits_{x \to -1} \frac{x^2+kx+m}{x^2-1} = 3$, then k + m = ……

(a) -4 (b) -5 (c) -8 (d) -9

Second Essay questions

1 Find each of the following :

(1) $\lim\limits_{x \to 5} \frac{x^2-25}{x-5}$ « 10 »

(2) $\lim\limits_{x \to 3} \frac{x^2-8x+15}{x-3}$ « −2 »

(3) $\lim\limits_{x \to 0} \frac{x^2}{3x^3-2x^2}$ « $-\frac{1}{2}$ »

(4) $\lim\limits_{x \to 2} \frac{5x-10}{4x-8}$ « $\frac{5}{4}$ »

(5) $\lim\limits_{x \to 4} \frac{4x^2-64}{x-4}$ « 32 »

(6) $\lim\limits_{x \to 4} \frac{2x-8}{x^2-x-12}$ « $\frac{2}{7}$ »

(7) $\lim\limits_{x \to -3} \frac{x^2+4x+3}{x^2-9}$ « $\frac{1}{3}$ »

(8) $\lim\limits_{x \to 2} \frac{x^2-2x}{x^2-x-2}$ « $\frac{2}{3}$ »

(9) $\lim\limits_{x \to -1} \frac{x^2-3x-4}{x^2-x-2}$ « $\frac{5}{3}$ »

(10) $\lim\limits_{x \to \frac{1}{2}} \frac{2x^2-5x+2}{2x-1}$ « $\frac{-3}{2}$ »

(11) $\lim\limits_{x \to 9} \frac{9-x}{x^2-81}$ « $-\frac{1}{18}$ »

(12) $\lim\limits_{x \to 1} \frac{3x^2+x-4}{x^2-1}$ « $\frac{7}{2}$ »

(13) $\lim\limits_{x \to \frac{3}{4}} \frac{16x^2-9}{8x-6}$ « 3 »

(14) $\lim\limits_{x \to 3} \frac{2x^2-5x-3}{2x^2-3x-9}$ « $\frac{7}{9}$ »

(15) $\lim\limits_{x \to 3} \frac{x^2+2x-15}{5x^2-45}$ « $\frac{4}{15}$ »

(16) $\lim\limits_{x \to 2} \frac{x^3-8}{3x^2-12}$ « 1 »

UNIT 3

● Understand ○ Apply ● Higher Order Thinking Skills

2 Find each of the following :

(1) $\lim\limits_{x \to 0} \frac{(x+2)^2-4}{x^2+x}$ « 4 »

(2) $\lim\limits_{x \to 0} \frac{(2x-1)^2-1}{5x}$ « $\frac{-4}{5}$ »

(3) $\lim\limits_{x \to 2} \frac{(x-3)^2-1}{2x^2-3x-2}$ « $\frac{-2}{5}$ »

(4) $\lim\limits_{x \to -3} \frac{(x+2)^3+1}{2x^2+6x}$ « $-\frac{1}{2}$ »

(5) $\lim\limits_{x \to 2} \frac{x^4+x^2-20}{x-2}$ « 36 »

(6) $\lim\limits_{x \to 2} \frac{(x^2-4)^2}{x-2}$ « zero »

(7) $\lim\limits_{x \to 1} \frac{2x-2}{(x-1)^5+2x-2}$ « 1 »

(8) $\lim\limits_{x \to -2} \frac{x+2}{x^4-16}$ « $-\frac{1}{32}$ »

(9) $\lim\limits_{x \to 9} \frac{x+\sqrt{x}-12}{x-9}$ « $\frac{7}{6}$ »

(10) $\lim\limits_{x \to 0} \frac{\frac{1}{2+x}-\frac{1}{2}}{x}$ « $-\frac{1}{4}$ »

(11) $\lim\limits_{x \to 2} \frac{x^4-2x^3-6x+12}{x^2+x-6}$ « $\frac{2}{5}$ »

(12) $\lim\limits_{x \to 3} \left(\frac{5}{x}+\frac{x^2-3x}{x-3}\right)$ « $\frac{14}{3}$ »

(13) $\lim\limits_{x \to -1} \left(\frac{x^2}{x^2-1}-\frac{3x+4}{x^2-1}\right)$ « $\frac{5}{2}$ »

(14) $\lim\limits_{x \to 1} \left(\frac{1}{x-1}-\frac{3}{x^3-1}\right)$ « 1 »

3 Find each of the following :

(1) $\lim\limits_{x \to 4} \frac{x^3-15x-4}{x-4}$ « 33 »

(2) $\lim\limits_{x \to 1} \frac{x^3-2x+1}{x^2+x-2}$ « $\frac{1}{3}$ »

(3) $\lim\limits_{x \to 4} \frac{x^4-21x^2+20x}{x-4}$ « 108 »

(4) $\lim\limits_{x \to -2} \frac{2x^3+3x^2+4}{x^3+8}$ « 1 »

(5) $\lim\limits_{x \to -2} \frac{x^2+4x+4}{x^3+x^2-8x-12}$ « $-\frac{1}{5}$ »

(6) $\lim\limits_{x \to 1} \frac{x^3-5x^2+7x-3}{x^3-2x^2+x}$ « -2 »

4 Find each of the following :

(1) $\lim\limits_{x \to 9} \frac{\sqrt{x}-3}{x-9}$ « $\frac{1}{6}$ »

(2) $\lim\limits_{x \to 5} \frac{\sqrt{x-1}-2}{x-5}$ « $\frac{1}{4}$ »

(3) $\lim\limits_{x \to -1} \frac{x+1}{\sqrt{x+5}-2}$ « 4 »

(4) $\lim\limits_{x \to 0} \frac{\sqrt{3-2x}-\sqrt{3}}{x}$ « $-\frac{1}{\sqrt{3}}$ »

(5) $\lim\limits_{x \to 0} \frac{\sqrt{2x+9}-3}{x^2+x}$ « $\frac{1}{3}$ »

(6) $\lim\limits_{x \to 5} \frac{x^2-5x}{\sqrt{x+4}-3}$ « 30 »

(7) $\lim\limits_{x \to 3} \frac{x^2-x-6}{\sqrt{5x-6}-3}$ « 6 »

(8) $\lim\limits_{x \to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{2x}$ « $\frac{1}{2}$ »

(9) $\lim\limits_{x \to 1} \dfrac{x-1}{\sqrt{3x-2}-\sqrt{x}}$ « 1 »

(10) $\lim\limits_{x \to -1} \dfrac{\sqrt{x^2+8}-3}{x+1}$ « $-\frac{1}{3}$ »

(11) $\lim\limits_{x \to 1} \dfrac{1-x}{2-\sqrt{x^2+3}}$ « 2 »

(12) $\lim\limits_{x \to 3} \dfrac{\sqrt{x+1}-2}{\sqrt{x-2}-1}$ « $\frac{1}{2}$ »

(13) $\lim\limits_{x \to 4} \dfrac{\sqrt{24-5x}-2}{3-\sqrt{x+5}}$ « 7.5 »

(14) $\lim\limits_{x \to 3} \dfrac{x-3}{x-\sqrt{x+6}}$ « $\frac{6}{5}$ »

5 If $\lim\limits_{x \to 2} \dfrac{f(x)-5}{x-2} = 1$, where f is a polynomial function , then find : $\lim\limits_{x \to 2} f(x)$ « 5 »

6 If $\lim\limits_{x \to 0} \dfrac{f(x)}{x^2} = 5$ **, then find :**

(1) $\lim\limits_{x \to 0} f(x)$

(2) $\lim\limits_{x \to 0} \dfrac{f(x)}{x}$ « zero , zero »

7 If $\lim\limits_{x \to -1} \dfrac{x^2-(a-1)x-a}{x+1} = 4$, then find a « −5 »

8 If $\lim\limits_{x \to 1} \left(\dfrac{x^2+ax+b}{x-1}\right) = 5$

, find the value of each of a and b « 3 , −4 »

9 **Trade :**

A company found that if it spent L.E. x for advertising one of its production , then its profit will be given by the function $f : f(x) = 0.2\,x^2 + 40\,x + 150$

Find the profit of the company when the expenditure for advertisement approaches to L.E. 100 « 6150 pounds »

Third Higher skills

1 **Choose the correct answer from those givens :**

(1) If f is a function satisfying that : $x(f(x)+1) = f(x) + x^2$

, then $\lim\limits_{x \to 1} f(x) = \text{..............}$

(a) 1 (b) 2 (c) 3 (d) zero

(2) If : $\lim\limits_{x \to 0} \frac{\sqrt{mx + k} - 3}{x} = 2$, then $\frac{m}{k} = \dots\dots$

(a) 8 (b) 4 (c) 3 (d) $\frac{4}{3}$

(3) If : $\lim\limits_{x \to m} \left(2 f(x) - 5 g(x)\right) = 10$, $\lim\limits_{x \to m} \left(f(x) + g(x)\right) = 6$

, then $\lim\limits_{x \to m} \frac{f(x)}{g(x)} = \dots\dots$

(a) $\frac{40}{7}$ (b) $\frac{2}{7}$ (c) 10 (d) 20

(4) If : n (x) is a function and $\lim\limits_{x \to 2} \frac{n(x) - 8}{x - 2} = 7$

, then $\lim\limits_{x \to 2} \frac{2x^2 - n(x)}{x - 2} = \dots\dots$

(a) 1 (b) 7 (c) 8 (d) 15

---

2 Find each of the following :

(1) $\lim\limits_{x \to 5} \left(\frac{x^2 - 4x - 5}{x^2 - 6x + 5}\right)^2$ « $\frac{9}{4}$ »

(2) $\lim\limits_{x \to 1} \sqrt{\frac{x^2 + 2x - 3}{x^2 - x}}$ « 2 »

(3) $\lim\limits_{x \to \sqrt{2}} \frac{x^2 - 2}{x^2 + \sqrt{2}\, x - 4}$ « $\frac{2}{3}$ »

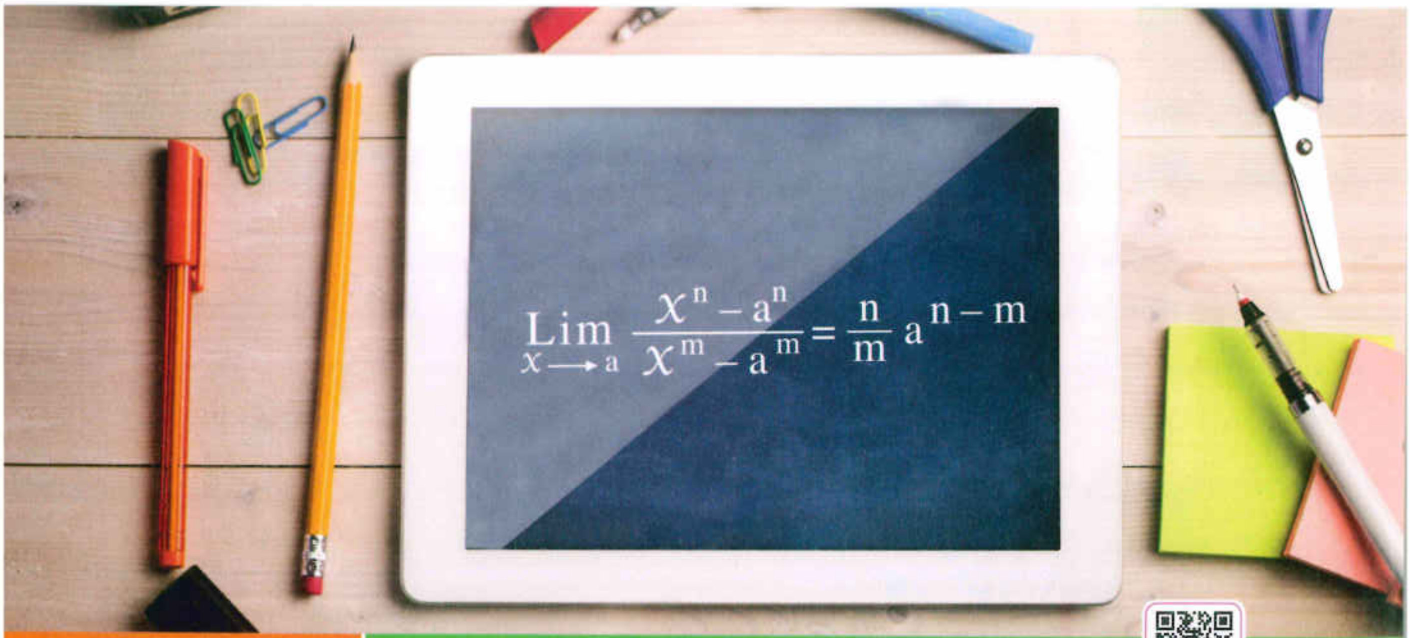
(4) $\lim\limits_{x \to 1} \frac{5x^{-3} + 2x^{-1} - 7}{3x^{-2} + x^{-3} - 4}$ « $\frac{17}{9}$ »

(5) $\lim\limits_{\tan x \to 3} \frac{\tan^2 x - 2\tan x - 3}{\tan^2 x - 4\tan x + 3}$ « 2 »

(6) $\lim\limits_{x \to 0} \frac{\cos^2 x - 5\cos x + 4}{\sin^2 x}$ « $\frac{3}{2}$ »

(7) $\lim\limits_{(x-4) \to 4} \frac{\left(\sqrt[3]{x} - 2\right)\left(\sqrt[3]{x^2} + 2\sqrt[3]{x} + 4\right)}{x^2 - 64}$ « $\frac{1}{16}$ »

(8) $\lim\limits_{x \to 1} \left(\frac{\sqrt{x^2 + 3} + 2}{x^2 - x} - \frac{2}{\sqrt{x^2 + 3} - 2}\right)$ « – 2 »



Exercise 14 Theorem (4) " The law "

Test yourself

From the school book · Understand · Apply · Higher Order Thinking Skills

First Multiple choice questions

Choose the correct answer from those given :

(1) $\lim_{x \to a} \frac{x^n - a^n}{x^m - a^m} = \dots\dots\dots$

(a) $\frac{m}{n}$ (b) $\frac{m}{n}(a)^{m-n}$ (c) $\frac{n}{m}(a)^{m-n}$ (d) $\frac{n}{m}(a)^{n-m}$

(2) $\lim_{y \to 2} \frac{y^5 - 32}{y - 2} = \dots\dots\dots$

(a) $31\, y^4$ (b) 32×2^4 (c) 64 (d) 5×2^4

(3) $\lim_{x \to 1} \frac{x^{10} - 1}{x - 1} = \dots\dots\dots$

(a) zero. (b) 1 (c) 9 (d) 10

(4) $\lim_{x \to 2} \frac{x^{-1} - 2^{-1}}{x^{-4} - 2^{-4}} = \dots\dots\dots$

(a) 2 (b) $\frac{1}{2}$ (c) $\frac{1}{32}$ (d) 8

(5) $\lim_{x \to -1} \frac{x^5 + 1}{x + 1} = \dots\dots\dots$

(a) 5 (b) 4 (c) – 5 (d) – 4

(6) $\lim_{x \to 2} \frac{x^5 - 32}{x^3 - 8} = \dots\dots\dots$

(a) 4 (b) $\frac{5}{3}$ (c) zero (d) $6\frac{2}{3}$

(7) $\lim_{x \to \frac{1}{2}} \frac{4x^2 - 1}{32x^5 - 1} = \dots\dots\dots$

(a) $\frac{2}{5}$ (b) $\frac{5}{2}$ (c) $\frac{2}{9}$ (d) $\frac{1}{8}$

(8) $\lim\limits_{x \to -2} \dfrac{x^7 + 128}{x^4 - 16} = \dots\dots$

(a) 9 (b) – 9 (c) – 14 (d) 14

(9) $\lim\limits_{x \to 16} \dfrac{\sqrt[4]{x^5} - 32}{x - 16} = \dots\dots$

(a) 5 (b) $\frac{5}{2}$ (c) $\frac{5}{4}$ (d) $\frac{5}{8}$

(10) $\lim\limits_{h \to 0} \dfrac{(x + h)^7 - x^7}{h} = \dots\dots$

(a) x^7 (b) $7x^6$ (c) zero. (d) 1

(11) $\lim\limits_{h \to 0} \dfrac{(2 - 3h)^7 - 128}{4h} = \dots\dots$

(a) 336 (b) – 336 (c) 448 (d) – 448

(12) $\lim\limits_{h \to 0} \dfrac{(3h - 1)^5 + 1}{5h} = \dots\dots$

(a) – 3 (b) $\frac{3}{5}$ (c) 3 (d) 5

(13) $\lim\limits_{x \to 1} \dfrac{x^6 - 64}{x - 2} = \dots\dots$

(a) $6(2)^5$ (b) 128 (c) $64(2)^5$ (d) 63

(14) $\lim\limits_{x \to 1} \dfrac{x^{\frac{13}{2}} - x^{\frac{1}{2}}}{x^{\frac{7}{2}} - x^{\frac{1}{2}}} = \dots\dots$

(a) $\frac{13}{7}$ (b) 1 (c) 2 (d) x

(15) $\lim\limits_{x \to 1} \dfrac{x^2 - x^{-2}}{x - x^{-1}} = \dots\dots$

(a) zero (b) 1 (c) 2 (d) – 2

(16) $\lim\limits_{x \to 0} \dfrac{(x + 2)^5 - 32}{x} = \dots\dots$

(a) 25 (b) 64 (c) 80 (d) 100

(17) $\lim\limits_{x \to 5} \dfrac{(x - 3)^7 - 128}{x - 5} = \dots\dots$

(a) 7 (b) 28 (c) 64 (d) 448

(18) $\lim\limits_{x \to 2} \dfrac{(x + 1)^4 - 81}{x - 2} = \dots\dots$

(a) 18 (b) 81 (c) – 108 (d) 108

(19) $\lim\limits_{x \to 1} \dfrac{x^a - x^c}{x^a - x^d} = \dots\dots$

(a) $\frac{a + c}{a + d}$ (b) $a\,x^{c - d}$ (c) $\frac{a - c}{a - d}$ (d) $\frac{c}{d}$

(20) $\lim\limits_{x \to 1} \dfrac{1 - \sqrt[n]{x}}{1 - \sqrt[m]{x}} = \dots\dots$

(a) 1 (b) $\frac{n}{m}$ (c) – 1 (d) $\frac{m}{n}$

(21) $\lim\limits_{x \to 9} \dfrac{3-\sqrt{x}}{27-\sqrt{x^3}} = \ldots\ldots\ldots\ldots$

(a) $\frac{1}{9}$ (b) $\frac{1}{27}$ (c) 3 (d) $-\frac{1}{27}$

(22) $\lim\limits_{x \to -2} \dfrac{(x+3)^5-1}{x^2-4} = \ldots\ldots\ldots\ldots$

(a) $\frac{5}{4}$ (b) $\frac{-5}{4}$ (c) $\frac{1}{4}$ (d) $\frac{-1}{4}$

(23) $\lim\limits_{x \to 2} \dfrac{x^5-32}{x^2+3x-10} = \ldots\ldots\ldots\ldots$

(a) 80 (b) $\frac{80}{7}$ (c) $\frac{7}{80}$ (d) $\frac{1}{80}$

(24) $\lim\limits_{x \to 1} \dfrac{\sqrt[5]{x}+2\sqrt{x}-3}{x-1} = \ldots\ldots\ldots\ldots$

(a) $\frac{6}{5}$ (b) 2 (c) 5 (d) 3

(25) If $f(x) = x^5$, $g(x) = x^2-4$, then $\lim\limits_{x \to 2} \dfrac{f(x)-32}{g(x)} = \ldots\ldots\ldots\ldots$

(a) -20 (b) 20 (c) ± 20 (d) 32

(26) If $\lim\limits_{x \to a} \dfrac{x^n-a^n}{x^m-a^m} = \dfrac{n}{m}$, then a = ……………

(a) 1 (b) n (c) m (d) $\frac{m}{n}$

(27) If $\lim\limits_{x \to 2} \dfrac{(x)^n-(2)^n}{x-2} = 32$, then n = ……………

(a) 3 (b) 4 (c) 9 (d) 12

(28) If $\lim\limits_{x \to a} \dfrac{x^8-a^8}{x^6-a^6} = 48$, then a = ……………

(a) 4 (b) 6 (c) ± 4 (d) ± 6

(29) $\lim\limits_{x \to 5} \dfrac{\sqrt[3]{x+3}-2}{x-5} = \ldots\ldots\ldots\ldots$

(a) $\frac{1}{12}$ (b) $\frac{2}{5}$ (c) $\frac{3}{5}$ (d) 1

(30) $\lim\limits_{x \to 0} \dfrac{(1+x)^{\frac{5}{2}}-1}{x} = \ldots\ldots\ldots\ldots$

(a) $\frac{5}{2}$ (b) zero. (c) $\frac{2}{5}$ (d) does not exist.

(31) $\lim\limits_{x \to 0} \dfrac{(x+2)^6-64}{x^2+16x} = \ldots\ldots\ldots\ldots$

(a) 6 (b) 12 (c) 16 (d) 64

(32) $\lim\limits_{x \to 1} \left(\dfrac{x^6-x^7+x^8-x^9}{x-1}\right) = \ldots\ldots\ldots\ldots$

(a) 30 (b) -2 (c) 3 (d) 9

UNIT 3

● Understand ○ Apply Higher Order Thinking Skills

Second Essay questions

1 Find each of the following :

(1) $\lim\limits_{x \to 2} \frac{x^3 - 8}{x - 2}$ « 12 »

(2) $\lim\limits_{x \to -5} \frac{x^4 - 625}{x + 5}$ « – 500 »

(3) $\lim\limits_{x \to a} \frac{x^5 - a^5}{x - a}$ « $5a^4$ »

(4) $\lim\limits_{x \to -3} \frac{x^4 - 81}{x^5 + 243}$ « $-\frac{4}{15}$ »

(5) $\lim\limits_{x \to 2} \frac{x^6 - 64}{x^7 - 128}$ « $\frac{3}{7}$ »

(6) $\lim\limits_{x \to 2} \frac{2x^6 - 128}{x^2 - 4}$ « 96 »

(7) $\lim\limits_{x \to -2} \frac{x^6 - 64}{3x + 6}$ « – 64 »

(8) $\lim\limits_{x \to -1} \frac{x^{10} + x}{x^7 - x}$ « $\frac{-3}{2}$ »

(9) $\lim\limits_{x \to 1} \frac{1 - x^9}{x^7 - 1}$ « $\frac{-9}{7}$ »

(10) $\lim\limits_{x \to -\frac{1}{2}} \frac{32x^5 + 1}{64x^6 - 1}$ « $\frac{-5}{6}$ »

(11) $\lim\limits_{x \to \frac{-2}{3}} \frac{243x^5 + 32}{27x^3 + 8}$ « $\frac{20}{3}$ »

(12) $\lim\limits_{2x \to 3} \frac{16x^5 - 81x}{2x^3 - 3x^2}$ « 72 »

2 Find each of the following :

(1) $\lim\limits_{x \to 2} \frac{x^{-7} - (2)^{-7}}{x - 2}$ « $\frac{-7}{256}$ »

(2) $\lim\limits_{x \to -1} \frac{x^{-4} - 1}{x^{-18} - 1}$ « $\frac{2}{9}$ »

(3) $\lim\limits_{x \to \sqrt{2}} \frac{x^7 - 8\sqrt{2}}{x^2 - 2}$ « $14\sqrt{2}$ »

(4) $\lim\limits_{x \to -\sqrt{2}} \frac{x^8 - 16}{x^5 + 4\sqrt{2}}$ « $\frac{-16\sqrt{2}}{5}$ »

(5) $\lim\limits_{x \to \frac{-3}{\sqrt{2}}} \frac{8x^6 - 729}{\sqrt{2}x + 3}$ « – 1458 »

(6) $\lim\limits_{x \to \frac{\pi}{6}} \frac{\sin^3 x - \frac{1}{8}}{\sin x - \frac{1}{2}}$ « $\frac{3}{4}$ »

(7) $\lim\limits_{x \to 2} \frac{x^{-5} - \frac{1}{32}}{x^{-7} - \frac{1}{128}}$ « $\frac{20}{7}$ »

(8) $\lim\limits_{x \to -2} \frac{8x^{-3} + 1}{x^{-8} - \frac{1}{256}}$ « – 96 »

(9) $\lim\limits_{x \to 2} \frac{x^{-8} - (16)^{-2}}{x - 2}$ « $-\frac{1}{64}$ »

(10) $\lim\limits_{x \to 4} \frac{x^{\frac{7}{2}} - 128}{x^{\frac{5}{2}} - 32}$ « $\frac{28}{5}$ »

(11) $\lim\limits_{x \to 1} \frac{\sqrt[7]{x} - 1}{x - 1}$ « $\frac{1}{7}$ »

(12) $\lim\limits_{x \to 4} \frac{\sqrt{x^3} - 8}{x^2 - 16}$ « $\frac{3}{8}$ »

(13) $\lim\limits_{x \to 16} \frac{\sqrt[4]{x^7} - 128}{x - 16}$ « 14 »

(14) $\lim\limits_{x \to 1} \frac{x\sqrt[5]{x} - 1}{x^2 - 1}$ « $\frac{3}{5}$ »

(15) $\lim\limits_{x \to 1} \dfrac{x^{\frac{21}{2}} - x^{\frac{1}{2}}}{x^{\frac{14}{3}} - x^{\frac{2}{3}}}$ «$\frac{5}{2}$»

(16) $\lim\limits_{x \to 4} \dfrac{\frac{1}{2}x^4 - 128}{x - 4}$ «128»

(17) $\lim\limits_{x \to -1} \dfrac{x^4 - \frac{1}{x^4}}{x^3 - \frac{1}{x^3}}$ «$\frac{-4}{3}$»

(18) $\lim\limits_{2x \to 1} \dfrac{5x^4 - \frac{5}{16}}{2x - 1}$ «$\frac{5}{4}$»

(19) $\lim\limits_{x \to 2} \dfrac{x^{10} - 1024}{x^2 - 3x + 2}$ «5120»

(20) $\lim\limits_{x \to 1} \dfrac{x^{17} - 1}{3x^2 + 2x - 5}$ «$\frac{17}{8}$»

3 **Find each of the following :**

(1) $\lim\limits_{x \to 0} \dfrac{(1 + x)^{10} - 1}{(1 + x)^7 - 1}$ «$\frac{10}{7}$»

(2) $\lim\limits_{x \to 6} \dfrac{(x - 5)^7 - 1}{x - 6}$ «7»

(3) $\lim\limits_{x \to -2} \dfrac{(x + 3)^5 - 1}{x + 2}$ «5»

(4) $\lim\limits_{x \to 0} \dfrac{(x + 2)^5 - 32}{x}$ «80»

(5) $\lim\limits_{h \to 0} \dfrac{(3 + h)^4 - 81}{6h}$ «18»

(6) $\lim\limits_{h \to 0} \dfrac{(1 + 4h)^8 - 1}{h}$ «32»

(7) $\lim\limits_{x \to 0} \dfrac{(1 - 2x)^5 - 1}{5x}$ «−2»

(8) $\lim\limits_{h \to 0} \dfrac{(x + 3h)^5 - x^5}{h}$ «$15x^4$»

(9) $\lim\limits_{h \to 0} \dfrac{(x - 2h)^{17} - x^{17}}{51h}$ «$-\frac{2}{3}x^{16}$»

(10) $\lim\limits_{x + 1 \to 0} \dfrac{(3x + 2)^9 + 1}{x + 1}$ «27»

(11) $\lim\limits_{x \to 0} \dfrac{\sqrt[3]{1 + 3x} - 1}{2x}$ «$\frac{1}{2}$»

(12) $\lim\limits_{x \to 1} \dfrac{\sqrt[3]{26 + x} - 3}{x - 1}$ «$\frac{1}{27}$»

(13) $\lim\limits_{x \to 7} \dfrac{\sqrt[5]{x + 25} - 2}{x - 7}$ «$\frac{1}{80}$»

(14) $\lim\limits_{x \to 2} \dfrac{(x - 4)^5 + 32}{x - 2}$ «80»

(15) $\lim\limits_{x \to -2} \dfrac{(x + 3)^5 - 1}{x^2 - 4}$ «$-\frac{5}{4}$»

(16) $\lim\limits_{x \to 1} \dfrac{x^{19} + x^8 - 2}{x - 1}$ «27»

(17) $\lim\limits_{x \to -1} \dfrac{x^7 + x^9 + 2}{x + 1}$ «16»

(18) $\lim\limits_{x \to 1} \dfrac{\sqrt{x} + \sqrt[3]{x} - 2}{x - 1}$ «$\frac{5}{6}$»

(19) $\lim\limits_{x \to 2} \dfrac{x^5 + x^2 - 36}{x - 2}$ «84»

(20) $\lim\limits_{x \to 5} \dfrac{(x - 3)^5 - 32}{x^2 - 5x}$ «16»

(21) $\lim\limits_{x \to 0} \dfrac{(2x - 3)^5 + 243}{3x(x - 2)}$ «−135»

4 **Find the value of a if :** $\lim\limits_{x \to a} \dfrac{x^{12} - a^{12}}{x^{10} - a^{10}} = 30$ «±5»

5 **Find the value of k if :** $\lim\limits_{x \to -1} \dfrac{x^{15} + 1}{x + 1} = \lim\limits_{x \to k} \dfrac{x^5 - k^5}{x^3 - k^3}$ «±3»

6 If $\lim\limits_{x \to 2} \frac{x^n - 64}{x - 2} = \ell$, find the value of each of : n and ℓ « 6 , 192 »

7 If $f(x) = \frac{1}{x^4}$, then find : $\lim\limits_{x \to 2} \frac{f(x) - f(2)}{x - 2}$ « $\frac{-1}{8}$ »

8 Find each of the following :

(1) $\lim\limits_{x \to 2} \left(\frac{x^3 - 8}{x^5 - 32} + \frac{x^4 - 16}{x^7 - 128}\right)$ « $\frac{31}{140}$ »

(2) $\lim\limits_{x \to 3} \left(\frac{x^5 - 243}{x^2 - 4} \times \frac{x - 2}{x - 3}\right)$ « 81 »

(3) $\lim\limits_{x \to -3} \left(\frac{x^4 - 81}{x^3 + 27}\right)^3$ « – 64 »

(4) $\lim\limits_{x \to 1} \frac{(x^{25} - 1)^2}{x^2 - 2x + 1}$ « 625 »

(5) $\lim\limits_{x \to 1} \frac{x^{12} - 2x^6 + 1}{x^2 - 2x + 1}$ « 36 »

(6) $\lim\limits_{x \to 2} \left[\frac{(x^4 - 16)^4}{(x^3 - 8)^3} \times \frac{1}{x^8 - 256}\right]$ « $\frac{16}{27}$ »

(7) $\lim\limits_{x \to 1} \frac{\left(\sqrt[3]{x} - 1\right)\left(\sqrt[5]{x^2} - 1\right)}{(x - 1)^2}$ « $\frac{2}{15}$ »

(8) $\lim\limits_{x \to 0} \frac{(1 + x)^{15} - (1 - x)^{15}}{(1 + x)^9 - (1 - x)^9}$ « $\frac{5}{3}$ »

(9) $\lim\limits_{x \to 1} \frac{x^{12} - 1}{x^3 - x^2 + x - 1}$ « 6 »

Third Higher skills

Find each of the following :

(1) $\lim\limits_{x \to 1} \frac{\sqrt{x^3} - \sqrt{x}}{\sqrt{x^{11}} - \sqrt{x}}$ « $\frac{1}{5}$ »

(2) $\lim\limits_{x \to 0} \frac{(1 + 5x)^{10} - 1}{(1 + 7x)^8 - 1}$ « $\frac{25}{28}$ »

(3) $\lim\limits_{x \to 0} \frac{\sqrt[3]{x + 8} - 2}{(x + 3)^5 - 243}$ « $\frac{1}{4860}$ »

(4) $\lim\limits_{x \to 2} \frac{\sqrt[3]{2x - 3} - 1}{\sqrt[5]{3x - 5} - 1}$ « $\frac{10}{9}$ »

(5) $\lim\limits_{x \to 2} \frac{(x^2 - 3)^5 - 1}{x - 2}$ « 20 »

(6) $\lim\limits_{x \to -1} \frac{(4x + 5)^6 + 7x + 6}{x + 1}$ « 31 »

(7) $\lim\limits_{x \to 1} \frac{(5x - 4)^{10} + 4x - 5}{x - 1}$ « 54 »

(8) $\lim\limits_{x \to 0} \frac{\sqrt[5]{x + 1} - \sqrt[4]{x + 1}}{x}$ « $\frac{-1}{20}$ »

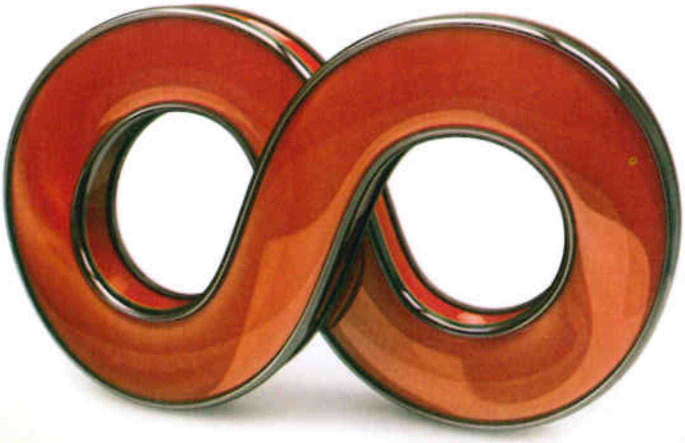
(9) $\lim\limits_{x \to 0} \frac{\sqrt[5]{x + 1} - (x + 1)^7}{x}$ « $-6\frac{4}{5}$ »

(10) $\lim\limits_{x \to 2} \frac{5\sqrt{x - 1} + \sqrt[3]{x - 1} - 6}{x - 2}$ « $\frac{17}{6}$ »

(11) $\lim\limits_{x \to 1} \frac{x^7 + x^9 + x^{11} - 3}{x - 1}$ « 27 »

(12) $\lim\limits_{x \to 2} \frac{x^4 (x - 1)^6 - 16}{x - 2}$ « 128 »

(13) $\lim\limits_{x \to 2} \frac{x^5 \sqrt[3]{x + 6} - 64}{x - 2}$ « $162\frac{2}{3}$ »



Exercise 15 Limit of the function at infinity

Test yourself

From the school book · Understand · Apply · Higher Order Thinking Skills

First Multiple choice questions

Choose the correct answer from those given :

(1) $\lim\limits_{x \to \infty} \left(\frac{3}{x^2} - 2\right) = \dots\dots\dots$

(a) 3 (b) 2 (c) -3 (d) -2

(2) $\lim\limits_{x \to \infty} \frac{3x}{4x+5} = \dots\dots\dots$

(a) ∞ (b) $\frac{3}{4}$ (c) $\frac{1}{5}$ (d) zero.

(3) $\lim\limits_{x \to \infty} \frac{2x^2+1}{x^2+1} = \dots\dots\dots$

(a) zero. (b) does not exist. (c) ∞ (d) 2

(4) $\lim\limits_{x \to \infty} \frac{x^2+5}{6} = \dots\dots\dots$

(a) zero. (b) $\frac{5}{6}$ (c) 1 (d) ∞

(5) $\lim\limits_{x \to \infty} \frac{\sqrt{x}-3}{x} = \dots\dots\dots$

(a) zero. (b) 1 (c) -2 (d) ∞

(6) $\lim\limits_{x \to \infty} (3x^{-5} + 4x^{-2} + 5) = \dots\dots\dots$

(a) 12 (b) ∞ (c) 5 (d) zero

(7) $\lim\limits_{x \to \infty} \frac{3x^{-3} + 4x^{-2} - 2}{7x^{-3} - x^{-2} + 6} = \dots\dots\dots$

(a) ∞ (b) zero (c) $\frac{3}{7}$ (d) $\frac{-1}{3}$

(8) $\lim\limits_{x \to \infty} \frac{x^7 - 2x^3}{2x^4 - 3x^2 - 1} = ...$

(a) zero. (b) 3 (c) ∞ (d) $\frac{1}{2}$

(9) $\lim\limits_{x \to \infty} \frac{3x^2}{x(2x-1)} = ...$

(a) $\frac{3}{2}$ (b) $\frac{2}{3}$ (c) zero. (d) 3

(10) $\lim\limits_{x \to \infty} \sqrt{\frac{3+2x}{4x-1}} = ...$

(a) $\frac{1}{2}$ (b) $\frac{3}{4}$ (c) $\frac{1}{\sqrt{2}}$ (d) $\frac{\sqrt{3}}{2}$

(11) 📖 $\lim\limits_{x \to \infty} \frac{x^3+5}{x(2x^2+3)} = ...$

(a) $\frac{5}{8}$ (b) 1 (c) $\frac{1}{2}$ (d) $\frac{5}{3}$

(12) 📖 $\lim\limits_{x \to \infty} \frac{5+x^{-2}}{3x^{-2}+1} = ...$

(a) $\frac{5}{4}$ (b) $\frac{5}{3}$ (c) 5 (d) $\frac{1}{3}$

(13) $\lim\limits_{x \to \infty} \frac{1}{x}\sqrt{8+9x^2} = ...$

(a) $2\sqrt{2}$ (b) 3 (c) $-2\sqrt{2}$ (d) -3

(14) $\lim\limits_{x \to \infty} \frac{\sqrt[3]{64x^3+7x-2}}{3x+2} = ...$

(a) 4 (b) 3 (c) $\frac{2}{3}$ (d) $\frac{4}{3}$

(15) $\lim\limits_{x \to \infty} \frac{\sqrt[3]{8x^3+1}}{|x|} = ...$

(a) 2 (b) 8 (c) zero (d) 4

(16) $\lim\limits_{x \to \infty} \frac{\sqrt{9x^2+3}}{\sqrt[3]{1-8x^3}} = ...$

(a) $\frac{-2}{3}$ (b) $\frac{-9}{8}$ (c) $\frac{3}{2}$ (d) $\frac{-3}{2}$

(17) $\lim\limits_{x \to \infty} \frac{4-3x^3}{\sqrt{x^6+9}} = ...$

(a) -3 (b) 1 (c) zero (d) ∞

(18) $\lim\limits_{x \to \infty} \frac{\sqrt{x+2}+\sqrt{x+3}}{\sqrt{2x+5}} = ...$

(a) $\frac{1}{2}$ (b) 1 (c) $\sqrt{2}$ (d) $\frac{1}{\sqrt{5}}$

(19) $\lim\limits_{x \to 4} \left(\frac{3x^2+2x+1}{x^2-3x+2}\right)^4 =$ ....

(a) 3 (b) 9 (c) 27 (d) 81

(20) $\lim\limits_{x \to \infty} x\left[\left(a+\frac{1}{x}\right)^n - a^n\right] =$ ....

(a) na (b) na^{n-1} (c) zero (d) does not exist.

(21) $\lim\limits_{x \to \infty} \frac{(12)^{\frac{1}{x}}}{x+7} =$ ....

(a) 1 (b) zero. (c) $\frac{12}{7}$ (d) ∞

(22) $\lim\limits_{x \to \infty} \frac{k^{\frac{1}{x}}}{3} =$ .... where k is a positive constant.

(a) $\frac{k}{3}$ (b) $\frac{1}{3}$ (c) 3 (d) 3 k

(23) If $\lim\limits_{x \to \infty} \frac{ax+6}{2x-7} = 4$, then a = ... where $a \in \mathbb{R}$

(a) 2 (b) 4 (c) 6 (d) 8

(24) If $\lim\limits_{x \to \infty} \frac{a^2x+7}{2x-5} = 8$, then a = ... where $a \in \mathbb{R}$

(a) 2 (b) zero. (c) ± 4 (d) ± 8

(25) If $\lim\limits_{x \to \infty} \frac{ax^2-5x}{2x+3x^2} = 3$, then a = ...

(a) 3 (b) 6 (c) 9 (d) 12

(26) If $m \in \mathbb{R}$ and $\lim\limits_{x \to \infty} \frac{(m+2)x^3-x+4}{2mx^3-2x+5} = 2$, then m = ...

(a) zero. (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) 1

(27) $\lim\limits_{x \to \infty} (3+2x-5x^2) =$ ....

(a) ∞ (b) $-\infty$ (c) zero (d) 7

(28) $\lim\limits_{x \to \infty} (x^3+7x^2+8) =$ ....

(a) ∞ (b) zero (c) $-\infty$ (d) 1

(29) $\lim\limits_{x \to \infty} \left(\sqrt{x^2+x} - x\right) =$ ....

(a) $\frac{1}{2}$ (b) zero. (c) $\sqrt{2}$ (d) does not exist.

(30) $\lim\limits_{x \to \infty} x^n = \text{zero if } n \in$ ....

(a) $\mathbb{N}$ (b) $\mathbb{Z}$ (c) $\mathbb{R}^+$ (d) $\mathbb{R}^-$

(31) $\lim\limits_{x \to \infty} \dfrac{\sqrt{4x + \sqrt{x + \sqrt{x}}}}{8\sqrt{x}} =$ ....

(a) 1 (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) $\frac{1}{8}$

(32) If : a > b , then $\lim\limits_{x \to \infty} \dfrac{x^a}{x^b} =$ ....

(a) zero (b) ∞ (c) 1 (d) a – b

(33) If : a < b < zero , then $\lim\limits_{x \to \infty} \dfrac{x^a}{x^b} =$ ....

(a) ∞ (b) $-\infty$ (c) zero (d) a – b

(34) If $f(x)$ is a polynomial function of the third degree and g (x) is a polynomial function of the fifth degree , then : $\lim\limits_{x \to \infty} \dfrac{g(x)}{x^2 f(x)} =$ ....

(a) $\pm\infty$ (b) zero.

(c) real number $\neq 0$ (d) has no existence.

Second Essay questions

1 Find each of the following :

(1) $\lim\limits_{x \to \infty} \dfrac{2x-5}{3x+8}$ « $\frac{2}{3}$ »

(2) $\lim\limits_{x \to \infty} \dfrac{2x-5}{3x^2+8}$ « zero »

(3) $\lim\limits_{x \to \infty} \dfrac{2x^2-5}{3x+8}$ « ∞ »

(4) $\lim\limits_{x \to \infty} \dfrac{4-2x}{3x-1}$ « $\frac{-2}{3}$ »

(5) $\lim\limits_{x \to \infty} \dfrac{5-6x-3x^2}{2x^2+x+4}$ « $-\frac{3}{2}$ »

(6) $\lim\limits_{x \to -\infty} \dfrac{x^3-2}{|x|^3+1}$ « 1 »

(7) $\lim\limits_{x \to \infty} \dfrac{6x-4x^3}{2-7x^3}$ « $\frac{4}{7}$ »

(8) $\lim\limits_{x \to \infty} \dfrac{4x^5+5}{8x^5+x^4-2}$ « $\frac{1}{2}$ »

(9) $\lim\limits_{x \to \infty} \dfrac{7x^2+1}{4x^3-8x+1}$ « zero »

(10) $\lim\limits_{x \to \infty} \dfrac{5x^7+2x-1}{6x^4+13}$ « ∞ »

(11) $\lim\limits_{x \to \infty} \dfrac{5-7x^8+3x^{14}}{7-6x^{14}+2x^6}$ « $-\frac{1}{2}$ »

(12) $\lim\limits_{x \to \infty} \left(\dfrac{7}{x^2} + \dfrac{2}{x} - 3\right)$ « -3 »

(13) $\lim\limits_{x \to \infty} \dfrac{5x^{-3}+4x^{-2}-3}{7x^{-3}-2x^{-2}+8}$ « $\frac{-3}{8}$ »

(14) $\lim\limits_{x \to \infty} \dfrac{5x^3-4x^2+2}{7-x+|2x|^3}$ « $\frac{5}{8}$ »

(15) $\lim\limits_{x \to \infty} (x^3+5x^2+1)$ « ∞ »

(16) $\lim\limits_{x \to \infty} \left[\sqrt{x^2+5x+7} + x\right]$ « ∞ »

(17) $\lim\limits_{x \to \infty} (5+x-x^2)$ « $-\infty$ »

2 Find each of the following :

(1) $\lim\limits_{x \to \infty} \frac{(3x+9)(2x+7)}{(x+1)(5x+4)}$ « $\frac{6}{5}$ »

(2) $\lim\limits_{x \to \infty} \frac{3x^2-4x+5}{(x+2)^2}$ « 3 »

(3) $\lim\limits_{x \to \infty} \frac{6x^2-5x}{(3-x)(2+x)}$ « −6 »

(4) $\lim\limits_{x \to \infty} \frac{8x^3-x+1}{(x+1)(2x^2-3)}$ « 4 »

(5) 📖 $\lim\limits_{x \to \infty} \frac{x^3-4x+5}{(2x-1)^3}$ « $\frac{1}{8}$ »

(6) $\lim\limits_{x \to \infty} \frac{(2x+3)(4x^2-5)}{(3x^2-8)(5x-3)}$ « $\frac{8}{15}$ »

(7) $\lim\limits_{x \to \infty} \frac{(2x+3)(5x-1)(x-2)}{x(x+1)(3x-1)}$ « $\frac{10}{3}$ »

(8) $\lim\limits_{x \to \infty} \frac{\left(7+\sqrt{x}\right)\left(3+\sqrt{x}\right)}{4x-3}$ « $\frac{1}{4}$ »

(9) $\lim\limits_{x \to \infty} \frac{(6x-5)(3x-4)^2}{(3x-2)^3}$ « 2 »

(10) 📖 $\lim\limits_{x \to \infty} \frac{(x+2)^3(3-2x^2)}{3x(x^2+7)^2}$ « $-\frac{2}{3}$ »

(11) $\lim\limits_{x \to \infty} \frac{(3x^2-1)^3}{18x^6+x^2+1}$ « $\frac{3}{2}$ »

3 Find each of the following :

(1) $\lim\limits_{x \to \infty} \frac{2x+3}{\sqrt{9x^2+25}}$ « $\frac{2}{3}$ »

(2) 📖 $\lim\limits_{x \to \infty} \frac{1}{x}\sqrt{3+4x^2}$ « 2 »

(3) $\lim\limits_{x \to \infty} \frac{2x+1}{\sqrt{4x^2+3x-4}}$ « 1 »

(4) $\lim\limits_{x \to \infty} \frac{2x-3}{\sqrt[3]{125x^3+5}}$ « $\frac{2}{5}$ »

(5) $\lim\limits_{x \to \infty} \frac{\sqrt[3]{8x^3+5x-2}}{3x+2}$ « $\frac{2}{3}$ »

(6) 📖 $\lim\limits_{x \to \infty} \frac{4-3x^3}{\sqrt{x^6+9}}$ « −3 »

(7) $\lim\limits_{x \to \infty} \frac{\sqrt{9x^2-3x+8}}{\sqrt[3]{3x^2+125x^3+2}}$ « $\frac{3}{5}$ »

(8) $\lim\limits_{x \to \infty} \frac{\sqrt{x^2+1}}{\sqrt[4]{x^4+2}}$ « 1 »

(9) $\lim\limits_{x \to \infty} \frac{\sqrt[3]{8x^3+1}}{\sqrt[5]{32x^5+x}}$ « 1 »

(10) $\lim\limits_{x \to \infty} \frac{\sqrt[3]{8x^2+1}}{\sqrt[6]{x^4-3}}$ « 2 »

(11) $\lim\limits_{x \to \infty} \frac{\sqrt{3x-2}-\sqrt{12x+7}}{\sqrt{27x-5}}$ « $-\frac{1}{3}$ »

(12) $\lim\limits_{x \to \infty} \frac{\sqrt{4x^2+7}+3x}{2x+9}$ « $\frac{5}{2}$ »

(13) $\lim\limits_{x \to \infty} \frac{\sqrt[4]{x^6+8}-\sqrt[6]{x^2+4}}{\sqrt{x^3+9}}$ « 1 »

(14) $\lim\limits_{x \to \infty} \frac{3x^2-x-7}{(x+1)\sqrt{25x^2+1}}$ « $\frac{3}{5}$ »

4 Find each of the following :

(1) 📖 $\lim\limits_{x \to \infty} \left(7+\frac{2x^2}{(x+3)^2}\right)$ « 9 »

(2) 📖 $\lim\limits_{x \to \infty} \left(\frac{x}{2x+1}+\frac{3x^2}{(x-3)^2}\right)$ « $\frac{7}{2}$ »

(3) 📖 $\lim\limits_{x \to \infty} \left(\frac{2}{3}-\frac{3x}{2x+7}\right)$ « $-\frac{5}{6}$ »

(4) $\lim\limits_{x \to \infty} \left(\frac{2x^3}{2x^2+1}-x\right)$ « zero »

(5) $\lim\limits_{x \to \infty} \left(\frac{x^2-1}{x+2} - \frac{x^2+1}{x-2}\right)$ « – 4 »

(6) $\lim\limits_{x \to \infty} \left(\sqrt{x^2-2} - \sqrt{x^2+x}\right)$ « $-\frac{1}{2}$ »

(7) $\lim\limits_{x \to \infty} \frac{1}{x}\left(\sqrt{4x^2+1} - \sqrt{x^2+1}\right)$ « 1 »

(8) $\lim\limits_{x \to \infty} \left(\sqrt{x^2+5x} - x\right)$ « $\frac{5}{2}$ »

(9) $\lim\limits_{x \to \infty} x\left(\sqrt{4x^2+1} - 2x\right)$ « $\frac{1}{4}$ »

(10) $\lim\limits_{x \to \infty} \frac{(2x-1)^5 (x^2+3)^6}{(x+1)^7 (x^2-5)^5}$ « 32 »

(11) $\lim\limits_{x \to \infty} \frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{4x+1}-\sqrt{4x-1}}$ « 2 »

5 **Find each of the following :**

(1) $\lim\limits_{x \to \infty} \left(\frac{3x^2-5x+1}{4x^2-7}\right)^{\frac{1}{x}}$ « 1 »

(2) $\lim\limits_{x \to \infty} \left(\frac{x+1}{\sqrt{x^2-1}} + a^{\frac{1}{x}}\right)$ where a is positive. « 2 »

(3) $\lim\limits_{x \to \infty} \left(\frac{2x-3}{\sqrt[3]{27x^3-15x+2}} + 8^{\frac{1}{x}}\right)$ « $1\frac{2}{3}$ »

(4) $\lim\limits_{x \to \infty} \left(\frac{5x^2}{1-x^2} - \sqrt[x]{13}\right)$ « – 6 »

6 **Find the value of each of a and n if :** $\lim\limits_{x \to \infty} \frac{4ax^n-4x+5}{3-9x+8x^2} = 3$ « 6 , 2 »

7 **Find the value of a if :** $\lim\limits_{x \to \infty} \frac{\sqrt[3]{ax^3+3}}{\sqrt{4x^2+7}} = -1$ « – 8 »

8 If $\lim\limits_{x \to \infty} \left(\sqrt{ax^2+3bx+5} - 2x\right) = 3$, find the value of each of : a and b « 4 , 4 »

9 **Find the value of each of a and b if :** $\lim\limits_{x \to \infty} f(x) = 5$ and $\lim\limits_{x \to -2} f(x) = 2$

where : $f(x) = \frac{2-ax^2}{3-bx+x^2}$ « – 5 , 2 »

10 If $\lim\limits_{x \to \infty} \left(\frac{x^2+1}{x+1} - ax - b\right) = 2$

, find the value of each of a and b « 1 , – 3 »

11 **Find the value of each of a and b if :** $\lim\limits_{x \to \infty} \frac{x^3+2x+1}{(a+1)x^4+(7-b)x^3+x^2} = \infty$ « – 1 , 7 »

Third **Higher skills**

Find each of the following :

(1) $\lim\limits_{x \to \infty} \dfrac{x^{-2} + x^{-4}}{(2x+3)^{-2}}$ « 4 »

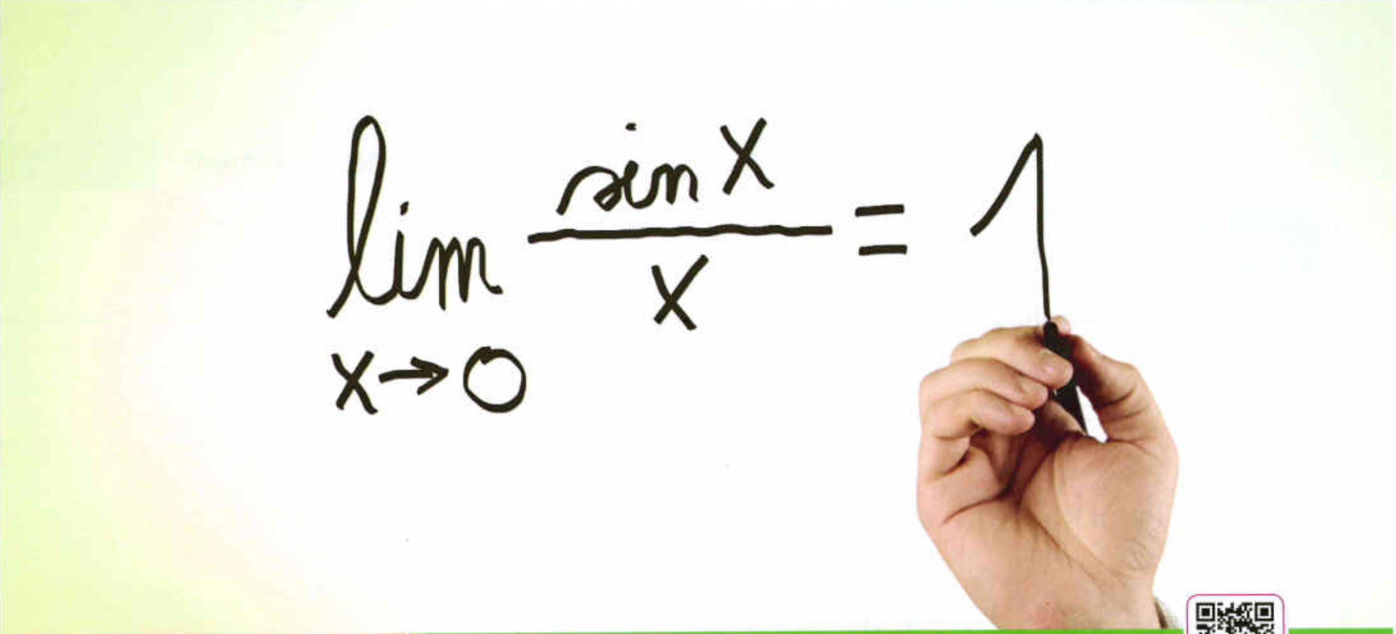
(2) $\lim\limits_{n \to \infty} n\left[\left(a + \frac{1}{n}\right)^4 - a^4\right]$ « $4a^3$ »

(3) $\lim\limits_{x \to \infty} x\left[\left(3 + \frac{1}{x}\right)^5 - 243\right]$ « 405 »

(4) $\lim\limits_{x \to \infty} \left(\sqrt{x^2+2} - x\right)\sqrt{2x^2+1}$ « $\sqrt{2}$ »

(5) $\lim\limits_{x \to \infty} \left(2x^{-1} - x^{-2}\right)\sqrt{4x^2+1}$ « 4 »

(6) $\lim\limits_{x \to \infty} \dfrac{5x\sqrt{x} + 16x - 3\sqrt{x^2-1}}{x + \sqrt{4x^3+1}}$ « $\frac{5}{2}$ »



Exercise 16 Limits of trigonometric functions

Test yourself

From the school book ● Understand ○ Apply Higher Order Thinking Skills

First Multiple choice questions

Choose the correct answer from those given :

(1) $\lim_{x \to 0} \tan x =$ ....

(a) zero (b) – 1 (c) $\frac{\pi}{2}$ (d) π

(2) $\lim_{x \to \frac{\pi}{2}} \sin 2x =$ ....

(a) zero (b) – 1 (c) 1 (d) $\frac{\pi}{2}$

(3) $\lim_{x \to 0} \frac{2x}{\sin 3x} =$ ....

(a) $\frac{2}{3}$ (b) $\frac{3}{2}$ (c) 6 (d) has no existence.

(4) $\lim_{x \to 0} \frac{\tan \pi x}{6x} =$ ....

(a) π (b) $\frac{\pi}{6}$ (c) $\frac{1}{6}$ (d) has no existence.

(5) $\lim_{x \to 0} \frac{\sin\left(\frac{3}{5}\sqrt[4]{x}\right)}{\sqrt[4]{x}} =$ ....

(a) zero (b) $\frac{3}{5}$ (c) $\frac{5}{3}$ (d) $\sqrt[4]{\frac{3}{5}}$

(6) $\lim_{(2x-3) \to 0} \frac{\tan(2x-3)}{2x-3} =$ ....

(a) zero (b) – 1 (c) 1 (d) does not exist.

(7) $\lim_{h \to 0} \frac{\sin 3h^2}{4h^2} =$ ....

(a) zero (b) 1 (c) $\frac{4}{3}$ (d) $\frac{3}{4}$

(8) $\lim\limits_{x \to 0} \dfrac{3x^5}{\tan 4x^5} = \dots\dots\dots$

(a) $\frac{3}{4}$ (b) $\frac{4}{3}$ (c) 1 (d) zero.

(9) 📖 $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{3x} = \dots\dots\dots$

(a) zero. (b) 1 (c) -1 (d) does not exist.

(10) $\lim\limits_{x \to 0} \dfrac{4 + 5x}{\cos 3x} = \dots\dots\dots$

(a) -4 (b) 4 (c) 3 (d) -3

(11) $\lim\limits_{x \to 0} \dfrac{\sin \frac{x}{3}}{2x} = \dots\dots\dots$

(a) $\frac{1}{3}$ (b) $\frac{1}{2}$ (c) $\frac{1}{6}$ (d) $\frac{1}{12}$

(12) 📖 $\lim\limits_{x \to 0} \dfrac{2x + \sin 3x}{\tan 5x} = \dots\dots\dots$

(a) 5 (b) $\frac{6}{5}$ (c) 1 (d) zero.

(13) $\lim\limits_{x \to 0} \dfrac{\sin 2x + 5 \sin 3x}{x} = \dots\dots\dots$

(a) 7 (b) 5 (c) 17 (d) 10

(14) $\lim\limits_{x \to 0} \dfrac{2x + \sin 3x}{5x + \tan 2x} = \dots\dots\dots$

(a) 1 (b) $\frac{5}{7}$ (c) $\frac{7}{5}$ (d) -1

(15) $\lim\limits_{x \to 0} \dfrac{\sin 3x}{5x^2 - 4x} = \dots\dots\dots$

(a) $\frac{3}{4}$ (b) $\frac{-3}{4}$ (c) 3 (d) $\frac{3}{5}$

(16) $\lim\limits_{x \to 0} \dfrac{x \sin 2x}{x^2} = \dots\dots\dots$

(a) zero. (b) 1 (c) 2 (d) 4

(17) $\lim\limits_{x \to 0} \dfrac{\sin 2x \cos 3x}{6x} = \dots\dots\dots$

(a) 1 (b) 3 (c) $\frac{1}{3}$ (d) zero.

(18) $\lim \dfrac{\sin 3x}{\tan \frac{1}{3}x} = \dots\dots\dots$

(a) 9 (b) $\frac{1}{3}$ (c) 3 (d) 1

(19) 📖 $\lim\limits_{x \to 0} \dfrac{\sin \frac{1}{2}x}{\sin \frac{3}{4}x} = \dots\dots\dots$

(a) $\frac{1}{6}$ (b) $\frac{3}{8}$ (c) $\frac{1}{2}$ (d) $\frac{2}{3}$

(20) $\lim\limits_{x \to 0} \frac{\tan^2 2x}{x \sin 3x} = \ldots\ldots$

(a) $\frac{4}{9}$ (b) $\frac{1}{2}$ (c) $\frac{2}{3}$ (d) $\frac{4}{3}$

(21) $\lim\limits_{x \to 0} \frac{x^2 + \sin^2 2x}{x \tan 2x} = \ldots\ldots$

(a) $\frac{2}{5}$ (b) $\frac{5}{2}$ (c) 1 (d) 5

(22) $\lim\limits_{x \to 0} \left(\frac{\sin 2x}{3x} + \frac{x \sin x^2}{\sin x^3}\right) = \ldots\ldots$

(a) 1 (b) $\frac{2}{3}$ (c) $\frac{4}{3}$ (d) $\frac{5}{3}$

(23) $\lim\limits_{x \to 0} \frac{\sin^2 9x + \tan 16x^2}{4x^2} = \ldots\ldots$

(a) $\frac{97}{4}$ (b) $\frac{97}{16}$ (c) $\frac{9}{4}$ (d) 4

(24) $\lim\limits_{x \to 0} 3x \csc 2x = \ldots\ldots$

(a) 6 (b) $\frac{3}{2}$ (c) $\frac{2}{3}$ (d) has no existence.

(25) $\lim\limits_{x \to 0} \frac{\sin 2x \tan 3x}{4x^2} = \ldots\ldots$

(a) $\frac{1}{2}$ (b) $\frac{3}{4}$ (c) $\frac{3}{2}$ (d) 6

(26) $\lim\limits_{x \to 0} \frac{1 - \tan x}{\sin x - \cos x} = \ldots\ldots$

(a) 1 (b) -1 (c) zero. (d) has no existence.

(27) $\lim\limits_{x \to 0} \frac{12 - 12\cos x}{x} = \ldots\ldots$

(a) zero (b) 12 (c) 1 (d) 24

(28) $\lim\limits_{x \to 0} \frac{\sin x - \sin x \cos x}{x^2} = \ldots\ldots$

(a) zero (b) 1 (c) 2 (d) 3

(29) $\lim\limits_{x \to 0} \frac{\cos x + 2x - 1}{3x} = \ldots\ldots$

(a) 1 (b) $\frac{2}{3}$ (c) $\frac{1}{3}$ (d) 0

(30) $\lim\limits_{x \to 0} \frac{1 - \cos x + \sin 3x}{1 - \cos x + \tan 2x} = \ldots\ldots$

(a) zero (b) $\frac{3}{2}$ (c) 1 (d) $\frac{4}{3}$

(31) $\lim\limits_{x \to 0} \frac{1 - \sec x}{\cos x - 1} = \ldots\ldots$

(a) 2 (b) 1 (c) 0 (d) -1

(32) If : $\lim\limits_{x \to 0} \frac{(a+3)x}{\sin ax} = \frac{2}{5}$, then : a = ……

(a) -5 (b) -3 (c) -1 (d) 3

(33) If : $\lim\limits_{x \to 0} \frac{\sin^2 a x - \tan 2 a x^2}{x^2} = -1$, then a = ...

(a) – 1 (b) 2 (c) 1 (d) 4

(34) If : $\lim\limits_{x \to 0} \frac{\sin a x}{b x} = \frac{1}{3}$, $\lim\limits_{x \to 0} \frac{\tan b x}{c x} = \frac{4}{3}$, then : $\frac{a}{c}$ = ...

(a) – 4 (b) $\frac{1}{4}$ (c) $\frac{4}{9}$ (d) $\frac{1}{12}$

(35) If : $\lim\limits_{x \to 0} \frac{\sin a x}{\sin b x} = \frac{3}{5}$, then : $\frac{a+b}{a-b}$ = ...

(a) – 4 (b) 8 (c) 2 (d) 4

(36) $\lim\limits_{x \to 0} \frac{(\sin x + \cos x)^2 - 1}{3 x}$ = ...

(a) 1 (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) zero.

(37) $\lim\limits_{x \to \frac{\pi}{2}} \frac{1 - \sin^3 x}{\cos^2 x}$ = ...

(a) 1 (b) $\frac{3}{2}$ (c) $\frac{3}{4}\pi$ (d) zero.

(38) $\lim\limits_{x \to 1} \frac{\tan (x - 1)}{\sqrt{x} - 1}$ = ...

(a) 2 (b) zero. (c) – 2 (d) $\frac{1}{2}$

(39) $\lim\limits_{x \to 1} \frac{\sin (x - 1)}{x^2 - 1}$ = ...

(a) $\frac{1}{2}$ (b) 1 (c) – 1 (d) zero.

(40) $\lim\limits_{x \to 3} \frac{\tan (x - 3)}{x^2 - x - 6}$ = ...

(a) $\frac{1}{5}$ (b) $\frac{8}{3}$ (c) 3 (d) – 3

(41) $\lim\limits_{x \to \frac{\pi}{2}} \frac{3 \cos x}{\pi - 2 x}$ = ...

(a) 3 (b) $\frac{3}{2}$ (c) $\frac{-3}{2}$ (d) – 3

(42) 📖 $\lim\limits_{x \to \pi} \frac{\sin x}{\pi - x}$ = ...

(a) 1 (b) π^2 (c) π (d) $-\pi$

(43) $\lim\limits_{x \to 0} \frac{1 - \cos x}{x^2}$ = ...

(a) 1 (b) does not exist. (c) 0 (d) $\frac{1}{2}$

(44) $\lim\limits_{x \to 0} \frac{\sin (\sin x)}{3 x}$ = ...

(a) zero. (b) 1 (c) $\frac{1}{3}$ (d) 3

(45) 📖 $\lim\limits_{x \to 0} \frac{\sin x}{x}$ = ... , where x is in degrees.

(a) 1 (b) $\frac{\pi}{180}$ (c) $\frac{180}{\pi}$ (d) π

UNIT 3

Understand | Apply | Higher Order Thinking Skills

Second Essay questions

1 Find each of the following :

(1) $\lim\limits_{x \to 0} \frac{\sin 2x}{\tan 5x}$ «$\frac{2}{5}$»

(2) $\lim\limits_{x \to 0} \frac{\sin \frac{x}{5}}{\tan \frac{2x}{3}}$ «$\frac{3}{10}$»

(3) $\lim\limits_{x \to 0} \frac{\sin 3x - \sin 2x}{5x}$ «$\frac{1}{5}$»

(4) $\lim\limits_{x \to 0} \frac{\sin x + \tan x}{\sin x}$ «2»

(5) $\lim\limits_{x \to 0} \frac{x \cos 3x}{\tan 2x}$ «$\frac{1}{2}$»

(6) $\lim\limits_{x \to 0} \left(\frac{5x}{\sin x} - \frac{\tan 2x}{x}\right)$ «3»

(7) $\lim\limits_{x \to 0} \frac{3x + \sin x}{2x + \tan 3x}$ «$\frac{4}{5}$»

(8) $\lim\limits_{x \to 0} \frac{2 \sin x + 4 \tan 2x}{2x + \sin 3x}$ «2»

(9) $\lim\limits_{x \to 0} \frac{\sin x - 3 \tan x}{5x \cos x}$ «$\frac{-2}{5}$»

(10) $\lim\limits_{x \to 0} \frac{3x}{5 \tan x - 2 \sin 2x}$ «3»

(11) $\lim\limits_{x \to 0} \frac{x + x \cos x}{\sin x \cos x}$ «2»

(12) $\lim\limits_{x \to 0} \frac{\tan 3x - \sin 4x}{x^2 + 2x}$ «$-\frac{1}{2}$»

(13) $\lim\limits_{x \to 0} \frac{x^2 + \sin 3x}{5x \cos 2x}$ «$\frac{3}{5}$»

(14) $\lim\limits_{x \to 0} \frac{\tan 2x + 2x \cos x}{\sin 3x}$ «$\frac{4}{3}$»

(15) $\lim\limits_{x \to 0} \frac{2}{x} \sin \frac{x}{7}$ «$\frac{2}{7}$»

(16) $\lim\limits_{x \to 0} \frac{1}{x}\left(\sin \frac{x}{5} + \tan \frac{x}{5}\right)$ «$\frac{2}{5}$»

(17) $\lim\limits_{x \to 0} \left(\frac{3}{5x} + 4\right) \sin 4x$ «$\frac{12}{5}$»

(18) $\lim\limits_{x \to 3} \frac{\sin (3x - 9)}{2x - 6}$ «$\frac{3}{2}$»

(19) $\lim\limits_{x \to 0} \frac{\sin 24x \times \cos 6x}{\tan 6x \times \cos 24x}$ «4»

(20) $\lim\limits_{x \to \frac{1}{2}} \frac{x \cos (-2x + 1)}{x^2 + x}$ «$\frac{2}{3}$»

(21) $\lim\limits_{x \to 0} \frac{\sin^2 x}{x}$ «zero»

2 Find each of the following :

(1) $\lim\limits_{x \to 0} \frac{\sin^2 3x}{5x^2}$ «$\frac{9}{5}$»

(2) $\lim\limits_{x \to 0} \frac{\sin^2 3x}{\tan^2 2x}$ «$\frac{9}{4}$»

(3) $\lim\limits_{x \to 0} \frac{\sin^2 \frac{x}{2}}{x \tan 2x}$ «$\frac{1}{8}$»

(4) $\lim\limits_{x \to 0} \frac{x \sin 2x}{\sin x \tan 3x}$ «$\frac{2}{3}$»

(5) $\lim\limits_{x \to 0} \frac{\tan^3 2x}{4x^3}$ «2»

(6) $\lim\limits_{x \to 0} \frac{x^3 \tan 2x}{\sin^4 2x}$ «$\frac{1}{8}$»

(7) $\lim\limits_{x \to 0} \frac{\sin 4x \tan^2 5x}{x^2 \sin x}$ «100»

(8) $\lim\limits_{x \to 0} \frac{x \tan 2x}{x^2 + \sin^2 3x}$ «$\frac{1}{5}$»

(9) $\lim\limits_{x \to 0} \dfrac{x^2 + \tan^2 2x}{2x^2 + \sin^2 3x}$ « $\frac{5}{11}$ »

(10) $\lim\limits_{x \to 0} \dfrac{x \tan x}{4x^2 - \sin^2 3x}$ « $\frac{-1}{5}$ »

(11) 📖 $\lim\limits_{x \to 0} \dfrac{2x^3 + x \sin 5x}{x^2 - \tan 3x^2}$ « $-\frac{5}{2}$ »

(12) $\lim\limits_{x \to 0} \dfrac{2x + 5 \tan x + 3 \sin x}{\sin 2x \cos 2x}$ « 5 »

(13) 📖 $\lim\limits_{x \to 0} \dfrac{x \sin 2x + \sin^2 2x}{\tan^2 3x + x^2}$ « $\frac{3}{5}$ »

(14) 📖 $\lim\limits_{x \to 0} \dfrac{\tan 3x^2 + \sin^2 5x}{x^2}$ « 28 »

(15) 📖 $\lim\limits_{x \to 0} \dfrac{\sin 5x^3 + \sin^3 5x}{2x^3}$ « 65 »

(16) 📖 $\lim\limits_{x \to 0} \left(\dfrac{2x^2 + \sin 3x}{2x^2 + \tan 6x}\right)^4$ « $\frac{1}{16}$ »

(17) $\lim\limits_{x \to 0} \dfrac{\tan^2 x + \tan^2 3x + \tan^2 5x}{x^2}$ « 35 »

(18) $\lim\limits_{x \to 0} \dfrac{\sin x \sin^2 2x \sin^3 4x}{x^6}$ « 256 »

(19) $\lim\limits_{x \to 0} \dfrac{\tan 6x - \sin(-3x)}{x(\cos 5x + \cos 2x)}$ « $\frac{9}{2}$ »

(20) $\lim\limits_{x \to 0} \dfrac{\sin x}{\sqrt{x+2} - \sqrt{2}}$ « $2\sqrt{2}$ »

(21) $\lim\limits_{x \to 3} \dfrac{\tan(x-3)}{x^3 - 27}$ « $\frac{1}{27}$ »

(22) 📖 $\lim\limits_{x \to 1} \dfrac{\sin(x-1)}{x^2 + x - 2}$ « $\frac{1}{3}$ »

3 **Find each of the following :**

(1) 📖 $\lim\limits_{x \to 0} \dfrac{3(1 - \cos x)}{x}$ « 0 »

(2) 📖 $\lim\limits_{x \to 0} \dfrac{\cos x - 1}{\sin x}$ « 0 »

(3) 📖 $\lim\limits_{x \to 0} \dfrac{1 - \cos x + \sin x}{1 - \cos x - \sin x}$ « −1 »

(4) 📖 $\lim\limits_{x \to 0} \dfrac{2 - \cos 3x - \cos 4x}{x}$ « 0 »

(5) 📖 $\lim\limits_{x \to 0} \dfrac{\sin x(1 - \cos x)}{x^2}$ « 0 »

(6) 📖 $\lim\limits_{x \to 0} \dfrac{x - x \cos x}{\sin^2 3x}$ « 0 »

(7) $\lim\limits_{x \to 0} \dfrac{1 - \cos^2 3x}{4x^2}$ « $\frac{9}{4}$ »

(8) $\lim\limits_{x \to 0} \dfrac{4x^2}{1 - \cos^2 \frac{1}{2} x}$ « 16 »

(9) $\lim\limits_{x \to 0} \dfrac{3 - 3\cos^2 4x}{8x^2}$ « 6 »

(10) $\lim\limits_{x \to 0} \dfrac{\sec^2 x - 1}{x^2}$ « 1 »

(11) 📖 $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x^2}$ « $\frac{1}{2}$ »

(12) $\lim\limits_{x \to 0} \dfrac{4 - 4\cos x}{\sin^2 3x}$ « $\frac{2}{9}$ »

(13) $\lim\limits_{x \to 0} \dfrac{1 - \cos 2x}{x \tan 2x}$ « 1 »

(14) 📖 $\lim\limits_{x \to 0} \dfrac{1 - \cos 3x}{\cos^2 2x - 1}$ « $-\frac{9}{8}$ »

4 **Find each of the following :**

(1) $\lim\limits_{x \to 0} \cot 3x \sin 5x$ « $\frac{5}{3}$ »

(2) 📖 $\lim\limits_{x \to 0} x(\csc 2x - \cot 3x)$ « $\frac{1}{6}$ »

(3) $\lim\limits_{x \to 0} 3x \tan 2x \csc^2 3x$ « $\frac{2}{3}$ »

(4) 📖 $\lim\limits_{x \to 0} 6x^2 \csc 2x \cot x$ « 3 »

(5) $\lim\limits_{x \to 0} \csc 4x \cos\left(\frac{\pi}{2} - x\right)$ « $\frac{1}{4}$ »

(6) 📖 $\lim\limits_{x \to 0} \dfrac{x}{\cos\left(\frac{\pi}{2} - x\right)}$ « 1 »

(7) $\lim\limits_{x \to 0} \frac{x \cot^2 2x}{\csc 3x}$ «$\frac{3}{4}$»

(8) $\lim\limits_{x \to \frac{\pi}{4}} \frac{\tan\left(2x - \frac{\pi}{2}\right)}{8x - 2\pi}$ «$\frac{1}{4}$»

(9) $\lim\limits_{x \to \frac{\pi}{4}} \frac{\sin\left(2x - \frac{\pi}{2}\right)}{\tan(4x - \pi)}$ «$\frac{1}{2}$»

(10) $\lim\limits_{x \to 0} \frac{\cos\left(\frac{\pi}{2} - x\right)}{\cot\left(\frac{\pi}{2} - 3x\right)}$ «$\frac{1}{3}$»

(11) $\lim\limits_{(x-\pi) \to 0} \frac{\sin x}{x - \pi}$ «-1»

(12) $\lim\limits_{x \to -\pi} \frac{\tan x}{x + \pi}$ «1»

(13) $\lim\limits_{x \to \frac{\pi}{2}} \frac{2x - \pi}{\cos x}$ «-2»

(14) $\lim\limits_{x \to \frac{\pi}{2}} \frac{\cot x}{\pi - 2x}$ «$\frac{1}{2}$»

(15) $\lim\limits_{x \to 1} \frac{\sin x\pi}{1 - x}$ «π»

(16) $\lim\limits_{x \to -1} \frac{1 + x}{\cos \frac{\pi}{2} x}$ «$\frac{2}{\pi}$»

(17) $\lim\limits_{x \to 1} (1 - x) \tan \frac{\pi x}{2}$ «$\frac{2}{\pi}$»

5 **If $f(x) = \frac{\sin x}{x}$, then find :**

(1) $\lim\limits_{x \to 0} f(x)$ «1»

(2) $\lim\limits_{x \to \frac{\pi}{2}} f(x)$ «$\frac{2}{\pi}$»

(3) $\lim\limits_{x \to \pi} f(x)$ «0»

6 **Find each of the following limits :**

(1) $\lim\limits_{x \to 5} \frac{\sin(x-5)}{x^2 - 25}$ «$\frac{1}{10}$»

(2) $\lim\limits_{x \to 5} \frac{\sin^2(x-5)}{(x-5)^2}$ «1»

(3) $\lim\limits_{x \to 5} \frac{\sin(x-5)^2}{x-5}$ «0»

Third Higher skills

1 **Choose the correct answer from those givens :**

(1) If : $\lim\limits_{x \to 0} \frac{\cos x - \cos^a x}{x^2} = 1$, then : a = ...

(a) 1 (b) zero (c) 2 (d) 3

(2) $\lim\limits_{x \to 0} x \sec\left(\frac{2x + \pi}{2}\right) =$ ....

(a) $\frac{\pi}{2}$ (b) zero (c) -1 (d) $\frac{1}{2}$

(3) $\lim\limits_{x \to \frac{\pi}{4}} \frac{1 - \tan x}{\cos x - \sin x} =$ ....

(a) 1 (b) -1 (c) 2 (d) $\sqrt{2}$

(4) $\lim\limits_{x \to \sqrt{5}} \frac{\sin (x^2 - 5)}{\tan \pi x^2} = \dots\dots\dots\dots$

(a) $\frac{5}{\pi}$ (b) $\frac{2}{\pi}$ (c) $\frac{1}{\pi}$ (d) $\frac{-5}{\pi}$

(5) $\lim\limits_{x \to 0} \frac{\sin x + \sin 2x + \sin 3x + \cdots + \sin 10x}{\tan x + \tan 2x + \tan 3x + \cdots + \tan 10x} = \dots\dots\dots\dots$

(a) zero (b) 1 (c) 10 (d) 55

2 Find each of the following :

(1) $\lim\limits_{x \to \frac{\pi}{2}} \frac{3 \tan^2 x - 4 \tan x + 7}{5 + 2 \tan^2 x}$ «$\frac{3}{2}$»

(2) $\lim\limits_{x \to -1} \frac{\sin (x^2 - x - 2)}{x + 1}$ « -3 »

(3) $\lim\limits_{x \to \infty} x \left(\tan \frac{4}{x} + \sin \frac{3}{x} \right)$ « 7 »

(4) $\lim\limits_{x \to 0} \frac{\sin (\sin x)}{5 \sin x}$ «$\frac{1}{5}$»

(5) $\lim\limits_{x \to 0} \frac{\sin (1 - \cos x)}{1 - \cos x}$ « 1 »

(6) $\lim\limits_{x \to 0} \frac{\tan (\tan 5x)}{4x}$ «$\frac{5}{4}$»

(7) $\lim\limits_{x \to 0} \frac{1}{x} \left(\frac{2 \tan x}{1 + \tan^2 x} \right)$ « 2 »

(8) $\lim\limits_{x \to \frac{\pi}{2}} \frac{\tan (\cos x)}{2x - \pi}$ «$-\frac{1}{2}$»

(9) $\lim\limits_{x \to \frac{\pi}{2}} \frac{1 - \sin x}{2x - \pi}$ « zero »

(10) $\lim\limits_{2x \to \pi} \left(\frac{\pi}{2} - x \right) \cot (\pi - 2x)$ «$\frac{1}{2}$»

(11) $\lim\limits_{x \to \frac{\pi}{2}} \frac{\cos x}{\cos 3x}$ «$-\frac{1}{3}$»



Exercise 17 Existence of the limit of a piecewise function

Test yourself

From the school book ● Understand ● Apply ● Higher Order Thinking Skills

First Multiple choice questions

Choose the correct answer from those given :

(1) If $f(x) = \begin{cases} -2 & , \quad x > 0 \\ 1 & , \quad x < 0 \end{cases}$, then :

First : $\lim\limits_{x \to -3} f(x) =$ ....

(a) -2 (b) 1 (c) -1 (d) does not exist.

Second : $\lim\limits_{x \to 4} f(x) =$ ....

(a) -2 (b) 1 (c) -1 (d) does not exist.

Third : $\lim\limits_{x \to 0} f(x) =$ ....

(a) -2 (b) 1 (c) -1 (d) does not exist.

(2) If $f(x) = \begin{cases} x^2 - 1 & , \quad x > 2 \\ 3x + 1 & , \quad x \leq 2 \end{cases}$, then $\lim\limits_{x \to 3} f(x) =$ ....

(a) does not exist. (b) 3 (c) 8 (d) 7

(3) $\lim\limits_{x \to 2} \frac{|x-2|}{x-2} =$ .... , $\lim\limits_{x \to 1} \frac{|x-2|}{x-2} =$ ....

(a) does not exist , zero (b) does not exist, does not exist.

(c) does not exist , -1 (d) 1 , -1

(4) If $f(x) = \begin{cases} 3x - 1 & , \quad x \neq 2 \\ 6 & , \quad x = 2 \end{cases}$, then $\lim\limits_{x \to 2} f(x) =$ ....

(a) -5 (b) 5 (c) 6 (d) does not exist.

(5) If $f(x) = \begin{cases} 3 - x & , \quad x < 1 \\ 4 & , \quad x = 1 \\ x^2 + 1 & , \quad x > 1 \end{cases}$, then $\lim\limits_{x \to 1} f(x) = \text{.............}$

(a) 4 (b) 2 (c) 1 (d) does not exist.

(6) If $f(x) = \begin{cases} \frac{\sin x}{x} & , \quad -\frac{\pi}{2} < x < 0 \\ \cos x & , \quad 0 < x < \frac{\pi}{2} \end{cases}$, then $\lim\limits_{x \to 0} f(x) = \text{.............}$

(a) $\frac{2}{\pi}$ (b) $\frac{\pi}{2}$ (c) 1 (d) 0

(7) If $f(x) = \begin{cases} \frac{x^2 - 4}{x - 2} & , \quad x \neq 2 \\ \tan \frac{\pi}{2x} & , \quad x = 2 \end{cases}$, then : $\lim\limits_{x \to 2} f(x) = \text{.............}$

(a) 1 (b) $\sqrt{2}$ (c) 4 (d) does not exist.

(8) If $f(x) = \begin{cases} \frac{2 \sin x}{\pi - x} & , \quad x < \pi \\ 1 - \cos x & , \quad x > \pi \end{cases}$, then $\lim\limits_{x \to \pi} f(x) = \text{.............}$

(a) 1 (b) 2 (c) does not exist. (d) zero.

(9) 📖 If $f(x) = \begin{cases} \frac{\sin(x - 1)}{x - 1} & , \quad x < 1 \\ \tan \frac{\pi x}{4} & , \quad 1 < x < 2 \end{cases}$, then $\lim\limits_{x \to 1} f(x) = \text{.............}$

(a) 1 (b) $\frac{1}{4}$ (c) $\frac{\pi}{4}$ (d) does not exist.

(10) If $f(x) = \begin{cases} \frac{\sin(2x - 6)}{x - 3} & , \quad x < 3 \\ 2 \sin \frac{\pi x}{6} & , \quad x > 3 \end{cases}$, then $\lim\limits_{x \to 3} f(x) = \text{.............}$

(a) 2 (b) $\frac{1}{2}$ (c) $\frac{\pi}{2}$ (d) does not exist.

(11) If $f(x) = \begin{cases} \frac{2x + \sin x}{\tan 2x + 4x} & , \quad 0 < x < \frac{\pi}{4} \\ 5x + \frac{1}{2} & , \quad x < 0 \end{cases}$, then $\lim\limits_{x \to 0} f(x) = \text{.............}$

(a) $\frac{3}{5}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) does not exist.

(12) If $f(x) = \begin{cases} x + 1 & \text{at} \quad x > a \\ 3x - 1 & \text{at} \quad x < a \end{cases}$, $\lim\limits_{x \to a} f(x)$ exist , then a = ...

(a) zero. (b) 1 (c) 2 (d) 3

(13) If $f(x) = \begin{cases} \sin 4x \cot 9x & , -\frac{\pi}{18} < x < 0 \\ 4x + a^2 & , x > 0 \end{cases}$ and $\lim\limits_{x \to 0} f(x)$ exists

, then a = ...

(a) $\frac{4}{9}$ (b) $\frac{2}{3}$ (c) $\pm \frac{2}{3}$ (d) $\pm \frac{4}{9}$

(14) If $f(x) = \begin{cases} a + \cos x & , x < 0 \\ \frac{\tan 2x}{a x} & , 0 < x < \frac{\pi}{4} \end{cases}$ and $\lim\limits_{x \to 0} f(x)$ exists , then a = ...

(a) zero or 1 (b) 1 or – 2 (c) 2 or 3 (d) 1 or 2

(15) If $f(x) = \begin{cases} \frac{\tan 2x}{\log_2 8^x} & , -\frac{\pi}{4} < x < 0 \\ x + \frac{2}{3} & , x > 0 \end{cases}$, then $\lim\limits_{x \to 0} f(x) =$ ....

(a) 2 (b) $\frac{2}{3}$ (c) $\frac{3}{2}$ (d) does not exist.

(16) If $f(x) = \begin{cases} \frac{\cot x}{\pi - 2x} & , 0 < x < \frac{\pi}{2} \\ \sin(\pi - x) & , x > \frac{\pi}{2} \end{cases}$, then $\lim\limits_{x \to \frac{\pi}{2}} f(x) =$ ....

(a) 2 (b) $\frac{\pi}{2}$ (c) π (d) does not exist.

(17) If $f(x) = \begin{cases} \frac{\sin^2 2x}{x^2} & , x < 0 \\ 2a + 3\cos x & , x > 0 \end{cases}$ and $\lim\limits_{x \to 0} f(x)$ exists , then a = ...

(a) $\frac{1}{2}$ (b) zero (c) 2 (d) $-\frac{1}{2}$

(18) If $f(x) = \begin{cases} \frac{3x - |x^3|}{x} & , x < 0 \\ 3a + 4x & , x > 0 \end{cases}$ and $\lim\limits_{x \to 0} f(x)$ exists

, then a = ...

(a) zero (b) $-\frac{1}{3}$ (c) $\frac{4}{3}$ (d) 1

(19) $\lim\limits_{x \to 0^-} \frac{\sqrt{1 - \cos^2 x}}{x} =$ ....

(a) 1 (b) – 1 (c) 2 (d) zero

(20) If $f(x) = \frac{2x - 6}{|x - 3|}$, then $\left(f(3^+)\right)^2 + \left(f(3^-)\right)^2 =$ ....

(a) 8 (b) zero. (c) 4 (d) – 4

(21) If $\lim\limits_{x \to a} \dfrac{x^2 - a^2}{x - a} = \lim\limits_{x \to -2} (|x| + 8)$, then a = ...

(a) 5 (b) – 5 (c) 4 (d) 10

(22) If $f(x) = \begin{cases} \dfrac{x^6 - 64}{x^4 - 16} & , \quad x < 2 \\ a\,x & , \quad x > 2 \end{cases}$

where $\lim\limits_{x \to 2} f(x)$ exists , then the value of a = ...

(a) 1 (b) 2 (c) 3 (d) 6

(23) If $f(x) = \begin{cases} a\cos x + \dfrac{15 \sin x}{x} & , \quad x > 0 \\ \dfrac{x^6 - 64}{x^3 - 8} & , \quad x < 0 \end{cases}$

and $f(x)$ has a limit when $x \longrightarrow 0$, then a = ...

(a) 1 (b) – 5 (c) – 7 (d) 7

(24) If $f(x) = \begin{cases} 3x^2 + a\,x - 2 & , \quad x > 3 \\ 2x + b & , \quad x < 3 \end{cases}$

has a limit at $x = 3$ and equals 16 , then a + b = ...

(a) 4 (b) 10 (c) – 13 (d) 7

(25) If the function f has a limit at $x \longrightarrow -3$ where

$f(x) = \begin{cases} \dfrac{x^2 + 2x - 3}{x + 3} & \text{at} \quad x < -3 \\ x + a & \text{at} \quad x > -3 \end{cases}$, then a = ...

(a) 3 (b) 4 (c) 2 (d) – 1

(26) If $f(x) = \begin{cases} 3x - 2 & , \quad x < -1 \\ a\,x + b & , \quad -1 < x < 3 \\ 6 - x & , \quad x > 3 \end{cases}$

has a limit at $x = -1$, $x = 3$, then a + b = ...

(a) 2 (b) – 3 (c) – 1 (d) zero.

(27) If the function $f : f(x) = \begin{cases} \dfrac{a\,x}{\sqrt{x + 4} - 2} & , \quad x > 0 \\ \dfrac{\sin 2x}{a\,x} & , \quad x < 0 \end{cases}$ has a limit at $x = 0$

, then a = ...

(a) $\pm \frac{1}{2}$ (b) $\pm \frac{1}{\sqrt{3}}$ (c) $\pm \frac{1}{\sqrt{2}}$ (d) $\frac{1}{4}$

UNIT 3

Understand   Apply   Higher Order Thinking Skills

Second Essay questions

1 If $f(x)=\begin{cases} x^2+1 & , x<3 \\ 3x+1 & , x\geq 3 \end{cases}$, find : $\lim\limits_{x\to 3} f(x)$ « 10 »

---

2 If $f(x)=\begin{cases} \dfrac{x^2-7x+12}{x-3} & , x>3 \\ 2x-7 & , x<3 \end{cases}$

, discuss the existence of : $\lim\limits_{x\to 3} f(x)$ « − 1 »

---

3 If $f(x)=\begin{cases} \dfrac{3x}{\sin x} & , -\pi<x<0 \\ \cos 3x & , x>0 \end{cases}$

, discuss the existence of : $\lim\limits_{x\to 0} f(x)$ « does not exist »

---

4 If $f(x)=\begin{cases} \dfrac{5x+\tan 2x}{6x+\sin x} & , 0<x<\dfrac{\pi}{4} \\ \cos x & , x<0 \end{cases}$

, discuss the existence of : $\lim\limits_{x\to 0} f(x)$ « 1 »

---

5 If $f(x)=\begin{cases} x-2 & , x<0 \\ x^2 & , 0\leq x\leq 2 \\ 2x & , x>2 \end{cases}$, discuss the existence of :

(1) $\lim\limits_{x\to 0} f(x)$ | (2) $\lim\limits_{x\to 1} f(x)$

(3) $\lim\limits_{x\to 2} f(x)$ « does not exist , 1 , 4 »

---

6 If $f(x)=\begin{cases} |x-3| & , x\neq 3 \\ 2 & , x=3 \end{cases}$, discuss the existence of : $\lim\limits_{x\to 3} f(x)$ « zero »

---

7 If $f(x)=\begin{cases} \dfrac{x}{\sqrt{x+1}-1} & , x>0 \\ \dfrac{2x}{\sin x} & , -\pi<x<0 \end{cases}$, find : $\lim\limits_{x\to 0} f(x)$ « 2 »

---

8 If $f(x)=\begin{cases} \dfrac{(x-3)^6-1}{x-4} & , x>4 \\ x+2 & , x<4 \end{cases}$, find : $\lim\limits_{x\to 4} f(x)$ « 6 »

9 If $\lim_{x \to 2} f(x) = 7$ where $f(x) = \begin{cases} x^2 + 3m &, x < 2 \\ 5x + k &, x > 2 \end{cases}$

, **find the values of :** m and k « 1 , −3 »

---

10 If $\lim_{x \to 0} f(x) = 2$ where $f(x) = \begin{cases} \frac{\sin 2x}{x} &, x > 0 \\ a \cos 2x &, x < 0 \end{cases}$

, **find the value of :** a « 2 »

---

11 If $\lim_{x \to a} |3x + 2| = 14$, **find the value of :** a « 4 or $\frac{-16}{3}$ »

---

12 **Discuss the existence of the limit of the function :**

$f : f(x) = \begin{cases} \frac{3x}{\tan x} &, -\frac{\pi}{3} < x < 0 \\ 3 \cos x &, 0 < x < \frac{\pi}{3} \end{cases}$ **When :**

(1) $x \to \frac{-\pi}{3}$ (2) $x \to \frac{\pi}{3}$ (3) $x \to 0$

« does not exist , does not exist , 3 »

---

13 If $f(x) = x^2 + \frac{\sqrt{x^2 - 6x + 9}}{x - 3}$, **discuss the existence of :** $\lim_{x \to 3} f(x)$ « does not exist »

---

14 If $f(x) = \frac{x^2 + 2\sqrt{x^2}}{x}$, **discuss the existence of :** $\lim_{x \to 0} f(x)$ « does not exist »

---

15 If $f(x) = \begin{cases} x|x| + 2 &, x < 0 \\ \frac{|x|}{x} + 1 &, x > 0 \end{cases}$, **find :** $\lim_{x \to 0} f(x)$ « 2 »

---

16 If the function f where $f(x) = \begin{cases} \frac{(x-1)^2}{|x-1|} &, x < 1 \\ 6x - 3m &, x > 1 \end{cases}$

has a limit at $x = 1$, find the value of m « 2 »

---

17 **Discuss the existence of** $\lim_{x \to 1} [(f + g)(x)]$ **where :**

$f(x) = \begin{cases} x^2 + 3 &, x \geq 1 \\ 4 - 2x &, x < 1 \end{cases}$, $g(x) = \begin{cases} -3x &, x \geq 1 \\ 2x^2 - 3x &, x < 1 \end{cases}$ « 1 »

18 **If the function** $f : f(x) = \begin{cases} \dfrac{a\,x}{\sqrt{x+4}-2} & , \quad x > 0 \\ \dfrac{1}{a\,x \csc 3x} & , \quad -\pi < x < 0 \end{cases}$ has a limit when $x = 0$

, find the value of : a « $\pm\frac{\sqrt{3}}{2}$ »

19 **Find the value of the right and left limit , then deduce the value of the limit (if exist) for each of the following functions at the given point :**

(1) $f(x) = \sqrt{x-5}$ when $x = 5$

(2) $f(x) = \sqrt{3-x}$ when $x = 3$

(3) $f(x) = \sqrt{x^2-1}$ when $x = -1 , x = 1$

(4) $f(x) = \sqrt{4-x^2}$ when $x = -2 , x = 2$

20 **Discuss the existence of the limit of each of the following functions at the given point :**

(1) $f(x) = \begin{cases} \dfrac{(x+2)^6 + 64}{x^2 + 32x} & , \quad -5 < x < 0 \\ \dfrac{-12x}{1-(\sin x + \cos x)^2} & , \quad 0 < x < \dfrac{\pi}{2} \end{cases}$ when $x = 0$ « 6 »

(2) $f(x) = \begin{cases} \dfrac{1-\cos x}{\sin^2 3x} & , \quad -\dfrac{\pi}{3} < x < 0 \\ -\dfrac{1}{2}x^2 \cos 6x\,(1-\csc^2 3x) & , \quad 0 < x < \dfrac{\pi}{3} \end{cases}$ when $x = 0$ « $\frac{1}{18}$ »

21 **Discuss the existence of each of the following « where x in radian » :**

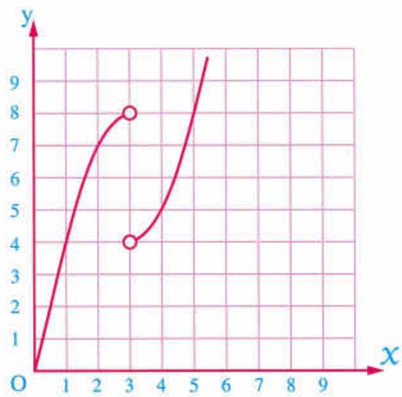
(1) $\lim\limits_{x \to 0} \dfrac{\sin|x|}{x}$ (2) $\lim\limits_{x \to 0} \dfrac{\cos|x|}{x+1}$ « not exist , 1 »

22 **Discuss the existence :** $\lim\limits_{x \to 0} \dfrac{\sqrt{\sec^2 x - 1}}{x}$ « not exist »

23 The opposite figure represents the curve of the function $f(x)$

If $g(x) = \begin{cases} x^2 + 1 & , \quad x < 3 \\ 4x + 2 & , \quad x > 3 \end{cases}$

Find : $\lim\limits_{x \to 3} \left(f(x) + g(x)\right)$ « 18 »



24 **Find the value of the right and left limit , then deduce the value of the limit for each of the following functions at the given point :**

(1) $f(x) = \frac{1}{x-2}$ when $x = 2$

(2) $f(x) = \frac{1}{|x-3|}$ when $x = 3$

(3) $f(x) = \begin{cases} x-2 & , \quad x > 1 \\ \frac{1}{x-1} & , \quad x < 1 \end{cases}$ when $x = 1$

25 If $f(x) = \begin{cases} 1 + \frac{x^2+2x}{|x+2|} & , \quad x \in]-3, 0[- \{-2\} \\ 2x+1 & , \quad x \in]0, 3[\end{cases}$

, find :

(1) $\lim_{x \to -3} f(x)$

(2) $\lim_{x \to -2} f(x)$

(3) $\lim_{x \to 0} f(x)$

(4) $\lim_{x \to 3} f(x)$

26 If $f(x) = \begin{cases} \frac{\sqrt[3]{2x-3}-1}{\sqrt[5]{3x-5}-1} & , \quad x > 2 \\ kx & , \quad x < 2 \end{cases}$ has a limit when $x = 2$

, find the value of : k « $\frac{5}{9}$ »

27 If $f(x) = \begin{cases} \frac{x^2+ax+b}{x^2-4x+3} & , \quad x < 3 \\ 4x & , \quad x > 3 \end{cases}$ has a limit when $x = 3$

, find the value of each of : a and b « 18 , – 63 »

Third Higher skills

Choose the correct answer from those given :

(1) If $f(x) = x^2$, then $\lim_{x \to 2} f(f(x)) =$ ....

(a) 2 (b) 4 (c) 16 (d) 32

(2) If : f is an odd function and $\lim_{x \to 3} f(x) = 7$

, which of the following statements is true ?

(a) $\lim_{x \to -3} f(x) = 7$

(b) $\lim_{x \to -3} f(x) = -7$

(c) $\lim_{x \to -3} f(x) = -3$

(d) $\lim_{x \to -3} f(x) = 0$

(3) If : f is an even function and $\lim\limits_{x \to 2} f(x) = 5$, which of the following statements is true ?

(a) $\lim\limits_{x \to -2} f(x) = 5$ (b) $\lim\limits_{x \to -2} f(x) = -7$

(c) $\lim\limits_{x \to -2} f(x) = 0$ (d) $\lim\limits_{x \to -2} f(x) = -2$

(4) If f is a one - to - one polynomial function and $\lim\limits_{x \to 2} f(x) = 3$, then $\lim\limits_{x \to 3} f^{-1}(x) = \cdots\cdots$

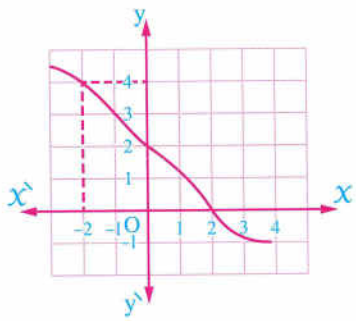
(a) -2 (b) -3 (c) 2 (d) 3

(5) If $\lim\limits_{x \to 1} f(x) = 3$, $\lim\limits_{x \to 2} f(x) = 5$, then $\lim\limits_{x \to 1} f(x+1) = \cdots\cdots$

(a) 3 (b) 4 (c) 5 (d) 6

(6) The opposite figure represents the curve of the function f, then

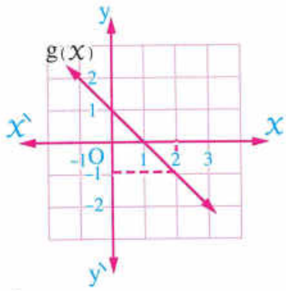
$$\lim_{x \to -2} \frac{[f(x)]^3 - 64}{f(x) - 4} = \cdots\cdots$$

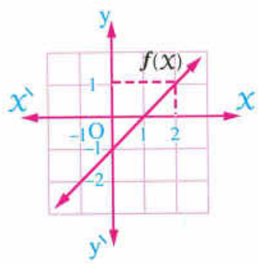


(a) zero (b) 16

(c) 32 (d) 48

(7) The opposite two figures represent the curves of the two functions f , g , then $\lim\limits_{x \to 1} \frac{f(x)}{g(x)} = \cdots\cdots$





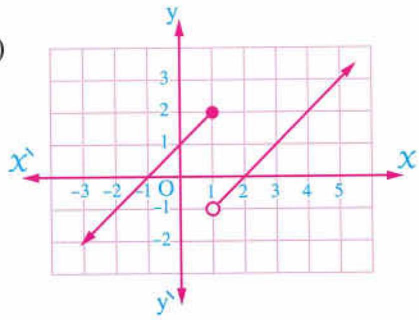
(a) 1 (b) -1

(c) zero. (d) does not exist.

(8) The opposite figure represents the curve of the function h (x)

, $g(x) = \begin{cases} x & , \quad x > 1 \\ x - 3 & , \quad x < 1 \end{cases}$, $f(x) = h(x) + g(x)$

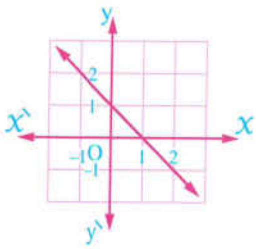
, then $\lim\limits_{x \to 1} f(x) = \cdots\cdots$



(a) does not exist. (b) 2

(c) -1 (d) zero.

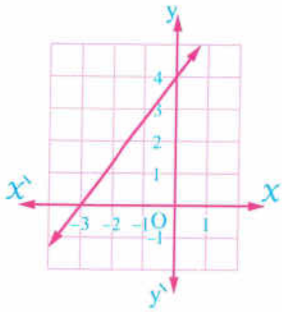
(9) The opposite figure represents the curve of the function f, then $\lim_{x \to 2} |f(x)| = \dots\dots$



(a) -1 (b) zero.

(c) 1 (d) does not exist.

(10) The opposite figure represents the curve of the function f, then $\lim_{x \to \infty} \frac{f(x)}{f^{-1}(x)} = \dots\dots$



(a) $\frac{4}{3}$ (b) $\frac{-4}{3}$

(c) $\frac{-16}{9}$ (d) $\frac{16}{9}$



Exercise 18 Continuity

Test yourself

From the school book ● Understand ● Apply ● Higher Order Thinking Skills

First Multiple choice questions :

Choose the correct answer from the given ones :

(1) The function $f(X)$ is continuous at $X = a$ if ....

(a) $f(a)$ exist.

(b) $f(a) = f(a^+) = f(a^-)$

(c) $f(X)$ has no limit at $X \longrightarrow a$

(d) a and c together.

(2) If $\lim_{X \to a^+} f(X) = \ell$, $\lim_{X \to a^-} f(X) = m$ and the function is continuous at $X = a$, then $\ell^2 + m^2 - 2\ell m =$ ....

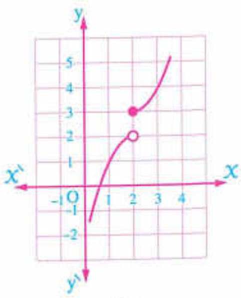
(a) zero.

(b) 1

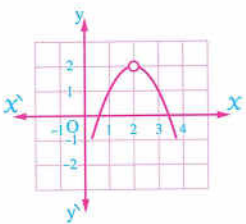
(c) – 1

(d) 2

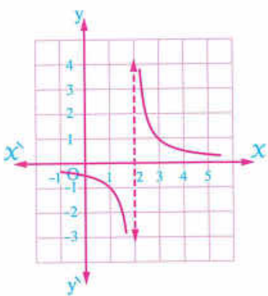
(3) The figure which represents the continuous function when $X = 2$ is the figure ....



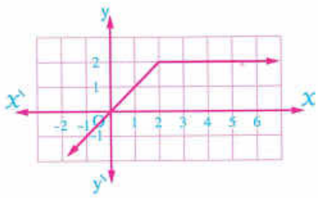
(a)



(b)

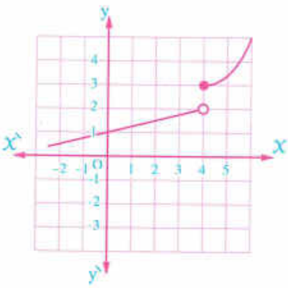


(c)

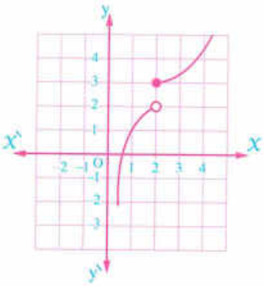


(d)

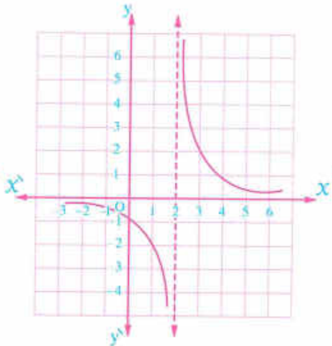
(4) Which of the following figures is continouous at $x = 2$



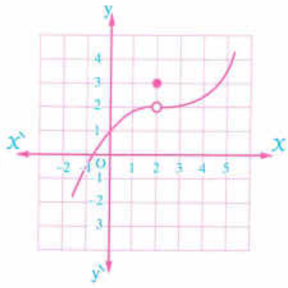
(a)



(b)

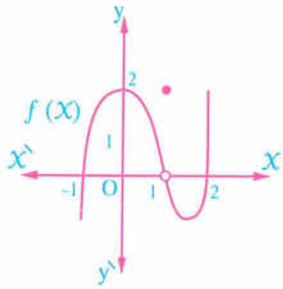


(c)



(d)

(5) In the opposite figure :



All the following statements are true except ....

(a) $\lim\limits_{x \to 1} f(x) = \text{zero}$.

(b) the function is not continuous at $x = 1$

(c) $f(1) = 2$

(d) we can not redefine the function to be contiouous.

(6) If the function $f : f(x) = \begin{cases} \frac{x^2 - 1}{x - 1} & , x \neq 1 \\ 2a & , x = 1 \end{cases}$ is continuous at $x = 1$

, then a = ...

(a) zero. (b) 1 (c) 2 (d) 4

(7) If $f : f(x) = \begin{cases} 2x & , \quad x \leq 1 \\ 3x - a & , \quad x > 1 \end{cases}$ is continuous at $x = 1$, then a = ...

(a) 1 (b) 2 (c) – 1 (d) – 2

(8) If $f : f(x) = \begin{cases} a x^2 - 3 & , \quad x \neq 2 \\ 2a & , \quad x = 2 \end{cases}$ is continuous at $x = 2$, then a = ...

(a) $\frac{1}{2}$ (b) $\frac{2}{3}$ (c) $\frac{3}{2}$ (d) 6

(9) If $f : f(x) = \begin{cases} x^2 - 2x & \text{at} \quad x \geq 1 \\ kx - 3 & \text{at} \quad x < 1 \end{cases}$ is continuous at $x = 1$, then k = ...

(a) zero (b) 1 (c) 2 (d) 3

(10) If $f : f(x) = \begin{cases} a - x^2 & , \quad x \leq -2 \\ x & , \quad x > -2 \end{cases}$ is continuous at $x = -2$, then a = ...

(a) – 2 (b) – 4 (c) 2 (d) 4

(11) If the function f is continuous at $x = 2$ where $f(x) = \begin{cases} a x^2 + 5 & , \text{ at } x \leq 2 \\ 9 - b x & , \text{ at } x > 2 \end{cases}$

, then $2a + b =$ ....

(a) 7 (b) 14 (c) 2 (d) – 2

(12) If $f : f(x) = \begin{cases} x^2 + b x + 3 & , \quad x < 1 \\ a x + b & , \quad x \geq 1 \end{cases}$ is continuous at $x = 1$, $f(1) = 7$

, then $\frac{a}{b} =$ ....

(a) $\frac{7}{3}$ (b) $\frac{4}{3}$ (c) $\frac{3}{4}$ (d) 7

(13) If $f : f(x) = \begin{cases} \frac{(x+2)^6 - 64}{2x} & , \quad x \neq 0 \\ k & , \quad x = 0 \end{cases}$ is continuous at $x = 0$

, then $k =$ ....

(a) 96 (b) 192 (c) 384 (d) 92

(14) If $f : f(x) = \begin{cases} \frac{\sqrt{x+3} - 2}{x - 1} & , \quad x \neq 1 \\ a & , \quad x = 1 \end{cases}$ is continuous at $x = 1$, then $a =$ ....

(a) 3 (b) 4 (c) $\frac{1}{4}$ (d) 12

(15) If $f : f(x) = \begin{cases} \frac{\sqrt{x} - a - 1}{\sqrt{x} - 3} & , \quad x \neq 9 \\ x - b & , \quad x = 9 \end{cases}$ is continuous on $\mathbb{R}$, then $\frac{a}{b} =$ ....

(a) $\frac{8}{3}$ (b) $\frac{1}{4}$ (c) $\frac{3}{4}$ (d) 14

(16) If the function $f : f(x) = \begin{cases} \frac{|x|}{x} + 6 & , \quad x < 0 \\ a^2 + \cos 3x & , \quad x \geq 0 \end{cases}$ is continuous at $x = 0$

, then $a =$ ....

(a) 2 (b) $\pm\sqrt{6}$ (c) ± 2 (d) $\pm\sqrt{5}$

(17) If $f(x) = \begin{cases} \frac{\tan(x-2)}{x-2} & , \quad x > 2 \\ \sin\left(\frac{1}{4}\pi x\right) & , \quad x < 2 \end{cases}$, then ....

(a) the function is continuous at $x = 2$

(b) the function has no limit at $x \longrightarrow 4$

(c) the function has a limit at $x \longrightarrow 2$

(d) $f(2^+) \neq f(2^-)$

(18) If the function $f : f(x) = \begin{cases} \dfrac{\sin 2x + \tan x}{1 + \sin 4x} & , x \in]0 , \frac{\pi}{3}[- \{\frac{\pi}{4}\} \\ k & , x = \frac{\pi}{4} \end{cases}$

is continuous at $x = \frac{\pi}{4}$, then k = ...

(a) $\frac{1}{2}$ (b) $\frac{3}{5}$ (c) 3 (d) 2

(19) If the function $f : f(x) = \begin{cases} \sin 9x \cot 4x & , x \in]\frac{-\pi}{8} , \frac{\pi}{8}[- \{0\} \\ k^2 & , x = 0 \end{cases}$

is continuous at $x = 0$, then k = ...

(a) $\frac{9}{4}$ (b) $\frac{3}{2}$ (c) $\pm\frac{3}{2}$ (d) $\pm\frac{2}{3}$

(20) The even continuous function at the point (a , b) is also continuous at the point ....

(a) (– a , b) (b) (– a , – b) (c) (a , – b) (d) otherwise.

(21) If $f(x) = \dfrac{x^2 - 25}{x - 5}$ where $x \neq 5$, then the value of $f(5)$ that makes the function continuous is ....

(a) 10 (b) 0 (c) 25 (d) undefined.

(22) If $f : f(x) = \dfrac{x^8 - a^8}{x^5 - a^5}$ at $x \neq a$, $f(a) = 200$ is continuous at $x = a$, then a = ...

(a) 5 (b) $\frac{8}{5}$ (c) 125 (d) $\frac{1}{5}$

(23) If $f(x) = \dfrac{x^2 - 5x + 6}{x^2 - 9}$ when $x \neq \pm 3$, then the value of $f(3)$ so that the function is continuous at this position is ....

(a) $\frac{1}{6}$ (b) 0 (c) $\frac{1}{9}$ (d) undefined.

(24) The function $f : f(x) = 4x^{-3} + \dfrac{x}{9 - x^2}$ is continuous for each $x \in$ ....

(a) $\mathbb{R}$ (b) $\mathbb{R} - \{0\}$ (c) $\mathbb{R} - \{3 , -3\}$ (d) $\mathbb{R} - \{3 , -3 , 0\}$

(25) If the function $f : f(x) = \dfrac{x + 1}{x^2 + a}$ is continuous on $\mathbb{R}$, then $a \in$ ....

(a) $\mathbb{R}$ (b) $\mathbb{R}^+$ (c) $\mathbb{R}^-$ (d) $\mathbb{R}^- \cup \{0\}$

(26) The function $f : f(x) = \dfrac{1}{\sqrt[3]{x + 2}}$ is continuous for each $x \in$ ....

(a) $\mathbb{R}$ (b) $\mathbb{R} - \{-2\}$ (c) $[-2 , \infty[$ (d) $]-2 , \infty[$

(27) The function $f : f(x) = \frac{x}{|x-1|-3}$ is continuous on ....

(a) $\mathbb{R} - \{3\}$ (b) $\mathbb{R} - \{-3, 3\}$ (c) $\mathbb{R} - \{-2, 4\}$ (d) $\mathbb{R}$

(28) The function $f : f(x) = \sqrt[3]{4 - x^2}$ is continuous on ....

(a) $[-2, 2]$ (b) $\mathbb{R} -]-2, 2[$ (c) $]4, \infty[$ (d) $\mathbb{R}$

(29) The function $f : f(x) = \sqrt[3]{2x - 5} + \sqrt[4]{x^2 + 4}$ is continuous for each $x \in$ ....

(a) $\mathbb{R}$ (b) $\mathbb{R}^+$ (c) $[\frac{5}{2}, \infty[$ (d) $[-4, \frac{5}{2}]$

(30) The function $f : f(x) = \sqrt{x - 1}$ is continuous on ....

(a) $[1, \infty[$ (b) $[-1, \infty[$ (c) $]-\infty, 1[$ (d) $]-\infty, 1]$

(31) The function $f : f(x) = 4 \times \sqrt{-x}$ is continuous for each $x \in$ ....

(a) $\mathbb{R}$ (b) $]-\infty, 4]$ (c) $]-\infty, 0]$ (d) $[0, \infty[$

(32) The function $f : f(x) = \frac{x + 2}{\sqrt{x} - 2}$ is continuous for each $x \in$ ....

(a) $[4, \infty[$ (b) $[0, \infty[$ (c) $[0, \infty[- \{4\}$ (d) $]-\infty, 2[$

(33) All the following functions are continuous on $\mathbb{R}$ except ....

(a) $f(x) = \sin x$ (b) $f(x) = x^2 - 2$

(c) $f(x) = \tan x$ (d) $f(x) = \frac{5x + 3}{x^2 + 4}$

(34) The function $f : f(x) = \tan 2x$ is continuous for each $x \in$ ....

(a) $\mathbb{R} - \{x : x = \frac{\pi}{2} + \pi n, n \in \mathbb{Z}\}$ (b) $\mathbb{R} - \{x : x = \frac{\pi}{4} + \frac{\pi n}{2}, n \in \mathbb{Z}\}$

(c) $\mathbb{R} - \{x : x = \frac{\pi}{4} + \pi n, n \in \mathbb{Z}\}$ (d) $\mathbb{R} - \{x : x = \frac{\pi}{2} + 2\pi n, n \in \mathbb{Z}\}$

(35) If $f : f(x) = \begin{cases} \frac{\sin a x}{2x} & , x \neq 0 \\ 5 & , x = 0 \end{cases}$ is continuous on $\mathbb{R}$, then a = ...

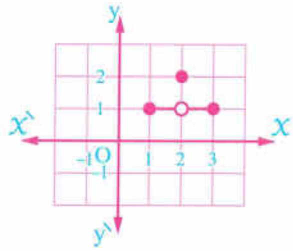
(a) 5 (b) 10 (c) 2 (d) zero.

(36) If $f : f(x) = \begin{cases} x + 1 & , 1 < x < 3 \\ x^2 + bx + c & , x \in \mathbb{R} -]1, 3[\end{cases}$, is continuous on $\mathbb{R}$

, then b – c = ...

(a) – 3 (b) 1 (c) 4 (d) – 7

(37) The opposite figure represents the curve of the function f , which of the following statements is true ?



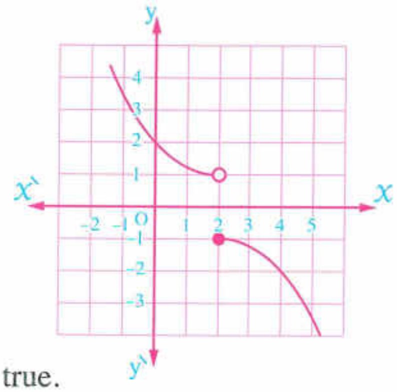
(a) f is continuous on the interval $[1 , 3]$

(b) f is continuous on the interval $]1 , 3[$

(c) $\lim_{x \to a} f(x)$ exists where $a \in [1 , 3]$

(d) $\lim_{x \to a} f(x)$ exists where $a \in]1 , 3[$

(38) The opposite figure represents the curve of the function f , $g(x) = |f(x)|$, then :



(I) $g(x)$ is continuous on $\mathbb{R}$

(II) $\lim_{x \to 2} f(x) = \pm 1$

(III) $\lim_{x \to 2} g(x) = 1$

(a) both (I) , (II) are true. (b) both (I) , (III) are true.

(c) All (I) , (II) , (III) are true. (d) (II) is true.

Second Essay questions

Exercises on continuity of a function at a point

1 **Discuss the continuity of each of the functions defined by the following rules at the indicated points :**

(1) $f(x) = x^2 + x - 3$ at $x = -1$

(2) $f(x) = \sqrt[3]{x - 1}$ at $x = 9$

(3) 📖 $f(x) = \frac{x^2 - 4}{x - 2}$ at $x = 1 , x = 2$

(4) 📖 $f(x) = 5 - |x - 3|$ at $x = 3$

(5) $f(x) = |x - 4| + |x + 2|$ at $x = 4 , x = -2$

(6) $f(x) = \begin{cases} 4x^2 + 3 & , x \le \frac{1}{2} \\ 5 - 2x & , x > \frac{1}{2} \end{cases}$ at $x = \frac{1}{2}$

(7) $f(x)=\begin{cases}2x & , x<1\\ 4-2x & , x>1\end{cases}$ at $x=-1$, $x=1$

(8) 📖 $f(x)=\begin{cases}x+4 & , x<-2\\ 1 & , -2\le x<-1\\ 2x+3 & , x\ge -1\end{cases}$ at $x=-2$, $x=-1$

(9) 📖 $f(x)=\begin{cases}x^2+3 & , x\ge 1\\ \dfrac{x^2+2x-3}{x-1} & , x<1\end{cases}$ at $x=1$

(10) $f(x)=\begin{cases}x|x| & , x\ne 0\\ 2 & , x=0\end{cases}$ at $x=0$

(11) $f(x)=\begin{cases}\dfrac{|x-3|}{x-3} & , x\ne 3\\ 2 & , x=3\end{cases}$ at $x=3$

(12) 📖 $f(x)=\begin{cases}\dfrac{x|x-2|-x+2}{|x-2|} & , x<2\\ 3 & , x\ge 2\end{cases}$ at $x=2$

(13) $f(x)=\begin{cases}\dfrac{x^7-128}{x^4-16} & , x\ne 2\\ 3x^2+2 & , x=2\end{cases}$ at $x=2$

(14) 📖 $f(x)=\begin{cases}\dfrac{1-\cos x}{x} & , x>0\\ 2\sin x & , x\le 0\end{cases}$ at $x=0$

(15) 📖 $f(x)=\begin{cases}\dfrac{\sin(x-2)}{x^2-4} & , x\in\,]-\infty,2[\,-\{-2\}\\ 1-\dfrac{3}{x^2} & , x\in[2,\infty[\end{cases}$ at $x=2$

(16) $f(x)=\begin{cases}x|x|+4 & , x\le 0\\ \dfrac{|x|}{x}+3 & , x>0\end{cases}$ at $x=0$

(17) $f(x)=\begin{cases}\dfrac{x^2+2x-3}{|x+3|} & , x\ne -3\\ 2 & , x=-3\end{cases}$ at $x=-3$, $x=1$

(18) $f(x)=\begin{cases}|x+3| & , x\le 2\\ 11-3x & , x>2\end{cases}$ at $x=-3$, $x=2$

(19) $f(x) = \begin{cases} \frac{5x^2 + \sin^2 2x}{x \tan 3x} & , -\frac{\pi}{6} < x < 0 \\ \frac{x+3}{x+1} & , x \geq 0 \end{cases}$ at $x = 0$

(20) $f(x) = \begin{cases} \frac{\sin x}{\pi - x} & , x \neq \pi \\ 1 & , x = \pi \end{cases}$ at $x = \pi$

2 **Find the values of k , a , b and c , so that each of the functions defined by the following rules is continuous at the indicated points :**

(1) $f(x) = \begin{cases} \frac{x^2 + 2x - 3}{x + 3} & , x \neq -3 \\ x + a & , x = -3 \end{cases}$ at $x = -3$ « − 1 »

(2) $f(x) = \begin{cases} \frac{x^2 - 5x + 6}{x^3 - 8} & , x \neq 2 \\ \frac{-2}{|a|} & , x = 2 \end{cases}$ at $x = 2$ « ± 24 »

(3) $f(x) = \begin{cases} \frac{\sqrt{x+3} - 2}{x^2 - 1} & , x \neq 1 \\ k & , x = 1 \end{cases}$ at $x = 1$ « $\frac{1}{8}$ »

(4) $f(x) = \begin{cases} \frac{\sin^2 2x}{x \tan 3x} & , x \in]\frac{-\pi}{6} , \frac{\pi}{6}[- \{0\} \\ a + 1 & , x = 0 \end{cases}$ at $x = 0$ « $\frac{1}{3}$ »

(5) $f(x) = \begin{cases} 3x - 2 & , x \leq -2 \\ a x + b & , -2 < x < 5 \\ x^2 - 12 & , x \geq 5 \end{cases}$ at $x = -2 , x = 5$ « 3 , − 2 »

(6) $f(x) = \begin{cases} a + b x & , x > 2 \\ 3 & , x = 2 \\ b - a x^2 & , x < 2 \end{cases}$ at $x = 2$ « $-\frac{1}{3}$, $\frac{5}{3}$ »

(7) $f(x) = \begin{cases} \frac{\cos 2x - 1}{x^2} & , x \neq 0 \\ k & , x = 0 \end{cases}$ at $x = 0$ « − 2 »

(8) 📖 $f(x) = \begin{cases} 2 - x^2 & , x \le c \\ x & , x > c \end{cases}$ at $x = c$ « 1 or – 2 »

(9) $f(x) = \begin{cases} \dfrac{1 - \cos x}{k x} & , x \ne 0 \\ \sin x & , x = 0 \end{cases}$ at $x = 0$ « $\mathbb{R} - \{0\}$ »

3 **Redefine (if possible) each of the functions defined by the following rules at the indicated point , such that each function becomes continuous at this point :**

(1) 📖 $f(x) = \dfrac{x^2 - x - 6}{x - 3}$ at $x = 3$

(2) $f(x) = \dfrac{x^3 - 1}{x^2 - 3x + 2}$ at $x = 1$

(3) 📖 $f(x) = \begin{cases} x^3 + 2x & , x > 1 \\ 5x - 1 & , x < 1 \end{cases}$ at $x = 1$

(4) 📖 $f(x) = \begin{cases} x^2 + 1 & , x \ge 2 \\ \dfrac{x^2 - 4}{x - 2} & , x < 2 \end{cases}$ at $x = 2$

(5) 📖 $f(x) = \begin{cases} \dfrac{3x + 1 - \cos x}{5x} & , x > 0 \\ \dfrac{3}{5} \cos x & , x < 0 \end{cases}$ at $x = 0$

(6) $f(x) = \dfrac{|x - 3|}{x - 3}$ at $x = 3$

(7) $f(x) = \dfrac{x^2 - 64}{\sqrt[3]{x} - 2}$ at $x = 8$

(8) $f(x) = \dfrac{|x| + x}{x}$ at $x = 0$

(9) $f(x) = \dfrac{2}{x - 5} - \dfrac{12}{x^2 - 4x - 5}$ at $x = 5$

(10) $f(x) = \dfrac{\sin x}{|x|}$ at $x = 0$

Exercises on continuity of a function on an interval

4 Discuss the continuity of each of the functions defined by the following rules :

(1) $f(x) = 7$

(2) $f(x) = x^3 - 2x^2 + 1$

(3) $f(x) = \frac{3x - 5}{x - 3}$

(4) $f(x) = \frac{x^2 - 9}{x^2 - 5x + 6}$

(5) $f(x) = \frac{x - 1}{x^4 - 13x^2 + 36}$

(6) $f(x) = \frac{x}{x^2 + x + 1}$

(7) $f(x) = \frac{4x - 1}{x^3 - x}$

(8) $f(x) = |x - 2| + |5 + x|$

(9) $f(x) = x^3 \sin 2x$

(10) $f(x) = \sin x - 3\cos(x + 1)$

(11) $f(x) = (3x + 4)^2 + \sin 2x$

(12) $f(x) = \frac{x}{|x| - 2}$

(13) $f(x) = \frac{3x - 1}{|x| + 1}$

(14) $f(x) = \frac{1}{\sqrt{x - 2}}$

(15) $f(x) = \frac{x^2 + \sin 3x}{x + 1}$

(16) $f(x) = \frac{\sqrt{x - 4}}{x - 6}$

(17) $f(x) = \frac{\sqrt{x - 3}}{x^2 - 4}$

(18) $f(x) = \frac{3x + 2}{x\sqrt{x + 2}}$

(19) $f(x) = \frac{x + 1}{\sqrt{x + 2} - 1}$

(20) $f(x) = \sqrt{x + 2} + \sqrt{3 - x}$

(21) $f(x) = \frac{1}{\sqrt{25 - x^2}}$

(22) $f(x) = \sqrt{5 - |x|}$

(23) $f(x) = \frac{x^3 + 1}{\sin x}$

(24) $f(x) = \frac{x + 3}{\cos x}$

(25) $f(x) = \frac{\sin 3x + \cos 5x}{1 - \cos x}$

(26) $f(x) = \frac{2 + x}{1 + \sin x}$

(27) $f(x) = \frac{\sin^2 x + \cos x}{x^2 - 9}$

(28) $f(x) = \tan^2 x - 1$

(29) $f(x) = \frac{\tan x}{x^2 - 9}$

5 Discuss the continuity of each of the functions defined by the following rules on its domain :

(1) $f(x) = \begin{cases} x^2 + 1 & , x \le 1 \\ 2x & , x > 1 \end{cases}$

(2) $f(x) = \begin{cases} x^2 & , x < 3 \\ 5x - 4 & , x \geq 3 \end{cases}$

(3) $f(x) = \begin{cases} x^2 - 3x + 2 & , x \leq 3 \\ 2 & , 3 < x \leq 4 \\ 6 - x^2 & , x > 4 \end{cases}$

(4) 📖 $f(x) = \begin{cases} 1 + \sin x & , 0 \leq x \leq \frac{\pi}{2} \\ 1 - \cos 2x & , x > \frac{\pi}{2} \end{cases}$

(5) $f(x) = \begin{cases} \sin x & , \frac{-\pi}{4} \leq x < \frac{3\pi}{4} \\ \cos x & , \frac{3\pi}{3} \leq x \leq 2\pi \end{cases}$

(6) 📖 $f(x) = \begin{cases} \frac{x \tan x + \sin^2 3x}{5x^2} & , -\frac{\pi}{4} < x < 0 \\ 2\cos 2x & , 0 \leq x < \frac{\pi}{4} \end{cases}$

(7) $f(x) = \begin{cases} \frac{(x+3)^4 - 81}{x} & , x \neq 0 \\ 108 & , x = 0 \end{cases}$

(8) $f(x) = \begin{cases} 5 - x|x| & , x \leq 0 \\ \frac{|x|}{x} + 4 & , x > 0 \end{cases}$

---

6 If the function $f : f(x) = \begin{cases} \frac{x + \sin 2x}{\sin \frac{1}{2}x} & , -\pi \leq x < 0 \\ x + k & , 0 \leq x \leq \pi \end{cases}$

is continuous on the interval $[-\pi, \pi]$, then find the value of : k « 6 »

---

7 📖 If the function $f : f(x) = \begin{cases} 4x & , x \leq -1 \\ ax + b & , -1 < x < 3 \\ -2x & , x \geq 3 \end{cases}$ is continuous on $\mathbb{R}$

, then find the value of each of a and b « $-\frac{1}{2}$, $-\frac{9}{2}$ »

---

8 If the function $f : f(x) = \begin{cases} \frac{\sqrt{x - a} - 1}{\sqrt{x} - 3} & , x \neq 9 \\ x - b & , x = 9 \end{cases}$

is continuous on $\mathbb{R}$, then what is the real value of each of a and b ? « 8 , 6 »

Third Higher skills

1 **Discuss the continuity of the functions defined by the following rules on their domains :**

(1) $f(x) = \begin{cases} |x-1| + 2x & , \quad x \le 2 \\ 4x - x^2 & , \quad x > 2 \end{cases}$

(2) $f(x) = \begin{cases} \sqrt{4-x^2} & , \quad |x| \le 2 \\ x^2 + 2x & , \quad |x| > 2 \end{cases}$

---

2 Find the value of a which makes the function $f : f(x) = \dfrac{x+3}{x^2 + a x + 9}$ continuous on $\mathbb{R}$ « $]-6 , 6[$ »

---

3 Find the value of a which makes the function $f : f(x) = \dfrac{2x+1}{x^2 + 6x + a}$ continuous on $\mathbb{R}$ « $]9 , \infty[$ »

---

4 If $f(x) = \begin{cases} -x & , \quad x < 0 \\ 1 & , \quad x \ge 0 \end{cases}$, $g(x) = \begin{cases} 1 & , \quad x < 0 \\ x & , \quad x \ge 0 \end{cases}$

Prove that : f , g are not continuous at $x = 0$ but their product $(f . g)$ is continuous at $x = 0$

---

5 If $f(x) = \begin{cases} -1 & \text{when } x \ne 4 \\ 1 & \text{when } x = 4 \end{cases}$, $g(x) = \begin{cases} 4x - 10 & \text{when } x \ne 4 \\ -6 & \text{when } x = 4 \end{cases}$

discuss the continuity of each of the following functions at $x = 4$

(1) $f(x)$	(2) $g(x)$	(3) $f(x) . g(x)$
(4) $\lvert f(x) \rvert$	(5) $g(x) - 6 f(x)$	(6) $g(f(x))$

UNIT 4

Trigonometry

Unit Exercises

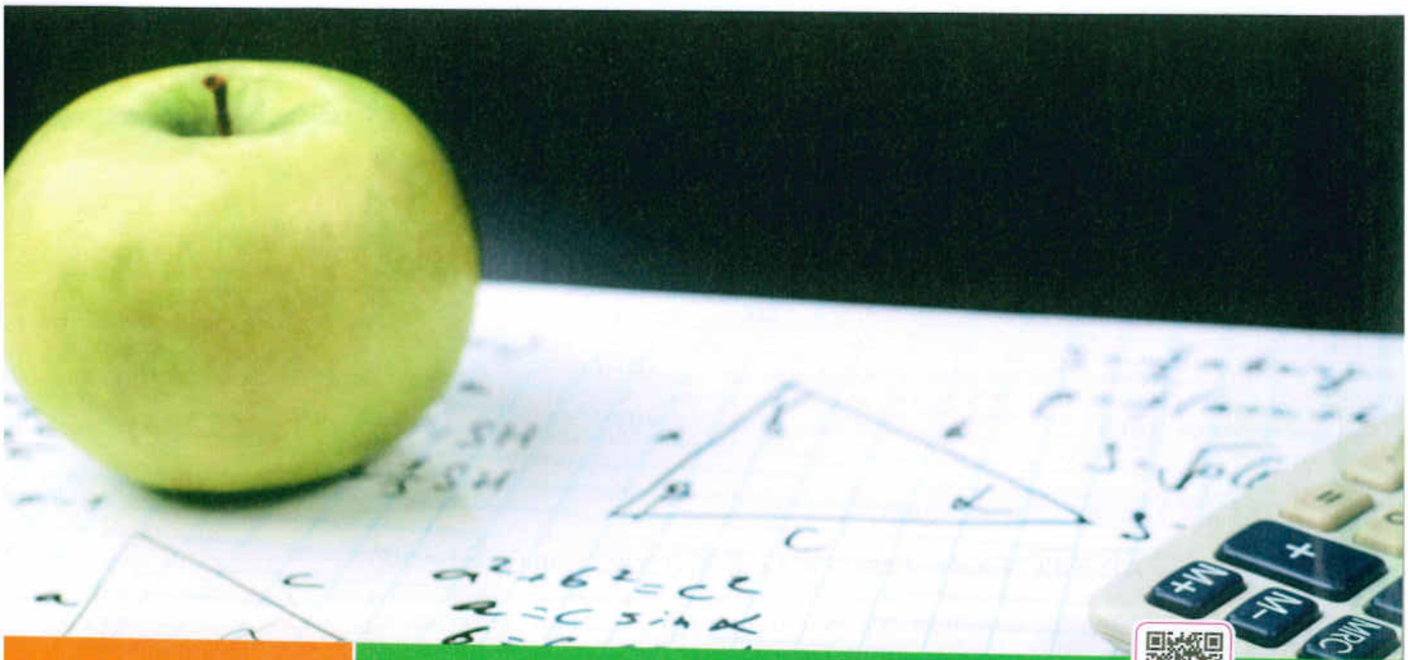
Exercise 19 — The sine rule

Exercise 20 — The cosine rule.

Exercise 21 — Solution of the triangle.

- **At the end of the unit :**

Life applications on unit four.



Exercise 19 The sine rule

Test yourself

From the school book · Understand · Apply · Higher Order Thinking Skills

First Multiple choice questions

Choose the correct answer from the given ones :

(1) In any triangle XYZ , XY : YZ = ...

(a) sin X : sin Y (b) sin Y : sin Z (c) sin Z : sin X (d) sin Z : sin Y

(2) In Δ ABC , if m (∠ A) = 30° , C = $15\sqrt{3}$ cm. , m (∠ C) = 60° , then a = ... cm.

(a) 30 (b) 45 (c) 15 (d) 60

(3) DEF is a triangle in which m (∠ D) = 80° and m (∠ E) = 60° , if f = 12 cm. , then d = ... cm.

(a) $\frac{12 \sin 80°}{\sin 40°}$ (b) $\frac{12 \sin 80°}{\sin 60°}$ (c) $\frac{12 \sin 40°}{\sin 80°}$ (d) $\frac{12 \cos 80°}{\cos 40°}$

(4) In Δ ABC , if a = 4 cm. , b = 7 cm. , m (∠ C) = 120° , then the area of the triangle = ... cm^2.

(a) $7\sqrt{3}$ (b) $14\sqrt{3}$ (c) 7 (d) 14

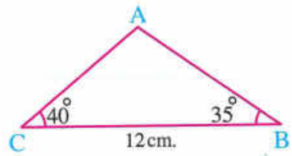
(5) XYZ is an equilateral triangle , the length of its side is $10\sqrt{3}$ cm. , then the length of the diameter of its circumcircle is .... cm.

(a) 5 (b) 10 (c) 15 (d) 20

(6) In Δ XYZ , $\frac{x}{\sin X} = 6$, then the length of the diameter of its circumcircle is .... length units.

(a) 6 (b) 12 (c) 3 (d) 9

(7) In the opposite figure :

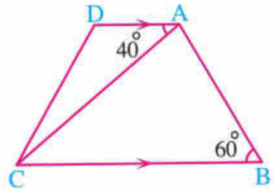


The length of $\overline{AB} \approx$ .... cm.

(a) 6 (b) 7

(c) 8 (d) 9

(8) In the opposite figure :



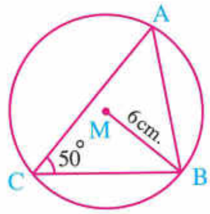
$\overline{AD} // \overline{BC}$, AB = 4 cm. , m (∠ DAC) = 40° , m (∠ B) = 60°

, then the length of $\overline{AC} \simeq$ .... cm.

(a) 5 (b) 3

(c) 2 (d) 4

(9) In the opposite figure :



M is the centre of the circle

BM = 6 cm. , then AB = ... cm.

(a) 6 sin 50° (b) 12 sin 50°

(c) 6 cos 50° (d) 12 cos 50°

(10) A circle with diameter of length 20 cm. , passes through the vertices of Δ ABC which is an acute-angled triangle in which BC = 10 cm. , then m (∠ A) = ...°

(a) 30 (b) 60 (c) 45 (d) 150

(11) In triangle ABC , m (∠ A) = 45° , the length of the radius of its circumcircle = 6 cm. , then a = ... cm.

(a) 13 (b) $6\sqrt{2}$ (c) 12 (d) $\sqrt{2}$

(12) If the length of a side in any triangle = 12 cm. and the measure of the opposite angle to this side = 55° , then the circumference of the circle that passes through the vertices of this triangle $\simeq$ .... cm.

(a) 36 (b) 42 (c) 46 (d) 52

(13) If the perimeter of triangle ABC equals 15 cm. , m (∠ A) = 53° , m (∠ B) = 47° , then the length of $\overline{AB} \simeq$ .... cm.

(a) 6 (b) 7 (c) 5 (d) 8

(14) In triangle ABC , a = 27 cm. , m (∠ B) = 82° , m (∠ C) = 56° , then its surface area $\simeq$ .... cm^2.

(a) 540 (b) 447 (c) 350 (d) 400

(15) In triangle ABC , m (∠ A) : m (∠ B) : m (∠ C) = 2 : 3 : 4 , AB = 12 cm. , then the length of $\overline{AC} \simeq$ .... cm.

(a) 10 (b) 11 (c) 16 (d) 18

(16) In triangle ABC , which of the following statements is true ?

(a) sin A + cos B = a + b (b) a sin B = b sin A

(c) a = b sin c (d) $\frac{a}{\sin A} = \frac{\sin B}{b}$

(17) In Δ XYZ , 2 r sin X = ... "where r is the radius length of its circumcircle"

(a) z (b) y (c) x (d) area of Δ XYZ

(18) 📖 If r is the length of the radius of the circumcircle of the triangle XYZ , then $\frac{y}{2 \sin Y}$ = ...

(a) r (b) 2 r (c) $\frac{1}{2}$ r (d) 4 r

(19) In acute-angled triangle ABC , $2a = \frac{b}{\sin B}$, then m (∠ A) = ...

(a) 30° (b) 45° (c) 60° (d) 75°

(20) In Δ ABC , sin A = 2 sin C , BC = 6 cm. , then AB = ... cm.

(a) 2 (b) 3 (c) 4 (d) 6

(21) If the radius length of circumcircle of Δ ABC equals 3 cm. and sin A + sin B + sin C = 2, then the perimeter of triangle ABC = ... cm.

(a) 6 (b) 9 (c) 12 (d) 24

(22) ABC is an equilateral triangle , its side length is 6 cm. and the area of its circumcircle equals k π cm^2 , then k = ...

(a) $2\sqrt{3}$ (b) $8\sqrt{3}$ (c) 12 (d) 24

(23) In any triangle ABC , $\frac{\sin (A + B)}{\sin A + \sin B}$ = ...

(a) 1 (b) $\frac{c}{a + b}$ (c) $\frac{a}{b + c}$ (d) $\frac{b}{a + c}$

(24) In Δ ABC , $\frac{a}{a + b} = \frac{\sin A}{\text{..........}}$

(a) sin B (b) sin C (c) sin A + sin B (d) sin A + sin C

(25) 📖 In Δ XYZ , if 3 sin X = 4 sin Y = 2 sin Z , then x : y : z = ...

(a) 2 : 3 : 4 (b) 6 : 4 : 3 (c) 3 : 4 : 6 (d) 4 : 3 : 6

(26) ABC is a triangle in which $\frac{\sin A}{3} = \frac{2 \sin B}{5} = \frac{\sin C}{4}$, then a : b : c = ...

(a) 6 : 5 : 8 (b) 8 : 5 : 6 (c) 7 : 2 : 4 (d) 3 : 5 : 4

(27) In Δ ABC : If $\frac{\sin A}{4} = \frac{\sin B}{9} = \frac{\sin C}{7}$, then the greatest angle in measure is ....

(a) $\angle$ A (b) $\angle$ B (c) $\angle$ C (d) right

(28) In triangle ABC , m ($\angle$ A) : m ($\angle$ B) : m ($\angle$ C) = 3 : 5 : 4

, then $c^2 : a^2$ = ...

(a) $\sqrt{6} : 2$ (b) 2 : 3 (c) 4 : 3 (d) 3 : 2

(29) In Δ ABC , $\frac{a}{b} \times \frac{\sin B}{\sin A}$ = ...

(a) $\frac{c}{\sin C}$ (b) $\frac{\sin C}{c}$ (c) 4r (d) 1

(30) In Δ ABC , if the radius of its circumcircle = 4 cm.

, then $\frac{a + b + c}{\sin A + \sin B + \sin C}$ = ...

(a) 4 (b) 2 (c) 8 (d) 16

(31) If Δ ABC is a right-angled at $\angle$ B and b = 10 cm.

, then $\frac{a}{\sin A} + \frac{c}{\sin C}$ = ... cm.

(a) 10 (b) 20 (c) 40 (d) 100

(32) If the radius of the circumcircle of Δ ABC equals r , then the perimeter of the triangle = ... (sin A + sin B + sin C)

(a) r (b) 2 r (c) $4 r^2$ (d) $8 r^3$

(33) In Δ ABC , a – b = 4 cm. , sin A = $\frac{3}{2}$ sin B , then a = ... cm.

(a) 4 (b) 6 (c) 8 (d) 12

(34) If the perimeter of Δ ABC is 24 cm. and sin A + sin B = 3 sin C , then C = ... cm.

(a) 4 (b) 6 (c) 8 (d) 9

(35) ABC is a triangle , sin B + sin C = 4 sin A and b + c = 2 a + 10 cm.

, then a = ... cm.

(a) 2 (b) 3 (c) 4 (d) 5

(36) In Δ ABC , AB = 8 cm. , BC = 12 cm. , m ($\angle$ A) – m ($\angle$ C) = 90°

, then tan C = ...

(a) $\frac{2}{3}$ (b) $\frac{3}{2}$ (c) $\frac{3}{4}$ (d) $\frac{4}{3}$

(37) If r is the radius length of the circumcircle of Δ ABC and a = r , then m ($\angle$ A) = ...

(a) 30° only. (b) 30° or 120° (c) 150° only. (d) 30° or 150°

(38) If the area of the triangle ABC is Δ and r is the radius length of the circumcircle of the triangle ABC , then : $\frac{4 r \Delta}{abc}$ = ...

(a) 1 (b) 2 (c) 4 (d) 8

(39) In Δ ABC , $\frac{2 b}{\sin B}$ = ... r (where r is the radius of its circumcircle)

(a) 1 (b) 2 (c) 4 (d) 8

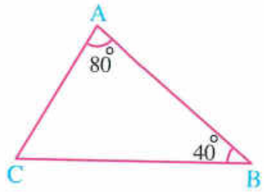
(40) ABC is a triangle , b = 12 cm. , the radius length of its circumcircle is r , then the area of the triangle = ... cm^2.

(a) $\frac{2 ac}{r}$ (b) $\frac{3 ac}{r}$ (c) $\frac{4 ac}{r}$ (d) $\frac{6 ac}{r}$

(41) If the triangle ABC is an isosceles right-angled triangle and r is the radius length of the circumcircle of the triangle ABC , then the area of Δ ABC = ... (in terms of r)

(a) $\frac{1}{2} r^2$ (b) $2 r^2$ (c) r^2 (d) $4 r^2$

(42) In the opposite figure :



If the perimeter of Δ ABC = 20 cm. ,

then the diameter length of its circumcircle ≈ .... cm.

(a) 2 (b) 4

(c) 6 (d) 8

(43) In Δ ABC , cos (B + C) = $\frac{3}{5}$, BC = 8 cm. , then the radius length of the circumcircle of Δ ABC = ... cm.

(a) 4 (b) 5 (c) 8 (d) 10

(44) If the area of a triangle is $\frac{a^2 \sin B \sin C}{k \sin A}$, then k = ... , k ≠ 0

(a) 1 (b) 2 (c) 3 (d) 4

(45) If r is the radius length of the circumcircle of triangle ABC and a = $\frac{1}{2}$ r , then m ($\angle$ A) = ... where A is an acute angle.

(a) $\sin^{-1}(1)$ (b) $\sin^{-1}(2)$ (c) $\sin^{-1}(\frac{1}{2})$ (d) $\sin^{-1}(\frac{1}{4})$

(46) If Δ is the area of triangle XYZ , S is half the perimeter of the triangle XYZ

, then $\frac{2}{z \sin X} + \frac{2}{x \sin Y} + \frac{2}{y \sin Z} = $ ....

(a) $\frac{S}{\Delta}$ (b) $\frac{2S}{\Delta}$ (c) $\frac{3S}{\Delta}$ (d) $\frac{4S}{\Delta}$

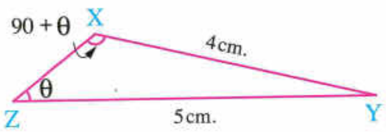
(47) In the opposite figure tan θ = ...

(a) $\frac{3}{5}$

(b) $\frac{4}{3}$

(c) $\frac{5}{4}$

(d) $\frac{4}{5}$



Second Essay questions

1 XYZ is a triangle in which : m (∠ X) = 80° , m (∠ Y) = 60° and z = 10 cm.

, find each of x and y to the nearest cm. « 15 cm. , 13 cm. »

2 ABC is a triangle in which c = 19 cm. , m (∠ A) = 112° and m (∠ B) = 33°

Find to the nearest hundredth each of b and the length of the radius of the circumcircle of the triangle. « 18.04 cm. , 16.56 cm. »

3 LMN is a triangle in which : m = 68.4 cm. , m (∠ M) = 100° and m (∠ N) = 40°

, find : (1) ℓ

(2) The length of the radius of the circumcircle of the triangle LNM

(3) The area of the triangle LMN « 44.64 cm. , 34.73 cm. , 981.34 cm.^{2} »

4 LMN is a triangle in which : m (∠ L) = 18° 52′ , m (∠ N) = 44° 17′ and NL = 35 cm.

Find :

(1) The length of each of $\overline{MN}$ and $\overline{LM}$

(2) The area of the circumcircle of Δ LMN « 12.7 cm. , 27.4 cm. , 1208.67 cm.^{2} »

5 ABC is a triangle in which : b = 10 cm. , m (∠ A) = 40° and m (∠ C) = 80°

Find the length of the greatest side of Δ ABC « 11 cm. »

6 ABC is a triangle in which : c = 4.5 cm. , m (∠ A) = 100° and m (∠ B) = 15°

Find the length of the smallest side of Δ ABC « 1.3 cm. »

7 ABC is a triangle in which : m (∠ A) = 60° and a = $7\sqrt{3}$ cm. Find the area and the circumference of the circumcircle of Δ ABC $\left(\pi = \frac{22}{7}\right)$ « 154 cm^2 , 44 cm. »

8 📖 ABC is a triangle in which m (∠ A) = 60° , m (∠ B) = 45°
Prove that : a : b : c = $\sqrt{6} : 2 : \sqrt{3} + 1$

9 ABC is a triangle in which : a = 13 cm. , m (∠ A) = $53^\circ\ 8'$, c = 15 cm. Find the radius length of the circumcircle of Δ ABC , then find m (∠ C) « 8.1 cm. , $67^\circ\ 23'\ 9''$ or $112^\circ\ 36'\ 51''$ »

10 ABC is a triangle in which m (∠ A) = 35° , a = 8 cm. and b = 6 cm.
Find : m (∠ B) « $25^\circ\ 28'\ 45''$ »

11 📖 Find the perimeter of the triangle ABC in which c = 8.7 cm. , m (∠ A) = $57^\circ\ 13'$ and m (∠ B) = $64^\circ\ 18'$ « 26.5 cm. »

12 ABC is a triangle in which m (∠ B) = 45° , m (∠ C) = 60° and the diameter length of the circumcircle of Δ ABC = 40 cm. Calculate the area and the perimeter of Δ ABC to the nearest whole number. « 473 cm^2 , 102 cm. »

13 ABC is an isosceles triangle in which : m (∠ A) = 120° and the length of the radius of the circumcircle of Δ ABC is 12 cm.
Find c and calculate the area of Δ ABC « 12 cm. , 62.4 cm^2 »

14 ABC is an isosceles triangle in which : a = b and m (∠ A) = 15° and the perimeter of Δ ABC is 25 cm. Find the area of the circumcircle of Δ ABC « 474 cm^2 »

15 📖 If the perimeter of Δ ABC = 40 cm. , m (∠ A) = 44° and m (∠ B) = 66° ,
Find the lengths of the sides of the triangle ABC « 10.9 cm. , 14.3 cm. , 14.8 cm. »

16 ABC is a triangle in which c = 12 cm. and m (∠ B) = 3 m (∠ A) = 60°
Find a and the area of Δ ABC to the nearest cm^2 « 4.2 cm. , 22 cm^2 »

17 📖 If the area of the triangle ABC is 450 cm^2 , m (∠ B) = 82° and m (∠ C) = 56° , find the value of a « 27 cm. »

18 ABC is an acute-angled triangle in which AC = 12 cm. , sin A = 0.6 and its area is 43.2 cm^2
Find the length of each of $\overline{AB}$ and $\overline{BC}$, Also find m (∠ B) « 12 cm. , 7.6 cm. , $71^\circ\ 34'$ »

19 Find the perimeter of the acute-angled triangle ABC if a = 7 cm. , b = 8 cm. and m (∠ A) = 60° « 20 cm. »

20 ABC is a right-angled triangle at B , let $D \in \overrightarrow{BC}$ and $D \notin \overline{BC}$ such that CD = 4 cm. , $m(\angle ADC) = 45^\circ\ 15'$ and $m(\angle CAD) = 18^\circ$

Find the length of $\overline{AB}$ to the nearest cm. « 8 cm. »

21 XYZ is a triangle in which YZ = 15 cm. , $m(\angle Y) = 30^\circ$ and $m(\angle Z) = 70^\circ$

Calculate the length of the perpendicular dropped from X to $\overline{YZ}$ « 7.16 cm. »

22 ABC is an obtuse-angled triangle at C in which : a = 8 cm. , c = 20 cm.

and $\tan A = \frac{1}{2\sqrt{2}}$ Find $m(\angle C)$ « $123^\circ\ 33'$ »

23 ABC is a triangle in which : b = 5 cm. , $\tan C = \frac{4}{3}$ and $m(\angle B) = 30^\circ$

find a , c and the area of the triangle to the nearest integer. « 10 cm. , 8 cm. , 20 cm^2 »

24 XYZ is a triangle in which sin X + sin Y + sin Z = 2.37 and its perimeter is 56.88 cm.

Find the length of the radius of the circumcircle of Δ XYZ « 12 cm. »

25 ABC is a triangle in which $m(\angle A) = 60^\circ$, $m(\angle B) = 45^\circ$, if $a + b = (\sqrt{6} + 2)$ cm. ,

then find each of : a and b « $\sqrt{6}$ cm. , 2 cm. »

26 ABC is a triangle in which sin A : sin B : sin C = 2 : 4 : 5 and c – b = 3 cm.

Find each of a and b « 6 cm. , 12 cm. »

27 ABC is a triangle in which $m(\angle A) : m(\angle B) : m(\angle C) = 3 : 4 : 3$, if a = 5 cm. ,

then find the perimeter of the triangle ABC « 15.9 cm. »

28 ABC is a triangle in which $m(\angle A) : m(\angle B) : m(\angle C) = 1 : 3 : 5$

Find the length of the smallest side of Δ ABC if its perimeter equals 16 cm. « 2.5 cm. »

29 ABC is a triangle in which $m(\angle A) = \frac{2}{3}\ m(\angle B) = \frac{1}{2}\ m(\angle C)$, the length of the radius of its circumcircle = 10 cm. Find the area of Δ ABC « 110 cm^2 »

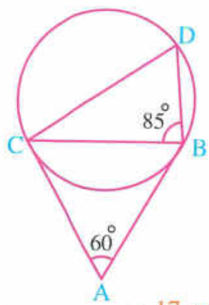
30 ABC is a triangle in which 6 sin A = 4 sin B = 3 sin C and its perimeter is 45 cm.

Find each of a and c « 10 cm. , 20 cm. »

31 If ABC is a triangle where : $\frac{a+b-c}{3} = \frac{a+c-b}{5} = \frac{b+c-a}{7}$

, then prove that sin A : sin B : sin C = 4 : 5 : 6

32 **In the opposite figure :**



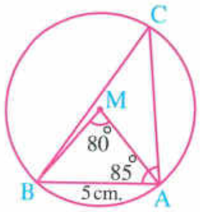
$\overline{AB}$ and $\overline{AC}$ are two tangent segments to the circle at B and C

, if m (∠ A) = 60° , m (∠ DBC) = 85°

and the area of the triangle ABC = $9\sqrt{3}$ cm^2

, then find the perimeter of the triangle DBC to the nearest centimetre. « 17 cm.»

33 **In the opposite figure :**



A circle M , AB = 5 cm. , m (∠ AMB) = 80°

and m (∠ CAB) = 85°

Find : (1) The perimeter of Δ ABC

(2) The area of the circle M « 19.12 cm. , 47.5 cm^2. »

34 ABCD is a parallelogram in which m (∠ A) = 50° , m (∠ DBC) = 70° and BD = 8 cm.

Find the perimeter of the parallelogram. « 38 cm. »

35 ABCD is a parallelogram in which AB = 18 cm. , m (∠ CAB) = 36° and m (∠ DBA) = 44°

Find the length of the diagonal $\overline{AC}$ and the area of the parallelogram. « 25.39 cm. , 269 cm^2. »

36 ABCD is a parallelogram. M is the point of intersection of its two diagonals.

Let AC = 20 cm. , m (∠ AMD) = 130° and m (∠ CAB) = 85°

Find the length of $\overline{BD}$ and the area of the parallelogram ABCD « 28.2 cm. , 216 cm^2. »

37 ABCD is a trapezium in which $\overline{AD}$ // $\overline{BC}$, AD = 20 cm. , m (∠ D) = 120° ,

m (∠ B) = 62° and m (∠ ACB) = 23° 25′

Find :

(1) The length of each of $\overline{AC}$ and $\overline{BC}$ to the nearest cm.

(2) The area of the trapezium ABCD to the nearest cm^2. « 29 cm. , 33 cm. , 305 cm^2. »

38 ABCD is a quadrilateral in which CD = 100 cm. , m (∠ BCA) = 36° ,

m (∠ BDA) = 55° , m (∠ BCD) = 85° and m (∠ CDA) = 87°

Find the lengths of $\overline{BD}$ and $\overline{AC}$ to the nearest centimetre. « 112 cm. , 144 cm. »

39 ABCD is a quadrilateral in which m (∠ ABC) = 90° , m (∠ BAD) = 80°

, AB = AD = 10 cm. , BD = BC Calculate the area of the quadrilateral ABCD « 102 cm^2. »

40 **In any triangle ABC , prove that :**

(1) $\frac{3a - 4b}{3 \sin A - 4 \sin B} = \frac{c}{\sin C}$

(2) The area of the triangle $= \frac{a^2 \sin B \sin C}{2 \sin A}$

(3) The area of the triangle $= \frac{abc}{4r}$

"where r is the radius length of the circumcircle of the triangle ABC"

Third Higher skills

1 **Choose the correct answer from the given ones :**

(1) If the radius length of the circumcircle of the triangle ABC equals 3 cm.

, then : $\frac{abc}{\sin A \sin B \sin C} =$ ....

(a) 3 (b) 6 (c) 27 (d) 216

(2) If ABC is a triangle , then : $a \csc A + b \csc B + c \csc C =$ ....

(a) 2r (b) 4r (c) 6r (d) 8r

(3) If $a = \sin B$, $b = \sin C$, $c = \sin A$, then the circumference of the circumcircle of Δ ABC equals ....

(a) 1 (b) $\frac{\pi}{2}$ (c) π (d) 2π

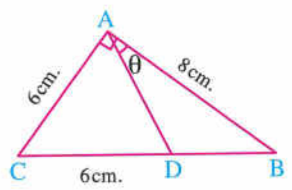
(4) In Δ ABC , $\frac{a \sin A + b \sin B + c \sin C}{a^2 + b^2 + c^2} =$ ....

(a) $\frac{1}{r^2}$ (b) $\frac{1}{2r}$ (c) 2r (d) r^2

(5) The area of Δ ABC is 24 cm^2 and the radius length of its circumcircle is 5 cm. , then $\sin A \sin B \sin (A + B) =$ ....

(a) $\frac{3}{25}$ (b) $\frac{6}{25}$ (c) $\frac{9}{25}$ (d) $\frac{12}{25}$

(6) In the opposite figure :

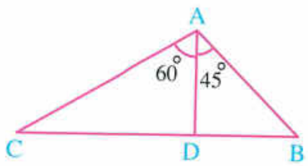


$\cot \theta =$ ....

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$

(c) 2 (d) 1

(7) In the opposite figure :

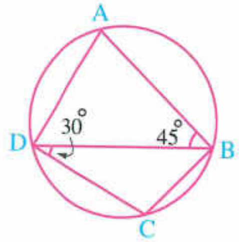


If CD = 2DB , then $\frac{\sin B}{\sin C}$ = ...

(a) $\sqrt{6}$ (b) $\frac{\sqrt{6}}{4}$

(c) $\frac{\sqrt{6}}{3}$ (d) $\frac{2\sqrt{6}}{3}$

(8) In the opposite figure :

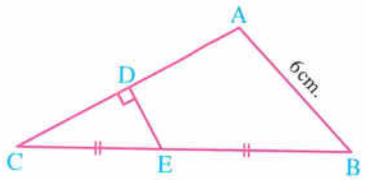


If AD = $4\sqrt{2}$ cm. , then BC = ... cm

(a) $2\sqrt{2}$ (b) 4

(c) $4\sqrt{2}$ (d) 8

(9) In the opposite figure :



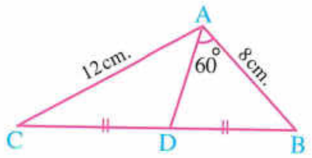
If tan (∠ DEC) = $\frac{3}{4}$,

then the radius length of

the circumcircle of Δ ABC = ... cm.

(a) 3 (b) 3.5 (c) 3.75 (d) 4

(10) In the opposite figure :



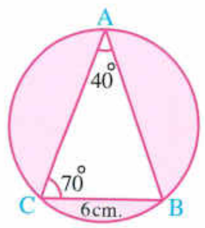
D is the midpoint of $\overline{BC}$ and

m (∠ BAD) = 60° , then :

tan (∠ DAC) = ...

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) 1 (d) $\sqrt{2}$

(11) In the opposite figure :



The area of the shaded part ≈ .... cm^2.

(a) 4.37 (b) 26.2

(c) 43.7 (d) 52.6

---

2 If ABCD is a cyclic quadrilateral , **prove that :** BC × sin (∠ ABD) = AD × sin (∠ CDB)

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3 **In the triangle ABC , prove that :**

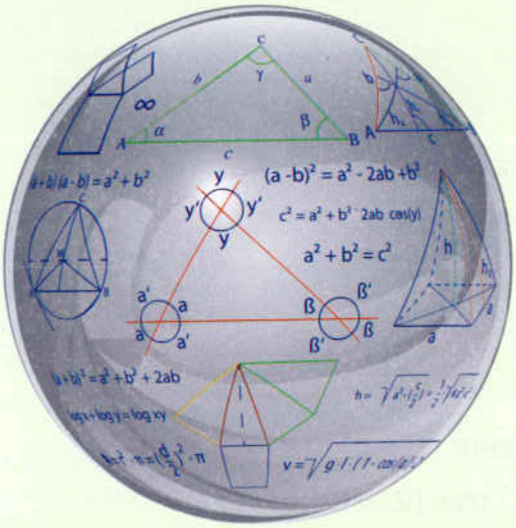
(1) $\sin A + \sin B + \sin C = \frac{4 S \Delta}{a b c}$

(2) $\frac{1}{c \sin A} + \frac{1}{a \sin B} + \frac{1}{b \sin C} = \frac{S}{\Delta}$

where S is half of the triangle's perimeter and Δ is the triangle's area.

---

4 Prove that the area of the circumcircle of Δ ABC equals $\frac{\pi a b}{4 \sin A \sin B}$



Exercise 20 The cosine rule

Test yourself

From the school book | Understand | Apply | Higher Order Thinking Skills

First Multiple choice questions

Choose the correct answer from those given :

(1) In Δ XYZ , the expression $\frac{x^2 + y^2 - z^2}{2xy}$ equals ....

(a) cos X (b) cos Y (c) cos Z (d) sin Z

(2) In Δ XYZ , $y^2 + z^2 - x^2 = 2yz \times$ ....

(a) cos X (b) sin Z (c) cos Z (d) sin X

(3) In Δ ABC , cos (A + B) = ...

(a) cos C (b) – cos C (c) sin C (d) – sin C

(4) If ABCD is a cyclic quadrilateral , then cos A + cos C = ...

(a) 1 (b) zero. (c) $\frac{1}{2}$ (d) – 1

(5) In Δ XYZ , $2xy\cos(X + Y)$ = ...

(a) $x^2 + y^2 - z^2$ (b) $y^2 + z^2 - x^2$ (c) $x^2 - z^2 + y^2$ (d) $z^2 - x^2 - y^2$

(6) In Δ LMN , ℓ = 5 cm. , m = 7 cm. , m ($\angle$ N) = 60°

, then n = ... cm. (to the nearest tenth)

(a) 6.2 (b) 5 (c) 4.3 (d) 3.5

(7) In Δ XYZ , x = 5 cm. , y = 3 cm. , m ($\angle$ Z) = $\frac{2}{3}\pi$

, then z = ...

(a) 7 (b) 8 (c) 9 (d) 4

(8) In Δ ABC , if m ($\angle$ A) + m ($\angle$ B) = 120° , a = 2 cm. , b = 3 cm. , then c = ... cm.

(a) 4 (b) 3 (c) $\sqrt{7}$ (d) $\sqrt{5}$

(9) In Δ ABC , a = 9 cm. , b = 15 cm. , m ($\angle$ C) = 106° , then its perimeter $\simeq$ .... cm.

(a) 44 (b) 24 (c) 34 (d) 28

(10) In Δ ABC , b = 2 cm. , c = 2.5 cm. , cos A = $\frac{2}{5}$, then Δ ABC is ....

(a) a right angled triangle. (b) an isosceles triangle.

(c) an equilateral triangle. (d) a scalene.

(11) In Δ XYZ , if $x = y$, then cos X = ...

(a) $\frac{2y^2}{z}$ (b) $\frac{z}{2y}$ (c) $\frac{z}{4x}$ (d) $\frac{y}{2x}$

(12) In Δ ABC , cos (A + B) = ...

(a) $\frac{a^2+b^2-c^2}{2ab}$ (b) $\frac{a^2+c^2-b^2}{2ab}$ (c) $\frac{b^2+c^2-a^2}{2bc}$ (d) $\frac{c^2-a^2-b^2}{2ab}$

(13) The measure of the greatest angle in triangle the lengths of its sides are 3 cm. , 5 cm. , 7 cm. equals ....°

(a) 110 (b) 150 (c) 100 (d) 120

(14) In Δ ABC , b = 4 cm. , a + c = 11 cm. , a – c = 1 cm. , then ....

(a) the triangle is an obtuse angled triangle.

(b) the triangle is a right angled triangle.

(c) m ($\angle$ B) = 2 m ($\angle$ A)

(d) m ($\angle$ A) = 2 m ($\angle$ B)

(15) In Δ ABC , c (a cos B + b cos A) = ...

(a) $2c^2$ (b) c^2 (c) a^2 (d) b^2

(16) In Δ ABC , if $\frac{\sin A}{\sin B}$ = 2 cos C , then ....

(a) b = c (b) a = c (c) a = b (d) a = b = c

(17) In Δ ABC , $a^2 + b^2 - c^2 + \sqrt{3}\,ab = 0$, then m ($\angle$ C) = ...°

(a) 30 (b) 150 (c) 60 (d) 120

(18) In Δ ABC , if m ($\angle$ C) = 60° , $a^2 + b^2 - c^2 = k\,a\,b$, then k = ...

(a) $\frac{1}{2}$ (b) 2 (c) 1 (d) – 1

(19) In triangle ABC , $c^2 = (a + b)^2 - ab$, then m ($\angle$ C) ....°

(a) 30 (b) 45 (c) 60 (d) 120

(20) In Δ ABC , 4 sin A = 3 sin B = 6 sin C , then m ($\angle$ C) = ...
(to the nearest degree)

(a) 89° (b) 29° (c) 57° (d) 82°

(21) In Δ ABC , $\frac{1}{2}$ sin A = $\frac{1}{3}$ sin B = $\frac{1}{4}$ sin C , then cos C = ...

(a) $\frac{-2}{3}$ (b) $\frac{2}{3}$ (c) $\frac{-1}{4}$ (d) $\frac{1}{4}$

(22) If ABC is a triangle in which : 5 sin A sin B = 6 sin B sin C = 9 sin C sin A ,
then m ($\angle$ C) $\simeq$ ....°

(a) 28 (b) 32 (c) 36 (d) 42

(23) If ABC is a triangle in which : 6 a = 4 b = 3 c , then the measure of the smallest angle in the triangle $\simeq$ ....

(a) 57° 28' (b) 41° 12' (c) 28° 57' (d) 36° 52'

(24) ABC is a triangle in which m ($\angle$ A) = 60° , b : c = 5 : 8 and the area of the circumcircle of the triangle ABC is 147 π cm^2. , then the perimeter of Δ ABC = ... cm.

(a) 21 (b) 34 (c) 54 (d) 60

(25) In the acute-angled triangle ABC , a = 8 cm. , b = 5 cm. , m ($\angle$ C) = 60°
, then m ($\angle$ A) $\simeq$ ....

(a) 83° 42' 12'' (b) 81° 47' 12'' (c) 38° 11' (d) 60° 23' 10''

(26) ABCD is a parallelogram in which AB = 8 cm. , BC = 11 cm. , BD = 9 cm. ,
then the length of $\overline{AC}$ = ... cm.

(a) 9 (b) 10 (c) 11 (d) 17

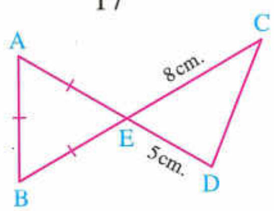
(27) ABCD is quadrilateral in which AB = 22 cm. , BC = 25 cm. , DC = 18 cm.
, m ($\angle$ ADB) = 65° , m ($\angle$ DBA) = 50° , then m ($\angle$ CBD) $\simeq$ ....

(a) 80° 75' (b) 42° 49' 19'' (c) 44° 28' 6'' (d) 85° 30'

(28) ABC is a triangle in which a = $\sqrt{2}$ cm. , b = $\sqrt{3}$ cm. , c = 2 cm.
, then $\frac{\cos A \cos B}{\cos (A + B)}$ = ...

(a) $\frac{8}{15}$ (b) $\frac{-15}{8}$ (c) $\frac{-17}{15}$ (d) $\frac{8}{17}$

(29) **In the opposite figure :**



CD = ... cm.

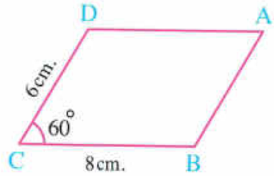
(a) 6 (b) 7

(c) 8 (d) 9

(30) In the opposite figure :

ABCD is a parallelogram

, then AC = ... cm.



(a) $2\sqrt{13}$ (b) $2\sqrt{37}$

(c) $2\sqrt{17}$ (d) 148

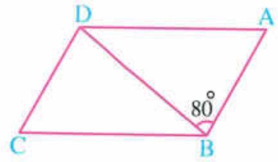
(31) In the opposite figure :

ABCD is a parallelogram

m (∠ ABD) = 80° , BD = 7 cm.

AB = 5 cm. , then the perimeter

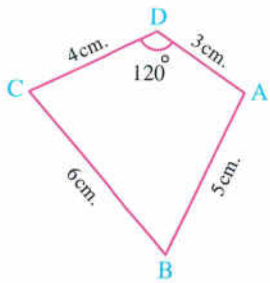
of parallelogram = ... to the nearest cm.



(a) 25 (b) 26 (c) 29 (d) 30

(32) In the opposite figure :

cos B = ...



(a) $\frac{1}{5}$

(b) $\frac{2}{5}$

(c) $\frac{3}{5}$

(d) $\frac{4}{5}$

(33) In the opposite figure :

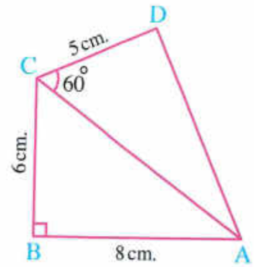
ABCD is a quadrilateral in which AB = 8 cm.

, BC = 6 cm. , m (∠ B) = 90°

, DC = 5 cm. and m (∠ ACD) = 60°

, then the area of the circumcircle of

the triangle ADC = ... cm^2.



(a) 9 π (b) 16 π (c) 25 π (d) 49 π

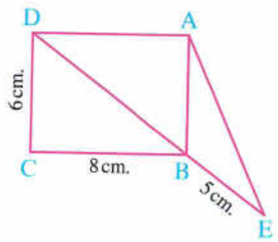
(34) In the opposite figure :

ABCD is a rectangle in which

DC = 6 cm. , BC = 8 cm.

and E ∈ $\overrightarrow{DB}$ where BE = 5 cm.

, then AE = ... cm.



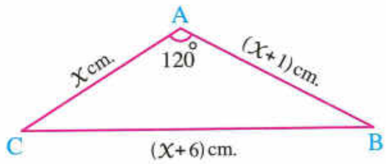
(a) $\sqrt{93}$ (b) $\sqrt{97}$ (c) 10 (d) $\sqrt{103}$

(35) **In the opposite figure :**

The value of X = ... cm.

(a) 7 (b) 8

(c) 9 (d) 10



Second Essay questions

1 XYZ is a triangle in which : m (∠ Z) = 95° , X = 13 cm. , y = 16 cm. Find z « 21.5 cm. »

2 ABC is a triangle in which : a = 3 cm. , c = 5 cm. and m (∠ B) = 36° 21′

Find b to the nearest cm. « 3 cm. »

3 Find the measures of the angles of the triangle ABC in which a = 7.6 cm.

, b = 5.8 cm. and c = 3.4 cm. « 108° 34′ , 46° 20′ , 25° 6′ »

4 ABC is a triangle in which a = 13 cm. , b = 14 cm. and c = 15 cm. Find m (∠ B) ,

then find the area of the triangle ABC to the nearest cm^2. « 59° 29′ , 84 cm^2. »

5 Find the measure of the smallest angle in Δ XYZ , where X = 18 cm. , y = 27 cm.

and z = 24 cm. Find also the area of the circumcircle of Δ XYZ « 40° 48′ , 596 cm^2. »

6 ABC is a triangle in which m (∠ C) = 96° 23′ , a = 7 cm. and b = 9 cm.

Find :

(1) c (2) The area of the triangle ABC to the nearest cm^2.

(3) The length of the radius of the circumcircle of the triangle ABC to the nearest cm.

« 12 cm. , 31 cm^2. , 6 cm. »

7 The perimeter of the triangle ABC is 52 cm. , a = 13 cm. and b = 17 cm.
Find the measure of the greatest angle in the triangle , then find the area of the triangle to the nearest centimetre square. « 93° 22′ , 110 cm^2. »

8 Find the measure of the greatest angle in Δ XYZ where X = 24.5 cm. , y = 18 cm.

and z = 10 cm. Find the circumference of the circumcircle of Δ XYZ $\left(\pi = \frac{22}{7}\right)$

« 119° 19′ , 88 cm. »

9 If the ratio among the lengths of the sides of the triangle XYZ is X : y : z = 4 : 5 : 6

, prove that the measure of the smallest angle of the triangle approximately equals 41° 25′

10 XYZ is a triangle in which sin X : sin Y : sin Z = 7 : 8 : 12

Find the measure of its greatest angle. « $106° \, 4'$ »

---

11 ABC is a triangle in which : a = 4 cm. , b = 5 cm. and $\cos C = \frac{-1}{2}$

Find c and the area of Δ ABC « 7.8 cm. , $5\sqrt{3}$ cm^2 »

---

12 ABC is a triangle in which : a = 16 cm. , c = 18 cm. , $\tan B = \frac{3}{4}$

Find the area of the triangle , then find its perimeter. « 86.4 cm^2 , 45 cm. »

---

13 ABC is a triangle in which : 2 sin A = 3 sin B = 4 sin C

Find the measure of the smallest angle. « $26° \, 23'$ »

---

14 📖 ABC is a triangle in which $\frac{1}{3}\sin A = \frac{1}{4}\sin B = \frac{1}{5}\sin C$ Find m (∠ C) and if the perimeter of the triangle = 24 cm. find its area. « 90° , 24 cm^2 »

---

15 ABC is a triangle in which D is the midpoint of $\overline{BC}$ Let m (∠ B) = 75° , m (∠ A) = 60° and a = 8 cm. Find the lengths of $\overline{AC}$ and $\overline{AD}$ « 8.92 cm. , 6.7 cm. »

---

16 ABC is a triangle in which : a = 8 cm. , b = 7 cm. and c = 9 cm. Let D ∈ $\overline{BC}$ such that BD = 4 cm. Calculate the length of $\overline{AD}$, Calculate also the length of the radius of the circumcircle of Δ ABC « 7 cm. , 4.7 cm. »

---

17 ABCD is a parallelogram in which : AC = 16 cm. , DB = 20 cm. and m (∠ AMB) = 50° , where M is the point of intersection of its diagonals.

Find AB and AD to the nearest cm. « 8 cm. , 16 cm. »

---

18 📖 ABCD is a parallelogram in which AB = 9 cm. , BC = 13 cm. and AC = 20 cm.

Find the length of $\overline{BD}$ « 10 cm. »

---

19 📖 If the perimeter of the parallelogram ABCD is 20 cm. , the ratio between the two adjacent side lengths is 2 : 3 and BD = 8 cm. , then find the length of $\overline{AC}$ « 6.3 cm. »

---

20 ABCD is a parallelogram in which : m (∠ A) = 60° , its perimeter is 44 cm. , the length of its small diagonal is 14 cm. and AB < AD **Find :** m (∠ ADB) , and then calculate the area of ▱ ABCD to the nearest cm^2 « $21° \, 47'$, 83 cm^2 »

21 ABCD is a trapezium in which : $\overline{AD} // \overline{BC}$, AD = 42 cm. , AB = 30 cm. , BC = 48 cm. and m (∠ A) = 100° **Find the length of each of :** $\overline{BD}$, $\overline{CD}$

« 55.7 cm. , 29.3 cm. »

---

22 ABCD is a quadrilateral in which AB = 9 cm. , BC = 5 cm. , CD = 8 cm. , DA = 9 cm. and AC = 11 cm.

Prove that : The figure ABCD is a cyclic quadrilateral.

---

23 ABCD is a cyclic quadrilateral in which AB = AD = 9 cm. , BC = 5 cm. , CD = 8 cm.

Find : AC « 11 cm. »

---

24 ABCD is a quadrilateral in which : AB = 6 cm. , BC = 14 cm. , CD = 10 cm. and AC = BD = 16 cm. Prove that : ABCD is a cyclic quadrilateral.

---

25 ABCD is a quadrilateral in which AB = 27 cm. , BC = 12 cm. , CD = 8 cm. , DA = 12 cm. and AC = 18 cm.

Prove that : $\overrightarrow{AC}$ bisects ∠ BAD , then find the area of the figure ABCD « 124 cm^2. »

---

26 ABCD is a quadrilateral in which : m (∠ DAB) = m (∠ DBC) = 90° , BD = 10 cm. , AD = 8 cm. and m (∠ DCB) = 30°

Find AC to the nearest cm. « 22 cm. »

---

27 ABC is a triangle in which : a = 3 b and m (∠ C) = 60° Find m (∠ B) and m (∠ A)

« 19° 6̀ , 100° 54̀ »

---

28 ABC is a triangle in which : a = 5 cm. , m (∠ B) = 120° and its area is $10\sqrt{3}$ cm^2.

Find each of c and b and also m (∠ A) « 8 cm. , 11.36 cm. , 22° 24̀ »

---

29 ABC is a triangle whose area is 64 cm^2. , m (∠ A) = 30° , b : c = 3 : 4

Find the perimeter of Δ ABC « 41.8 cm. »

---

30 ABC is a triangle in which : a = 6 cm. , b = 10 cm. , the area of the triangle is 20 cm^2.

If ∠ ACB is obtuse , find each of : m (∠ C) , the length of $\overline{AB}$ « 138° 11̀ , 15 cm. »

---

31 If $\sin A = \frac{2}{3} \sin B = \frac{1}{2} \sin C$, c – a = 4 cm.

, **find each of :** b and m (∠ A) « 6 cm. , 28° 57̀ »

32 If $\sin A : \sin B : \sin C = 3 : 5 : 7$

, prove that : $\cos A : \cos B : \cos C = 13 : 11 : -7$

33 ABC is a triangle of perimeter 70 cm. , $a = 26$ cm. and $m(\angle A) = 60°$

Find its area. « $105\sqrt{3}$ cm.^{2} »

34 ABC is a triangle whose perimeter is 34 cm. , $a = 12$ cm. and $b - c = 6$ cm.

Find the measure of its smallest angle , then calculate its area. « $34° 46' 19''$, 47.9 cm.^{2} »

35 A triangle , its side lengths are 14 , 10 and X in centimetres , if the measure of the greatest angle of the triangle is 120° Find X given that ($X < 10$) « 6 »

36 In Δ ABC : $m(\angle C) = 120°$, $a = b - 2$, $c = b + 2$ Find each of a , b , c « 3 , 5 , 7 »

37 In Δ ABC , if : $(a + b + c)(a + b - c) = 3ab$, then **prove that :** $m(\angle C) = 60°$

38 ABC is a triangle in which $(a + b + c)(a + b - c) = kab$

prove that : $k \in]0 , 4[$, then **find :** $m(\angle C)$ when $k = 1$ « 120° »

39 In the parallelogram ABCD **, prove that :** $(AC)^2 + (BD)^2 = 2(AB)^2 + 2(BC)^2$

40 ABC is a triangle in which D is the midpoint of $\overline{BC}$

Prove that : $(AB)^2 + (AC)^2 = 2(AD)^2 + 2(BD)^2$ and if AB = 5 cm. , AC = 8 cm.

and BC = 12 cm. **Find :** AD « $\frac{\sqrt{34}}{2}$ cm. »

41 **Prove that :** $\cot B + \cot C = \frac{a^2}{2\Delta}$ where Δ represents the area of Δ ABC

42 ABC is a triangle in which : $b^2 = (c - a)^2 + ca$ **Find :** $m(\angle B)$ « 60° »

43 XYZ is a triangle in which $X^2 = y^2 + z^2 - yz \cot X$

Find : $m(\angle X)$ « 90° , 30° or 150° »

44 In Δ ABC : $\cos B = \frac{c}{2a}$ **, prove that :** Δ ABC is an isosceles triangle.

45 ABCDE is a regular pentagon of side length 18.26 cm.

Find the length of its diagonal $\overline{AC}$ « 29.5 cm. »

46 **Discover the error :**

In the triangle ABC , if a = 7 cm. , b = 10 cm. , c = 5 cm. and m (∠ A) ≃ 40.54°

, **find :** m (∠ B)

Karim's answer

$\because \cos B = \frac{a^2 + c^2 - b^2}{2\,ac}$

$\therefore \cos B = \frac{(7)^2 + (5)^2 - (10)^2}{2 \times 7 \times 5}$

$\simeq -0.3714$

$\therefore m(\angle B) \simeq 111.8°$

Ziad's answer

$\because \frac{b}{\sin B} = \frac{a}{\sin A}$

$\therefore \frac{10}{\sin B} = \frac{7}{\sin 40.54°}$

$\therefore \sin B = \frac{10 \sin 40.54°}{7}$

$\simeq 0.9285$

$\therefore m(\angle B) \simeq 68.2°$

Which of the two answers is correct ? Why ?

Third Higher skills

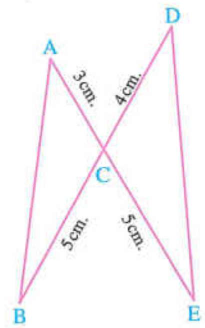
1 **Choose the correct answer from those givens :**

(1) If A (0 , 1) , B (3 , 4) , C (1 , 3) are the vertices of a triangle,

then cos (∠ ACB) = ...

(a) $\frac{4}{5}$ (b) $\frac{-4}{5}$ (c) $\frac{3}{5}$ (d) $\frac{-3}{5}$

(2) In the opposite figure :



To find the length of $\overline{DE}$ its needed to know ....

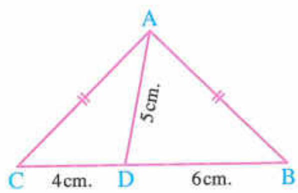
(a) the length of $\overline{AB}$

(b) the area of Δ ABC

(c) the perimeter of Δ ABC

(d) any of the previous.

(3) In the opposite figure :

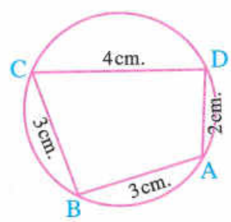


AB = ... cm.

(a) 6 (b) 7

(c) 8 (d) 9

(4) In the opposite figure :

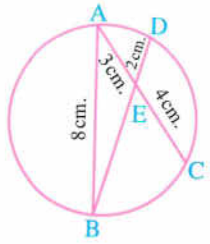


cos A = ...

(a) $-\frac{1}{6}$ (b) $-\frac{1}{3}$

(c) $\frac{1}{\sqrt{2}}$ (d) $\frac{1}{2}$

(5) In the opposite figure :

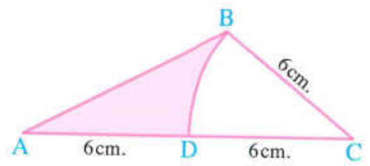


m (∠ BAE) ≈ ....

(a) 34° 39`  (b) 39° 34`

(c) 18° 34`  (d) 34° 18`

(6) In the opposite figure :



If m (∠ C) = 0.4^{rad} , then the perimeter of the shaded region ≈ .... cm.

(a) 15.3 (b) 13.4

(c) 6.9 (d) 21.3

(7) If the area of Δ ABC = 12 cm^2 , then $(b^2 + c^2 - a^2)$ tan A = ...

(a) 12 (b) 24 (c) 48 (d) 96

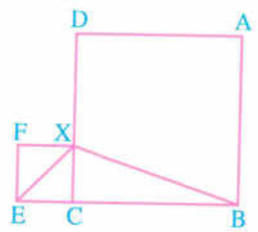
(8) In Δ ABC , if m (∠ A) = 60° , then : $(1 + \frac{a}{c} + \frac{b}{c})(1 + \frac{c}{b} - \frac{a}{b})$ = ...

(a) zero (b) 1 (c) 2 (d) 3

(9) In Δ ABC , if $\frac{a^3 + b^3 + c^3}{a + b + c} = a^2$, then m (∠ A) = ...

(a) 30° (b) 60° (c) 45° (d) 150°

(10) In the opposite figure :



ABCD , XCEF are two squares

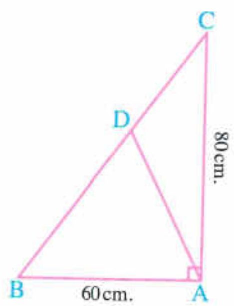
if : BC = 3 CE , then

sin (∠ BXE) = ...

(a) $\frac{1}{\sqrt{5}}$ (b) $\frac{2}{\sqrt{5}}$

(c) $\frac{-1}{\sqrt{5}}$ (d) $\frac{-2}{\sqrt{5}}$

(11) In the opposite figure :



ABC is a right-angled triangle at A , AB = 60 cm. , AC = 80 cm. , $\overline{AD}$ is drawn to divides Δ ABC into two triangles with equal perimeters , then AD = ... cm.

(a) $24\sqrt{5}$ (b) $5\sqrt{24}$ (c) $35\frac{2}{3}$ (d) $45\frac{1}{3}$

2 In Δ ABC : If $\frac{\cos B}{a} = \frac{\cos A}{b}$

Prove that : Δ ABC is a right-angled or an isosceles triangle.

---

3 ABC is a right-angled triangle at $\angle$ B. If E is a point inside the triangle such that :

EA = 10 cm. , EB = 6 cm. , m ($\angle$ AEB) = m ($\angle$ BEC) = m ($\angle$ CEA)

Find the length of $\overline{EC}$ « 33 cm. »

---

4 ABC is a right angled triangle at $\angle$ B If M and N belong to $\overline{AC}$ such that : AM = MN = NC , BM = 3 cm. , BN = 4 cm. Find the perimeter of Δ ABC to the nearest cm. « 16 cm. »

---

5 In any triangle ABC , **prove that :**

(1) $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2\,a\,b\,c}$

(2) $a^2 + b^2 + c^2 = 2\,(bc \cos A + ac \cos B + ab \cos C)$

(3) $\frac{\tan A}{\tan C} = \frac{a^2 + b^2 - c^2}{b^2 + c^2 - a^2}$



Exercise 21 Solution of the triangle

From the school book • Understand • Apply • Higher Order Thinking Skills

First Multiple choice questions

Choose the correct answer from those given :

(1) Solving the triangle means ....

(a) to find the lengths of its sides.

(b) to find the measures of its angles.

(c) to find the relation between the lengths of its sides and the measures of its angles.

(d) to find the lengths of its sides and the measures of its angles.

(2) The perimeter of Δ ABC , in which b = 11 cm. , m ($\angle$ A) = 67° , m ($\angle$ C) = 46° equals .... (to the nearest cm.)

(a) 22 (b) 38 (c) 31 (d) 27

(3) By solving the triangle ABC in which a = 5 cm. , b = 7 cm. , m ($\angle$ C) = 65° , then c $\simeq$ .... cm. (to the nearest tenth)

(a) 4.4 (b) 2.1 (c) 6.7 (d) 8.2

(4) By solving Δ ABC in which a = 2 cm. , b = $4\sqrt{2}$ cm. , c = $2\sqrt{5}$ cm. , then

First : cos A = ...

(a) $\frac{3}{\sqrt{10}}$ (b) $\frac{4}{5}$ (c) $\frac{2}{\sqrt{10}}$ (d) $\frac{\sqrt{10}}{5}$

Second : m ($\angle$ C) = ...

(a) $32^\circ\ 18'$ (b) $27^\circ\ 43'$ (c) 135° (d) 45°

(5) The number of possible solutions of Δ ABC in which m ($\angle$ C) = 115° , c = 12 cm. , a = 9 cm. is ....

(a) 1 (b) 2 (c) 3 (d) zero.

(6) The number of possible solutions of Δ ABC in which a = 8 cm. , b = 10 cm. , m ($\angle$ A) = 42° is ....

(a) 1 (b) 2 (c) infinite number. (d) zero.

(7) The number of possible solutions of Δ ABC in which m ($\angle$ A) = 60° , b = 3 cm. , a = 5 cm. is ....

(a) 1 (b) 2 (c) 0 (d) infinte number.

(8) The number of possible solutions of Δ XYZ in which x = 5 cm. , y = 6 cm. , m ($\angle$ X) = 70° equals ....

(a) zero. (b) 2 (c) 1 (d) 3

(9) In Δ XYZ , x = 30 cm. , y = 20 cm. , m ($\angle$ X) = 100° , then these conditions verify ....

(a) unique solution. (b) two solutions.
(c) three solutions. (d) no solution.

(10) In Δ ABC , a = 20 cm. , b = 25 cm. , m ($\angle$ A) = 40° , then these conditions verify ....

(a) unique solution. (b) two solutions.
(c) three solutions. (d) no solution.

(11) If Δ XYZ , m ($\angle$ X) = 100° , x = 3 cm. , y = 4 cm. , then these conditions verify ....

(a) unique solution. (b) two solutions.
(c) three solutions. (d) no solution.

(12) If Δ ABC , a = 15 cm. , m ($\angle$ B) = 30° , has a unique solution , then b could be .... cm.

(a) 8 (b) 7 (c) 7.5 (d) 8.5

(13) If Δ XYZ , x = 10 cm. , m ($\angle$ Y) = 50° has two solutions , then y could be .... cm.

(a) 6 (b) 11 (c) 7.66 (d) 8

(14) If the following conditions valid no triangle XYZ where x = 17 cm. , m ($\angle$ Y) = 92° , then y could be .... cm.

(a) 20 (b) 25 (c) 18 (d) 16

(15) If the following conditions valid no triangle LMN where ℓ = 35 cm. , m ($\angle$ M) = 75° , then m could be .... cm.

(a) 45 (b) 75 (c) 33 (d) 40

(16) By solving Δ ABC in which a = 15 cm. , cos B = $\frac{1}{2}$, tan C = $\frac{1}{\sqrt{3}}$, then the perimeter of Δ ABC = ... r "where r is the radius of the circumcircle"

(a) $\left(2+\sqrt{2}\right)$ (b) $\left(\frac{\sqrt{3}}{2}+2\right)$ (c) $\left(3+\sqrt{3}\right)$ (d) 2

Second Essay questions

Exercises on solving a triangle knowing a side length and the measures of two angles

1 Solve the triangle LMN in which m = 17 cm. , $m(\angle L) = 33^\circ\ 16'$ and $m(\angle N) = 44^\circ\ 19'$

« 9.5 cm. , 12.2 cm. , $102^\circ\ 25'$ »

2 Solve the triangle ABC in which : AB = 9 cm. and $m(\angle A) = 2\ m(\angle B) = 80^\circ$, then calculate its area to the nearest cm^2.

« 10.2 cm. , 6.7 cm. , 60° , the area ≃ 30 cm^2 »

3 Solve the triangle XYZ in which : XY = 40 cm. , $m(\angle X) = 75^\circ\ 12'$ and $m(\angle Y) = 48^\circ\ 15'$, then find the height of the triangle drawn from Z to $\overline{XY}$

« 46.4 cm. , 35.8 cm. , $56^\circ\ 33'$, the height ≃ 34.6 cm. »

Exercises on solving a triangle knowing the lengths of two sides and the measure of the included angle

4 Solve the triangle ABC in which $m(\angle A) = 153^\circ\ 12'$ and b = c = 6 cm.

« 11.67 cm. , $13^\circ\ 24'$, $13^\circ\ 24'$ »

5 Solve Δ LMN in which : $\ell = 12.5$ cm. , n = 7.25 cm. and $m(\angle M) = 1.2^{rad}$

« 11.96 cm. , $76^\circ\ 53'$, $34^\circ\ 22'$ »

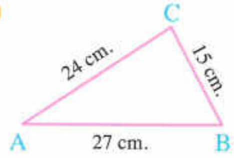
6 Solve Δ LMN in which : LM = 48.5 cm. , MN = 46 cm. and cos M = – 0.6

« 84.53 cm. , $25^\circ\ 48'$, $126^\circ\ 52'$, $27^\circ\ 20'$ »

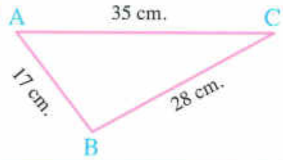
Exercises on solving a triangle knowing the lengths of the three sides

7 **Solve the triangle ABC in each of the following figures :**

(1)



(2)



8 Solve the triangle ABC in which : a = 13 cm. , b = 14 cm. and c = 15 cm.

« $53^\circ\ 8'$, $59^\circ\ 29'$, $67^\circ\ 23'$ »

9 Solve the triangle ABC in which a = 5 cm. and b = 2 c = 8 cm.

« $30^\circ\ 45'$, $125^\circ\ 6'$, $24^\circ\ 9'$ »

Exercises on solving a triangle knowing two side lengths and the measure of the opposite angle to one of them

10 Solve the triangle ABC in which a = 10 cm. , b = 9 cm. and $m(\angle B) = 57^\circ$

11 Solve the triangle ABC in which $m(\angle A) = 50^\circ$, a = 4 cm. and b = 3 cm.

12 Solve the triangle ABC in which m (∠ C) = 116° , c = 12 cm. and a = 10 cm.

13 **Show if the following conditions satisfy the existence of one triangle or more or don't satisfy the existence of any triangle at all, then find the possible solutions, approximated the side lengths to the nearest tenth and the angles measures to the nearest degree :**

(1) a = 15 cm. , b = 10 cm. and m (∠ A) = 120°

(2) a = 4 cm. , c = 16 cm. and m (∠ C) = 115°

(3) a = 12 cm. , b = 15 cm. and m (∠ A) = 100°

(4) a = 20 cm. , b = 28 cm. and m (∠ A) = 42°

(5) a = 5 cm. , b = 7 cm. and m (∠ A) = 60°

(6) a = 12 cm. , c = 7 cm. and m (∠ A) = 27°

(7) a = $4\sqrt{3}$ cm. , b = 6 cm. and m (∠ B) = 60°

(8) a = 6 cm. , b = 8 cm. and m (∠ A) = 47°

14 **Open problem :** ABC is a triangle in which m (∠ B) = 58° and a = 42 cm. Find b which makes the triangle ABC has no solution. Explain that.

Miscellaneous exercises

15 Solve the isosceles triangle ABC in which : m (∠ A) = 110° and a = 8 cm.

« 4.9 cm. , 4.9 cm. , 35° , 35° »

16 Solve the triangle ABC in which : a = 21 cm. , cos B = $\frac{3}{5}$ and tan C = $\frac{5}{12}$

« 17.3 cm. , 8.3 cm. , 104° 15′ , 53° 8′ , 22° 37′ »

17 Solve the triangle ABC in which : a = 5 cm. , m (∠ B) = 120° and its area is $10\sqrt{3}$ cm^2.

« 11.36 cm. , 8 cm. , 22° 24′ , 37° 36′ »

18 Solve the triangle ABC in which m (∠ A) : m (∠ B) : m (∠ C) = 4 : 5 : 6 and its perimeter equals 50 cm.

« 14.5 cm. , 16.9 cm. , 18.6 cm. , 48° , 60° , 72° »

19 Solve the triangle ABC in which sin A : sin B : sin C = 3 : 4 : 6 and its perimeter equals 52 cm.

« 12 cm. , 16 cm. , 24 cm. , 26° 23′ , 36° 20′ , 117° 17′ »

20 Solve the acute-angled triangle ABC in which a = 21 cm. , b = 25 cm. and the diameter length of its circumcircle = 28 cm.

« 26 cm. , 48° 35′ , 63° 14′ , 68° 11′ »

21 Solve the triangle ABC in which : c = 5 cm. , m (∠ A) = 82° and the radius length of its circumcircle = 8 cm.
« 15.8 cm. , 15.7 cm. , 79° 47' 24'' , 18° 12' 36'' »

22 Solve the triangle ABC in which : a = 7 cm. , m (∠ B) = 40° and circumference of its circumcircle = 44 cm. $\left(\pi = \frac{22}{7}\right)$
« 9 cm. , 13 cm. , 30° , 110° »

23 Solve the triangle XYZ in which : m (∠ X) = 82° , m (∠ Z) = 56° and its area = 900 cm^2.
« 56 cm. , 38 cm. , 47 cm. , 42° »

24 Solve the triangle ABC in which : m (∠ A) = 35° , m (∠ B) = 75° and a + 3 c = 25 cm.
« 4.2 cm. , 7.1 cm. , 6.9 cm. , 70° »

25 Solve the triangle ABC in which : a = 13 cm. , m (∠ B) = 42° and the length of the radius of the circumcircle of the triangle ABC is 8 cm.
« 10.71 cm. , 15.9 cm. , 54° 20' , 83° 40' or 10.71 cm. , 3.42 cm. , 125° 40' , 12° 20' »

26 **In each of the following , can the triangle ABC be formed ? If so , solve the triangle :**

(1) a = 3.2 cm. , b = 7.63 cm. and c = 6.4 cm.
(2) a = 12 cm. , b = 21 cm. and m (∠ C) = 95°
(3) a = 1 cm. , b = 5 cm. and c = 4 cm.
(4) m (∠ A) = 42° , a = 7 cm. and b = 10 cm.

Third Higher skills

Choose the correct answer from the given ones :

(1) If ABC is a triangle in which a = 3 cm. , b = 8 cm. , sin A = $\frac{5}{13}$, then the number of triangles could be drawn satisfying these conditions is ....
(a) zero (b) 1
(c) 2 (d) the information is not enough.

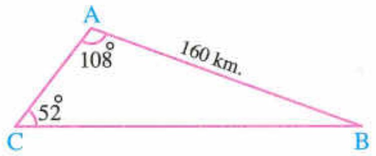
(2) In Δ ABC , AC = 8 cm. , m (∠ A) = 40° , and 8 sin 40° < BC < AC , then ....
(a) no triangle can be drawn. (b) a triangle can be drawn.
(c) two triangles can be drawn.
(d) an infinite number of triangles can be drawn.

(3) In Δ ABC , AC = 8 cm. , m (∠ A) = 40° and BC ≤ 8 sin 40° , then ....
(a) no triangle can be drawn. (b) a triangle can be drawn.
(c) two triangles can be drawn.
(d) an infinite number of triangles can be drawn.

Life Applications on Unit Four

From the school book

1 Geography :



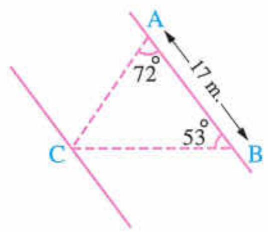
The opposite figure represents the positions of three towns A , B and C

Find to the nearest kilometre :

(1) The distance between A and C (2) The distance between B and C « 69 km. , 193 km. »

---

2 In the opposite figure :



The two places A and B are located on the same edge of a stream and the distance between them is 17 m. , the place C lies on the opposite edge such that m (∠ BAC) = 72° and m (∠ ABC) = 53° **Find :**

(1) The distance between the two places A and C to the nearest metre.

(2) The distance between the two edges to the nearest hundredth if the two edges are parallel. « 17 m. , 15.76 m. »

---

3 Art gallaries :

A picture was hanged on a wall of a gallery with a string tied by two rings on the above horizontal edge of the picture and it passes through a nail on the wall , if the length of the string on each side of the nail is 30 cm. and the measure of the angle between the two parts of the string is 50° , then find the distance between the two rings on the edge of the picture to the nearest centimeter. « 25 cm. »

---

4 Agriculture :

A farmer wanted to fence a triangular piece of land lengths of two sides of it 98 m. , 64 m. and the measure of the included angle between them is 52°

What is the length of that fence ? « 239 m. »

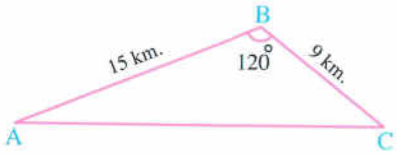
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5 Two ships A and B moved at the same moment from a port , if A moved in the direction 20° south of east for 24 km. while B moved in the direction 55° north of east for 10 km. in the same time.

Calculate the distance between them at the end of this time. « 23.5 km. »

6 **Distances :**

Kareem wanted to cover the distance from city A to city C passing by city B using his motor bike with uniform speed 36 km./h. , then returns from city C to city A with uniform speed 42 km/h. **Find :**



(1) The distance between city C and city A

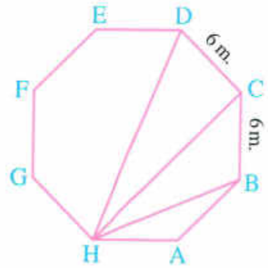
(2) The total time in minutes for the whole journey.

« 21 km. , 70 minutes »

7 **Architectural design :**

An architect designed a building at the form of regular octagon , the length of its side is 6 meter.

Find the lengths of the diagonals $\overline{HB}$, $\overline{HC}$, $\overline{HD}$



« 11.1 m. , 14.5 m. , 15.7 m. »

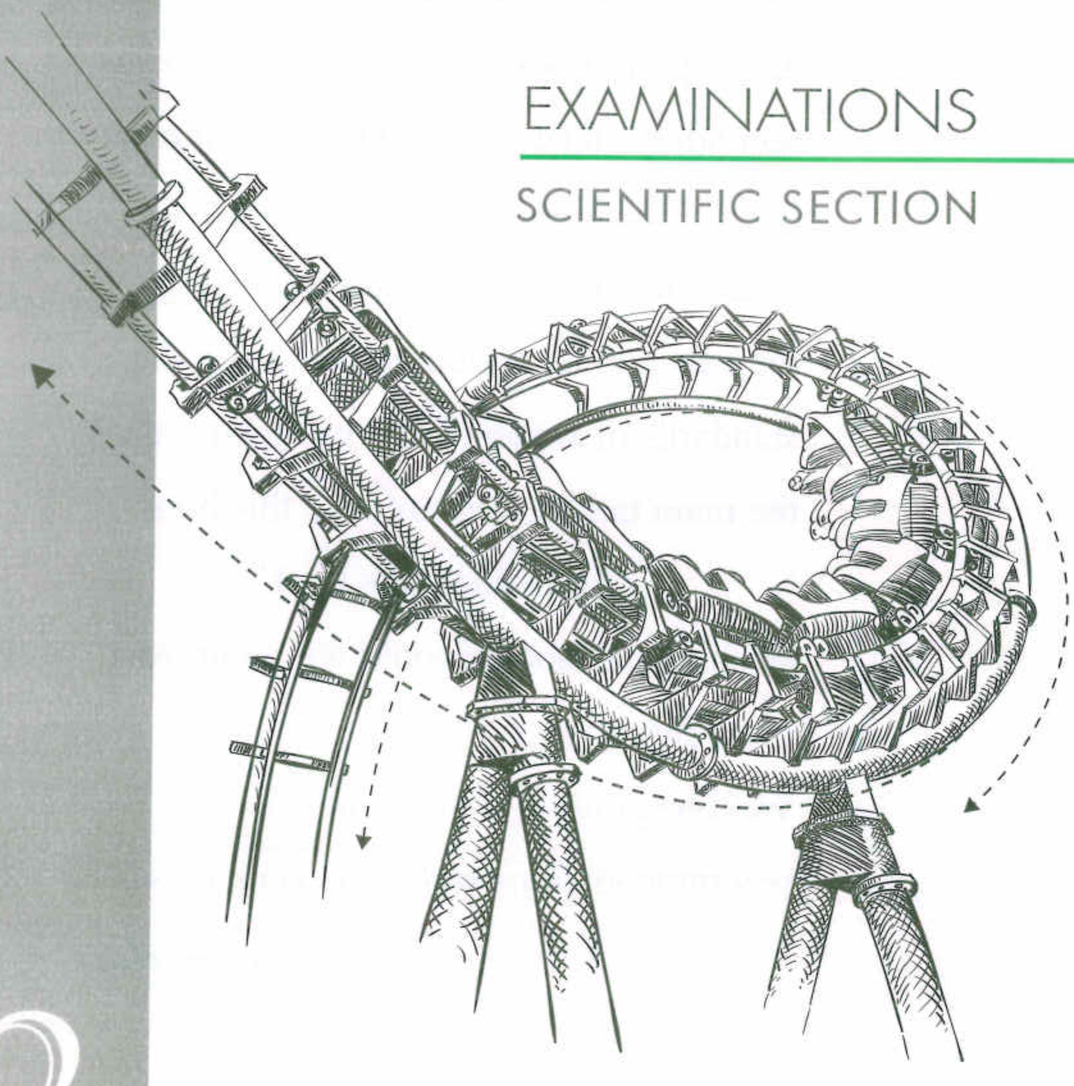
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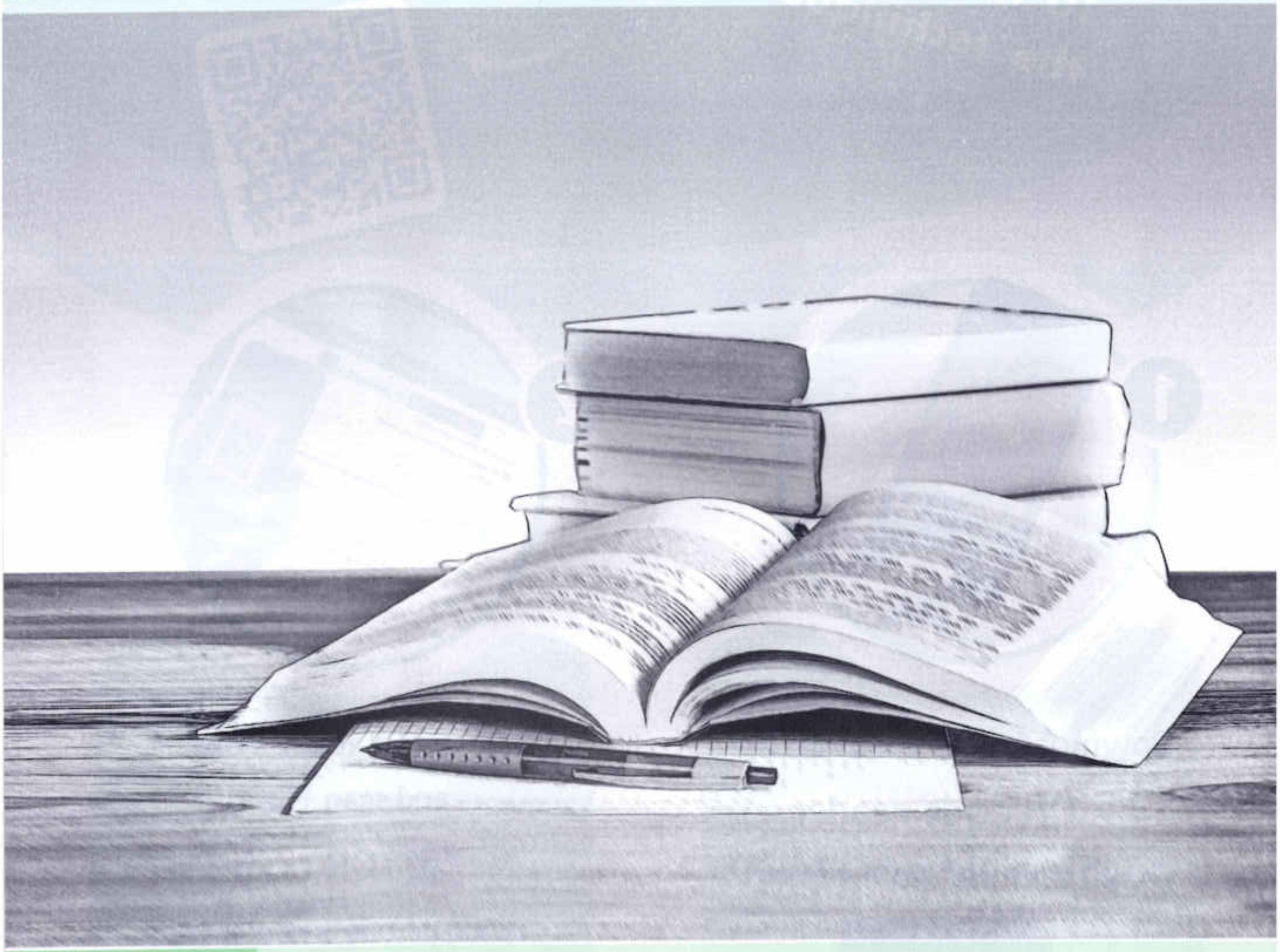
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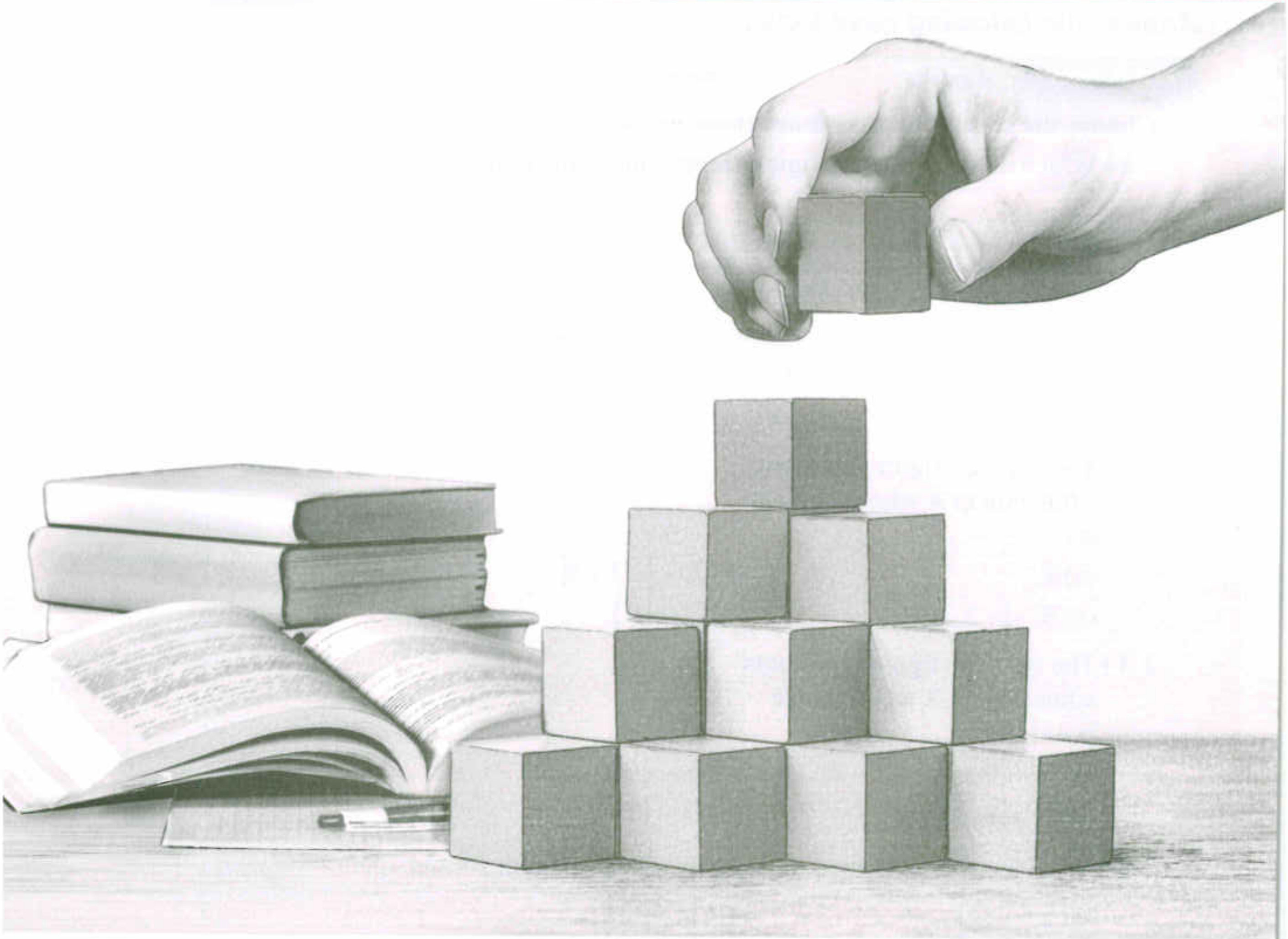
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CONTENTS



- Accumulative quizzes
- School book examinations
- Final examinations
- Guide answers

Accumulative Quizzes



- **First :** Accumulative quizzes on algebra
- **Second :** Accumulative quizzes on calculs
- **Third :** Accumulative quizzes on trigonometry

First — Accumulative quizzes on algebra

Total mark .... / 10

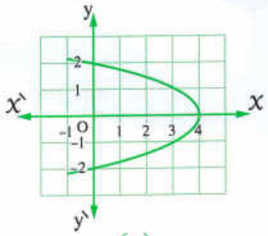
Quiz 1 on lesson 1 – unit 1

Answer the following questions :

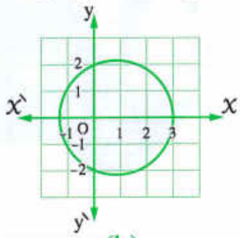
First question *4 marks* **1 mark for each item**

Choose the correct answer from those given :

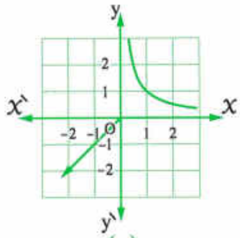
(1) Which of the following figures represents a function in x ?



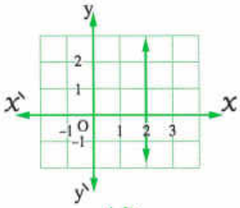
(a)



(b)

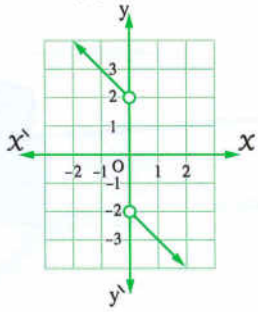


(c)



(d)

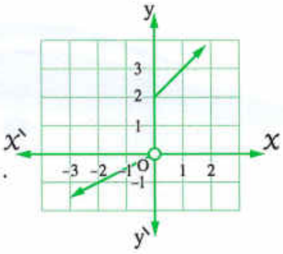
(2) The opposite figure represents a function in x whose domain is ....



(a) $\mathbb{R}$ (b) $\mathbb{R} -]-2 , 2[$

(c) $\mathbb{R} - [-2 , 2]$ (d) $\mathbb{R} - \{0\}$

(3) The opposite figure represents a function in x whose range is ....



(a) $\mathbb{R} - [0 , 2]$ (b) $\mathbb{R} - \{0\}$

(c) $\mathbb{R} - [0 , 2[$ (d) $\mathbb{R} -]0 , 2]$

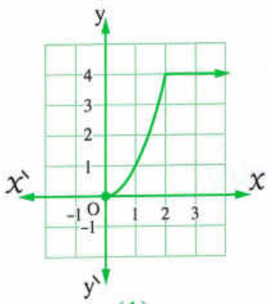
(4) If $f(x) = \sqrt{4 - x^2}$, then the domain of the function $f =$ ....

(a) $[-2 , 2]$ (b) $]-2 , 2[$

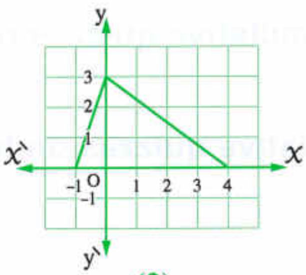
(c) $[-2 , 2[$ (d) $]-2 , 2]$

Second question *6 marks* **2 marks for each item**

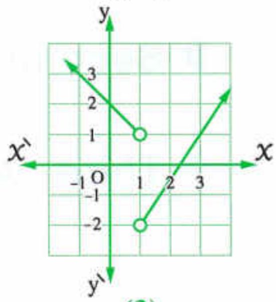
Investigate the monotony of the functions represented by the following figures :



(1)



(2)



(3)

Total mark

...... / 10

Quiz 2 till lesson 2 – unit 1

Answer the following questions :

First question *6 marks* **1 mark for each item**

Choose the correct answer from those given :

(1) If $f(X) = \frac{1}{X}$, $g(X) = \sqrt{X}$, then the domain of $(f . g) =$ ....

(a) $\mathbb{R} - \{0\}$ (b) $\mathbb{R}$ (c) $\mathbb{R}^+$ (d) $[0 , \infty[$

(2) If $f(X) = X + 1$, $g(X) = X^2$, then $(f \circ g)(2) =$ ....

(a) 3 (b) 4 (c) 5 (d) 9

(3) The domain of the function $f : f(X) = \sqrt{5 - X}$ equals ....

(a) $\mathbb{R} - \{5\}$ (b) $\mathbb{R}^+$ (c) $]-\infty , 5]$ (d) $[5 , \infty[$

(4) If $f(X) = \sqrt{X}$, $g(X) = X^2$, then the domain of $(f \circ g) =$ ....

(a) $[0 , \infty[$ (b) $\mathbb{R}$ (c) $\mathbb{R}^+$ (d) $\mathbb{R}^-$

(5) If $f(X) = \sqrt{X - 1}$, $g(X) = \sqrt{1 - X}$, then the domain of $(f + g)$ is ....

(a) $[1 , \infty[$ (b) $]-\infty , 1]$ (c) $[-1 , \infty[$ (d) $\{1\}$

(6) If the relation between X , $f(X)$, $g(X)$ is as shown in the given table for some values of X , then the value of X that satisfies that $g(f(X)) = -1$ is ....

X	$f(X)$	$g(X)$
−1	−2	4
zero	0	3
1	2	2
2	4	1
3	6	0
4	8	−1

(a) 8 (b) 3

(c) 2 (d) 4

Second question *4 marks* **2 marks for each item**

If $f(X) = \frac{1}{X}$, $g(X) = X + 3$, **find :**

(1) $(f \circ g)(X)$

(2) $(g \circ f)(X)$

and state the domain in each case.

Total mark

10

Quiz 3 till lesson 3 – unit 1

Answer the following questions :

First question ***6 marks*** **1 mark for each item**

Choose the correct answer from those given :

(1) If f is an even function in the interval $[a , b]$, then b = ...

(a) a (b) – a (c) 2 a (d) a^3

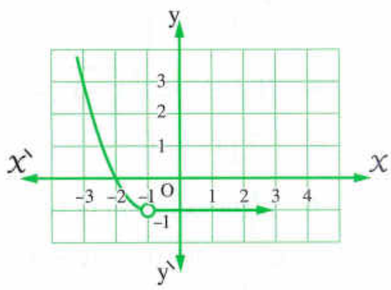
(2) The domain of the function $f : f(X) = \frac{5}{\sqrt[3]{X-8}}$ is ....

(a) $\mathbb{R}$ (b) $\mathbb{R} - \{2\}$ (c) $\mathbb{R} - \{8\}$ (d) $[8 , \infty[$

(3) The odd function from the given ones is $f : f(X) =$ ....

(a) $X \sin X$ (b) $\cos X$ (c) 5 (d) $X \cos X$

(4) The range of the function represented in the opposite figure is ....



(a) $\mathbb{R} - \{-1\}$

(b) $[-1 , \infty[$

(c) $]-1 , \infty[$

(d) $\mathbb{R}$

(5) If f is a one - to - one function and the point (2 , 3) belongs to the function f , which of the following points could belong to the function f ?

(a) (5 , 3) (b) (2 , – 1) (c) (3 , 2) (d) All the previous.

(6) If $f(X) = X^3$, $g(X) = X^2 + 1$, which of the following is an odd function ?

(I) $(f \times g)$ (II) $(f \circ g)$ (III) $(g \circ f)$

(a) (I) only. (b) (II) and (III) (c) (I) and (II) (d) (I) and (III)

Second question ***4 marks***

If $f_1(X) = X^5$, $f_2(X) = \sin X$, find $(f_1 + f_2)(X)$ hence find the type of $(f_1 + f_2)$ whether it is even , odd or otherwise.

Total mark

.... / 10

Quiz 4 till lesson 4 – unit 1

Answer the following questions :

First question *3 marks*

Graph the function $f : f(x) = \begin{cases} |x| & , \quad x \le 0 \\ x^2 & , \quad x > 0 \end{cases}$ from the graph , deduce the range of the function and determine its type whether it is even , odd or otherwise and investigate its monotony.

Second question *2 marks*

Find the domain of the function $f : f(x) = \frac{2x + 1}{x - 2}$ and prove that f is one - to - one.

Third question *3 marks*

If $f(x) = x^2 - 1$, $g(x) = x + 1$, graph the function $\frac{f}{g}$, determine the domain and the range of the function then investigate its monotony.

Fourth question *2 marks*

Graph the function $f : f(x) = \begin{cases} x - 1 & , \quad 2 < x \le 4 \\ -1 & , \quad -2 \le x \le 2 \end{cases}$

from the graph , deduce the range and investigate the monotony.

Total mark

.... / 10

Quiz 5 till lesson 5 – unit 1

Answer the following questions :

First question *6 marks* **1 mark for each item**

Choose the correct answer from those given :

(1) The curve of the function $f : f(x) = x^2 + 4$ is the same curve of the function $g : g(x) = x^2$ by translation of a magnitude 4 units in the direction of ....

(a) $\overrightarrow{Ox}$ (b) $\overleftarrow{Ox}$ (c) $\overrightarrow{Oy}$ (d) $\overleftarrow{Oy}$

(2) The function which is one - to - one from the functions defined by the following rules is ....

(a) $f_1(x) = x + 2$ (b) $f_2(x) = x^2$

(c) $f_3(x) = |x|$ (d) $f_4(x) = 5$

(3) If f is a function where $f(x) = \frac{1}{x}$, then the point of symmetry of the function $f(x + 1)$ is ....

(a) $(1, 0)$ (b) $(0, 1)$ (c) $(-1, 0)$ (d) $(-1, 1)$

(4) If $f(x) = \sqrt{x + 4}$, $g(x) = x^2 - 4$, then $(f \circ g)(x) =$ ....

(a) $|x|$ (b) x^2 (c) $x^2 + 4$ (d) 2

(5) The domain of a real function $f(x)$ is $[-2, 3]$, then the domain of the function $g(x) = f(x - 2)$ is ....

(a) $[-2, 3]$ (b) $[-4, 1]$ (c) $[0, 5]$ (d) $\mathbb{R}$

(6) If $f(x)$ is an odd function, then $|f(x)|$ is ....

(a) odd. (b) even.

(c) both odd and even. (d) neither odd nor even.

Second question *4 marks*

Graph the function $f : f(x) = |4 - x^2|$ from the graph, deduce the range of the function and determine its type whether it is even, odd or otherwise and investigate its monotony.

Total mark .... / 10

Quiz 6 till lesson 6 – unit 1

Answer the following questions :

First question *6 marks* **1 mark for each item**

Choose the correct answer from those given :

(1) The domain of the function $f : f(X) = \frac{5}{\sqrt{X-4}}$ is ....

(a) $[4, \infty[$ (b) $]4, \infty[$ (c) $]-\infty, 4]$ (d) $]-\infty, -4[$

(2) The function f where $f(X) = \begin{cases} 2 & , X > 0 \\ -2 & , X < 0 \end{cases}$ is symmetric about the point ....

(a) $(2, 0)$ (b) $(-2, 0)$ (c) $(0, 0)$ (d) $(2, -2)$

(3) The area bounded by the two curves of the functions $f : f(X) = |X + 3| - 2$, $g : g(X) = \text{zero}$ is .... area units.

(a) 2 (b) 3 (c) 4 (d) 5

(4) The function which is one - to - one from the functions defined by the following rules is ....

(a) $f(X) = X^2$ (b) $f(X) = |X|$ (c) $f(X) = \frac{1}{X}$ (d) $f(X) = 3$

(5) The solution set of the inequality $|X - 2| \leq -4$ in $\mathbb{R}$ is ....

(a) $]-2, 6[$ (b) $[-2, 6]$ (c) $\mathbb{R}$ (d) $\varnothing$

(6) The curve of the function $f : f(X) = -X^3$ is moved 4 units to the right and 2 units downward and the resulted curve is $g(X)$, then $g(-2) =$ ....

(a) -218 (b) -20 (c) 6 (d) 214

Second question *4 marks* **2 marks for each item**

Find in $\mathbb{R}$ the solution set of each of the following :

(1) $\sqrt{X^2 - 6X + 9} + 2X = 9$

(2) $\frac{1}{|2X - 3|} > 2$

Total mark

10

Quiz 7 till lesson 1 – unit 2

Answer the following questions :

First question *6 marks* **1 mark for each item**

Choose the correct answer from those given :

(1) If f is an odd function in the interval $[a , b]$, then b = ...

(a) a (b) – a (c) 2 a (d) a^3

(2) The symmetric point of the function $f : f(X) = (X-2)^3 + 1$ is ....

(a) (2 , 1) (b) (– 2 , 1) (c) (2 , – 1) (d) (– 2 , – 1)

(3) The solution set of the equation : $\sqrt{X^3} = 8$ in $\mathbb{R}$ is ....

(a) $\{2\}$ (b) $\{4\}$ (c) $\{16\}$ (d) $\{64\}$

(4) The solution set of the equation : $X^{\frac{2}{3}} = 25$ in $\mathbb{R}$ is ....

(a) $\{5\}$ (b) $\{5 , -5\}$ (c) $\{125\}$ (d) $\{125 , -125\}$

(5) If $7^{X+1} = 3^{2X+2}$, then X = ...

(a) – 1 (b) 1 (c) 4 (d) zero

(6) If $f_1(X) = 3^X$, $f_2(X) = 9^X$, then the value of X that satisfies that $f_1(2X-1) + f_2(X+1) = 756$ is ....

(a) 2 (b) 4 (c) 6 (d) 7

Second question *4 marks* **2 marks for each item**

Find in $\mathbb{R}$ the solution set of each of the following equations :

(1) $X^{\frac{4}{3}} - 10\,X^{\frac{2}{3}} + 9 = 0$

(2) $|X+2| = 3X - 10$

Total mark

Quiz 8 till lesson 2 – unit 2

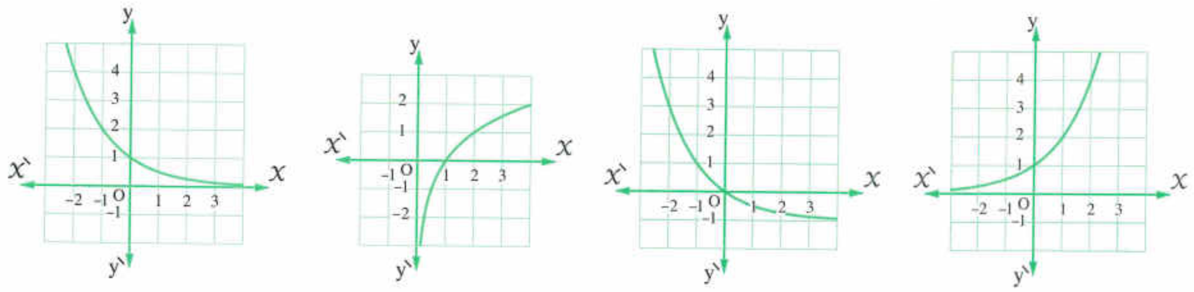
.... / 10

Answer the following questions :

First question *6 marks* **1 mark for each item**

Choose the correct answer from those given :

(1) The graphical representation of the function f where $f(\mathcal{X}) = 2^{\mathcal{X}}$ is ....



(a) (b) (c) (d)

(2) $f(\mathcal{X}) = \frac{1}{\mathcal{X}}$, then the symmetric point of the function whose rule $g(\mathcal{X}) = f(\mathcal{X} + 1)$ is ....

(a) (1 , 0) (b) (0 , 1) (c) (– 1 , 0) (d) (– 1 , 1)

(3) The sum of the roots of the equation : $\mathcal{X}^4 = 16$ equals ....

(a) 2 (b) – 2 (c) zero (d) ± 2

(4) If $f(\mathcal{X}) = a^{\mathcal{X}}$, $a > 1$, then $f(\mathcal{X}) > 1$ at ....

(a) $\mathcal{X} \in \mathbb{R}$ (b) $\mathcal{X} \in \mathbb{R}^+$ (c) $\mathcal{X} \in \mathbb{R}^-$ (d) $\mathcal{X} \in \mathbb{Z}$

(5) The function $f : f(\mathcal{X}) = (2a)^{\mathcal{X}}$ is decreasing when $a \in$ ....

(a) $]0 , 1[$ (b) $]1 , \infty[$ (c) $]0 , 2[$ (d) $]0 , \frac{1}{2}[$

(6) If $f(\mathcal{X}) = 3\mathcal{X} + 1$, $g(\mathcal{X}) = \mathcal{X}^2 - 3$, then $(f \circ g)(2) =$ ....

(a) 6 (b) 4 (c) 3 (d) – 4

Second question *4 marks*

The number of bees in a cell increases at a rate 25% each week , given that the number of bees was 60 bees , write an exponential function represents the number of bees after t weeks then estimate the number of bees after 6 weeks.

Total mark

10

Quiz 9 till lesson 3 – unit 2

Answer the following questions :

First question *6 marks* **1 mark for each item**

Choose the correct answer from those given :

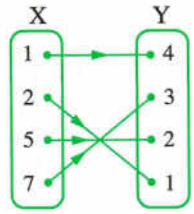
(1) The opposite figure represents the function :

$f : X \longrightarrow Y$

, then $f^{-1}(4) =$ ....

(a) 1 (b) 2

(c) 5 (d) 7



(2) If $3^{f(x)} = 2x - 1$, then $f^{-1}(0) =$ ....

(a) 1 (b) -1 (c) 2 (d) 5

(3) The curve of $f : f(x) = |x + 3|$ is the same curve of $g : g(x) = |x|$ by translation of magnitude 3 units in the direction of ....

(a) $\overrightarrow{Ox}$ (b) $\overleftarrow{Ox}$ (c) $\overrightarrow{Oy}$ (d) $\overleftarrow{Oy}$

(4) The domain of the function $f : f(x) = \frac{1}{|x| - 3}$ is ....

(a) $\{3, -3\}$ (b) $[-3, 3]$ (c) $\mathbb{R} - [-3, 3]$ (d) $\mathbb{R} - \{-3, 3\}$

(5) If the curve of the function f intersects the curve of the function f^{-1} at $\left(a, \frac{4}{a}\right)$, then a = ...

(a) 2 (b) ± 2 (c) 4 (d) ± 4

(6) If $f(x) = x^2$, $g(x) = x - 3$, then the solution set of the equation $g(f(x)) = g^{-1}(x)$ in $\mathbb{R}$ is ....

(a) $\{2, -3\}$ (b) $\{3\}$ (c) $\{3, -2\}$ (d) $\{2, 3\}$

Second question *4 marks* **2 marks for each item**

If f is a function such that $f(x) = \frac{2x + 3}{x + 1}$

, find : (1) The domain and the range of the function.

(2) $f^{-1}(x)$ and determine its domain and range.

Total mark

...... / 10

Quiz 10 till lesson 4 – unit 2

Answer the following questions :

First question *6 marks* **1 mark for each item**

Choose the correct answer from those given :

(1) If $\log_{X} 4 = 2$, then $X =$ ....

(a) 4 (b) ± 2 (c) 2 (d) -2

(2) The domain of the function $f : f(X) = \log_{1-X} X$ is ....

(a) $X > 0$ (b) $X < 1$ (c) $0 < X < 1$ (d) $0 \le X \le 1$

(3) If $f(X) = 6X$, then $f^{-1}(X) =$ ....

(a) $6X$ (b) $\frac{6}{X}$ (c) $\frac{X}{6}$ (d) $X - 6$

(4) The solution set of the equation : $|X - 3| + 1 = 0$ in $\mathbb{R}$ is ....

(a) $\mathbb{R}$ (b) $\{-1\}$ (c) $\varnothing$ (d) $\{4\}$

(5) If $f(X) = \log_a (2X + 4)$ and $f^{-1}(5) = 14$, then a = ...

(a) 1 (b) 2 (c) 3 (d) 4

(6) The product of the two roots of the equation :

$X^2 - 3|X| - 10 = 0$ equals ....

(a) -25 (b) -15 (c) 10 (d) 25

Second question *4 marks* **2 marks for each item**

Solve in $\mathbb{R}$ each of the following equations :

(1) $\log_4 \log_2 \log_3 (X + 1) = 0$

(2) $\left(\frac{3}{2}\right)^{|X-2|} = 3\,\frac{3}{8}$

Total mark

...... / 10

Quiz 11 till lesson 5 – unit 2

Answer the following questions :

First question *6 marks* **1 mark for each item**

Choose the correct answer from those given :

(1) The expression $\frac{3 \log 2}{\log 4 + \log 3}$ is equivalent to ....

(a) $\log_3 2$ (b) $\log_7 2$ (c) $\log_{12} 8$ (d) $\log_7 8$

(2) If the curve of $y = \log_4 (1 - a X)$ passes through $\left(\frac{1}{4}, -\frac{1}{2}\right)$, then a = ...

(a) 2 (b) 3 (c) 4 (d) 8

(3) If $f(X) = 3X + 1$, $g(X) = X^2 - 5$, then $(g \circ f)(-3) =$ ....

(a) – 5 (b) 5 (c) 59 (d) – 95

(4) If $y = \sqrt[3]{X}$ for each $X \geq 0$, then the inverse function of y = ...

(a) $\frac{1}{3} X^3$ (b) X^3 (c) $X^3 - 1$ (d) $X^{-\frac{1}{3}}$

(5) If L , M are the roots of the equation : $3X^2 - 16X + 12 = 0$

, then the value of $\log_2 L + \log_2 M =$ ....

(a) 2 (b) 4 (c) 12 (d) 16

(6) $\frac{1}{1 + \log_b a + \log_b c} + \frac{1}{1 + \log_c a + \log_c b} + \frac{1}{1 + \log_a b + \log_a c} =$ ....

(a) $\log_a bc$ (b) $\log_b ac$ (c) $\log_c ab$ (d) 1

Second question *4 marks*

If $\log (X + y) = \frac{1}{2} (\log X + \log y) + \log 2$

, prove that : $X = y$

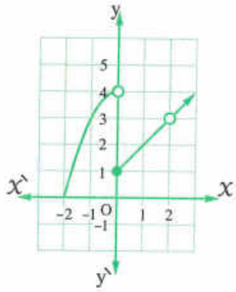
Second Accumulative quizzes on calculus

Quiz 1 on lesson 1 – unit 3

Total mark
....
10

Answer the following questions :

First question *4 marks* **1 mark for each item**



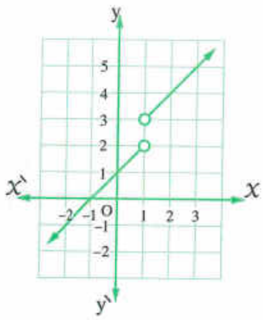
From the opposite figure , find :

(1) $f\ (\text{zero}^{+})$ (2) $f\ (\text{zero}^{-})$

(3) $f\ (2)$ (4) $\lim\limits_{x \to 2} f\ (x)$

Second question *6 marks* **2 marks for each item**

Choose the correct answer from those given :

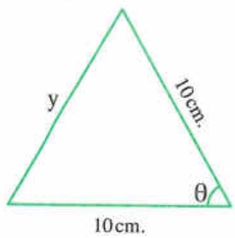


(1) The opposite figure represents the graph of the function f , then

$\lim\limits_{x \to 1} f\ (x) =$ ....

(a) 2 (b) 3

(c) 1 (d) not exist.



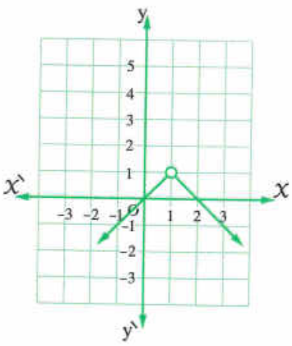
(2) **In the opposite figure :**

At $\theta \longrightarrow \frac{\pi}{2}$, then : $y \longrightarrow$ .... cm.

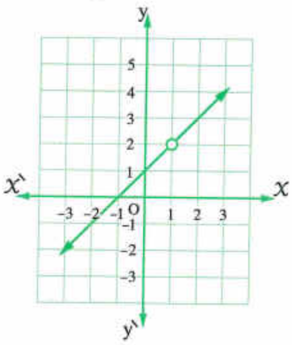
(a) zero (b) 5

(c) 10 (d) $10\sqrt{2}$

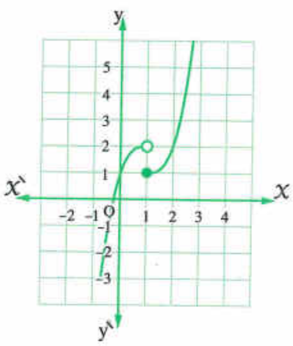
(3) Which of the following functions has no limit at $x = 1$?



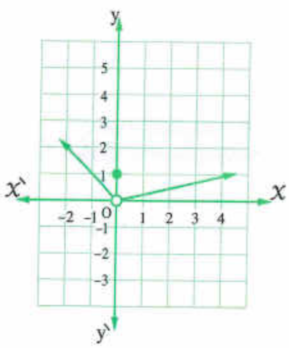
(a)



(b)



(c)



(d)

Total mark

10

Quiz 2 till lesson 2 – unit 3

Answer the following questions :

First question *2 marks* $\frac{1}{2}$ mark for each item

Choose the correct answer from those given :

(1) $\lim\limits_{x \to 0} \frac{1+x}{4x-1} = \dots\dots\dots$

(a) -1 (b) $\frac{1}{4}$ (c) $-\frac{1}{4}$ (d) 1

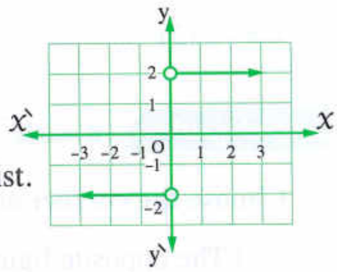
(2) $\lim\limits_{x \to 3} \frac{x^2-9}{x-3} = \dots\dots\dots$

(a) -6 (b) zero (c) 3 (d) 6

(3) The opposite figure represents $f(x)$

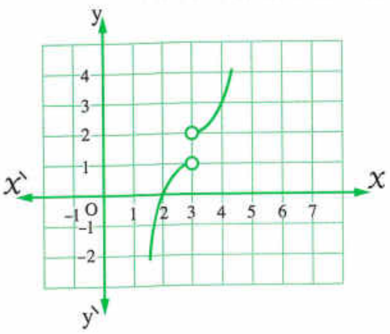
, then : $\lim\limits_{x \to 2} f(x) = \dots\dots\dots$

(a) 0 (b) -2 (c) 2 (d) not exist.

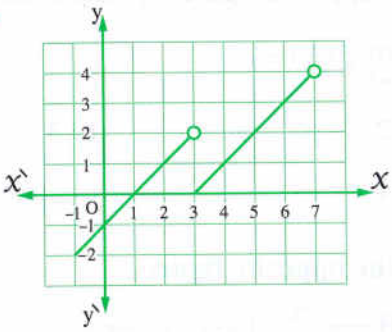


(4) Which of the following functions has a limit at $x = 3$?

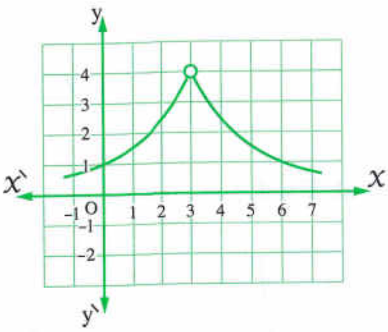
(a)



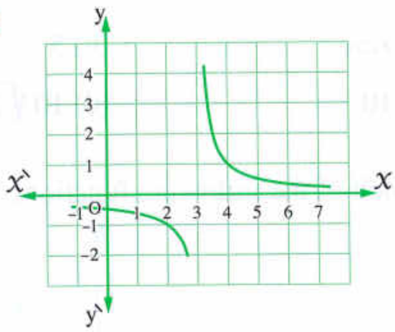
(b)



(c)



(d)



Second question *8 marks* 2 marks for each item

Find :

(1) $\lim\limits_{x \to 0} \frac{\sqrt{x+4}-2}{x^2+x}$

(2) $\lim\limits_{x \to 1} \frac{3x^2+x-4}{x^2-1}$

(3) $\lim\limits_{x \to 4} \frac{x^3-15x-4}{x-4}$

(4) $\lim\limits_{x \to 1} \left(\frac{1}{x-1} - \frac{3}{x^3-1}\right)$

Total mark
....
10

Quiz 3 till lesson 3 – unit 3

Answer the following questions :

First question *6 marks* **1 mark for each item**

Choose the correct answer from those given :

(1) $\lim\limits_{x \to 1} \dfrac{x^7 - 1}{x - 1} = \dots\dots$

(a) 7 (b) 8 (c) 6 (d) zero

(2) $\lim\limits_{x \to a} \dfrac{x^n - a^n}{x^m - a^m} = \dots\dots$

(a) $\dfrac{m}{n}$ (b) $\dfrac{m}{n} (a)^{m-n}$ (c) $\dfrac{n}{m} (a)^{m-n}$ (d) $\dfrac{n}{m} (a)^{n-m}$

(3) $\lim\limits_{x \to 0} \dfrac{(x + h)^5 - x^5}{h} = \dots\dots$

(a) x^5 (b) $5x^4$ (c) zero (d) 1

(4) **In the opposite figure :**

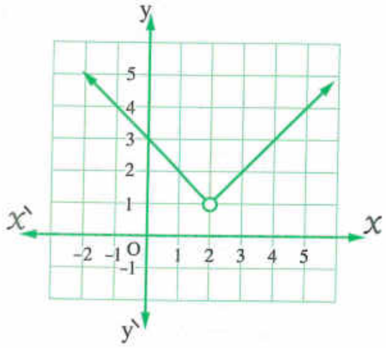
$\lim\limits_{x \to 2} f(x) = \dots\dots$

(a) 1

(b) – 1

(c) 2

(d) not exist.



(5) If n is a function and $\lim\limits_{x \to 2} \dfrac{n(x) - 8}{x - 2} = 7$

, then $\lim\limits_{x \to 2} \dfrac{2x^2 - n(x)}{x - 2} = \dots\dots$

(a) 1 (b) 4 (c) 8 (d) 15

(6) $\lim\limits_{x \to 1} \dfrac{x^6 - 64}{x - 2} = \dots\dots$

(a) $6(2)^5$ (b) 128 (c) $64(2)^5$ (d) 63

Second question *4 marks* **2 marks for each item**

Find :

(1) $\lim\limits_{x \to 2} \dfrac{x^7 - 128}{x^2 + 3x - 10}$

(2) $\lim\limits_{x \to -2} \dfrac{(x + 3)^5 - 1}{x + 2}$

Total mark

....

10

Quiz 4 till lesson 4 – unit 3

Answer the following questions :

First question *2 marks* $\frac{1}{2}$ **mark for each item**

Choose the correct answer from those given :

(1) $\lim\limits_{x \to \infty} \frac{x^{-3}+3x^{-2}+1}{x^{-2}+x^{-1}+3} = \text{...............}$

(a) 2 (b) 1 (c) 3 (d) $\frac{1}{3}$

(2) $\lim\limits_{x \to 3} \frac{x^3-27}{x^2-9} = \text{...............}$

(a) $\frac{3}{2}$ (b) $4\frac{1}{2}$ (c) 3 (d) 27

(3) If $\lim\limits_{x \to 2} \frac{x^2-4a}{x-2}$ exists , then a = ...

(a) – 1 (b) 1 (c) 2 (d) 4

(4) $\lim\limits_{x \to \infty} (4+3x-x^3) = \text{...............}$

(a) 4 (b) 2 (c) ∞ (d) $-\infty$

Second question *8 marks* **2 marks for each item**

Find :

(1) $\lim\limits_{x \to \infty} \frac{1}{x}\sqrt{3+4x^2}$

(2) $\lim\limits_{x \to \infty} x\left(\sqrt{4x^2+1}-2x\right)$

(3) $\lim\limits_{x \to \infty} \frac{x^7+5}{3x^4-8}$

(4) $\lim\limits_{x \to 4} \frac{(x-3)^7-1}{x-4}$

Total mark

...... / 10

Quiz 5 till lesson 5 – unit 3

Answer the following questions :

First question *6 marks* **1 mark for each item**

Choose the correct answer from those given :

(1) $\lim_{x \to 2} 2x \csc 4x = \ldots\ldots$

(a) 2 (b) 4 (c) $\frac{1}{2}$ (d) zero

(2) $\lim_{x \to \infty} \frac{x^3+5}{x(2x^2+3)} = \ldots\ldots$

(a) $\frac{5}{8}$ (b) 1 (c) $\frac{1}{2}$ (d) $\frac{5}{3}$

(3) $\lim_{x \to 0} \frac{2x + \sin 3x}{\tan 5x} = \ldots\ldots$

(a) 5 (b) $\frac{6}{5}$ (c) 1 (d) zero

(4) $\lim_{x \to 0} \frac{5 - 5\cos x}{x} = \ldots\ldots$

(a) 25 (b) 5 (c) zero (d) 2

(5) If a < b < zero , then $\lim_{x \to \infty} \frac{x^a}{x^b} = \ldots\ldots$

(a) ∞ (b) $-\infty$ (c) zero (d) a – b

(6) $\lim_{x \to 0} \frac{\sin x + \sin 2x + \sin 3x + \sin 4x}{\tan x + \tan 2x + \tan 3x + \tan 4x} = \ldots\ldots$

(a) zero (b) 1 (c) 4 (d) 10

Second question *4 marks* **2 marks for each item**

Find :

(1) $\lim_{x \to 0} \frac{x^2 + \sin 3x^2}{x \tan 2x}$

(2) $\lim_{x \to 0} \frac{\sqrt{x+4} - 2}{x^2 + x}$

Total mark 10

Quiz 6 till lesson 6 – unit 3

Answer the following questions :

First question *6 marks* **1 mark for each item**

Choose the correct answer from those given :

(1) If $f(x) = x^2$, then $\lim\limits_{x \to 2} f(f(x)) = \ldots\ldots$

(a) 2 (b) 4 (c) 16 (d) 32

(2) $\lim\limits_{x \to 9} \frac{3 - \sqrt{x}}{27 - \sqrt{x^3}} = \ldots\ldots$

(a) $\frac{1}{9}$ (b) $\frac{1}{27}$ (c) 3 (d) $-\frac{1}{27}$

(3) If $f(x) = \begin{cases} a \cos x + \frac{15 \sin x}{x} & , \quad x > 0 \\ \frac{x^2 - 64}{x^3 - 8} & , \quad x < 0 \end{cases}$

and f has a limit at $x = 0$, then a = ...

(a) -7 (b) 8 (c) -1 (d) 1

(4) If $\lim\limits_{x \to \infty} \frac{a x + 6}{2 x - 7} = 4$, then a = ... where $a \in \mathbb{R}$

(a) 2 (b) 4 (c) 6 (d) 8

(5) If f is an even function and $\lim\limits_{x \to 2} f(x) = 5$

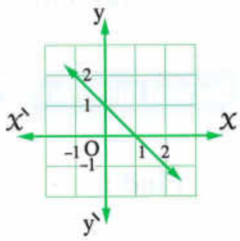
, which of the following statements is true ?

(a) $\lim\limits_{x \to -2} f(x) = 5$ (b) $\lim\limits_{x \to -2} f(x) = -5$

(c) $\lim\limits_{x \to -2} f(x) = \text{zero}$ (d) $\lim\limits_{x \to -2} f(x) = -2$

(6) The opposite figure represents the curve of the function f

, then $\lim\limits_{x \to 2} |f(x)| = \ldots\ldots$



(a) -1 (b) zero

(c) 1 (d) not exist.

Second question *4 marks*

Investigate the existence of $\lim\limits_{x \to 3} f(x)$, given that $f(x) = \begin{cases} \frac{x^2 - 7x + 12}{x - 3} & , \quad x > 3 \\ 2x - 7 & , \quad x < 3 \end{cases}$

Total mark

....

10

Quiz 7 till lesson 7 – unit 3

Answer the following questions :

First question *6 marks* 1 mark for each item

Choose the correct answer from those given :

(1) The function $f : f(x) = 4x^{-3} + \frac{x}{x^2 - 9}$ is continuous for every $x \in$ ....

(a) $\mathbb{R}$ (b) $\mathbb{R} - \{0\}$ (c) $\mathbb{R} - \{3, -3\}$ (d) $\mathbb{R} - \{3, -3, 0\}$

(2) If $f : f(x) = \begin{cases} \frac{x^8 - a^8}{x^5 - a^5} & , \quad x \neq a \\ 200 & , \quad x = a \end{cases}$ is continuous at $x = a$, then a = ...

(a) 5 (b) $\frac{8}{5}$ (c) 125 (d) $\frac{1}{5}$

(3) The function $f : f(x) = \frac{x + 2}{\sqrt{x - 2}}$ is continuous for every $x \in$ ....

(a) $[4, \infty[$ (b) $[0, \infty[$ (c) $[0, \infty[- \{4\}$ (d) $]-\infty, 2[$

(4) If the function $f : f(x) = \begin{cases} \frac{|x|}{x} + 6 & , \quad x < 0 \\ a^2 + \cos 3x & , \quad x \geq 0 \end{cases}$

is continuous at $x = 0$, then a = ...

(a) 2 (b) $\pm\sqrt{6}$ (c) ± 2 (d) $\pm\sqrt{5}$

(5) If f is a one-to-one polynomial function and $\lim_{x \to 2} f(x) = 3$

, then : $\lim_{x \to 3} f^{-1}(x) =$ ....

(a) -2 (b) -3 (c) 2 (d) 3

(6) $\lim_{x \to 0} \frac{x \sin 2x + \sin 3x^2}{\tan^2 3x + x^2} =$ ....

(a) $\frac{1}{2}$ (b) 2 (c) zero (d) $\frac{11}{10}$

Second question *4 marks*

Discuss the continuity of the function f, where $f(x) = \begin{cases} x^2 + 3 & , \quad x \geq 1 \\ \frac{x^2 + 2x - 3}{x - 1} & , \quad x < 1 \end{cases}$

at $x = 1$

Third Accumulative quizzes on trigonometry

Quiz 1 on lesson 1 – unit 4

Total mark 10

Answer the following questions :

First question *6 marks* $1\frac{1}{2}$ mark for each item

Choose the correct answer from those given :

(1) In Δ ABC , if a = 6 cm. , m ($\angle$ B) = 2 m ($\angle$ A) = 80° , then c = ... cm.

(a) $\frac{6 \sin 40°}{\sin 60°}$ (b) $\frac{\sin 60°}{6 \sin 40°}$ (c) $\frac{\sin 40°}{6 \sin 60°}$ (d) $\frac{6 \sin 60°}{\sin 40°}$

(2) ABC is an equilateral triangle , its side length equals $8\sqrt{3}$ cm. , then the length of the diameter of its circumcircle equals .... cm.

(a) 8 (b) $16\sqrt{3}$ (c) 16 (d) $4\sqrt{3}$

(3) In Δ XYZ , if $\frac{2x}{\sin X}$ = 8 cm. , then the area of its circumcircle equals .... cm².

(a) 16π (b) 8π (c) 4π (d) 64π

(4) In Δ ABC , $\frac{\sin (A + B)}{\sin A + \sin B}$ = ...

(a) 1 (b) $\frac{c}{a+b}$ (c) $\frac{a}{a+c}$ (d) $\frac{b}{a+c}$

Second question *4 marks*

ABC is a triangle in which : c = 19 cm. , m ($\angle$ A) = 112° , m ($\angle$ B) = 33°

Find to the nearest 2 decimal places each of b and the radius length of its circumcircle.

Total mark

10

Quiz 2 till lesson 2 – unit 4

Answer the following questions :

First question *6 marks* **1 mark for each item**

Choose the correct answer from those given :

(1) In Δ XYZ , if $\frac{\sin X}{3} = \frac{\sin Y}{4} = \frac{\sin Z}{5}$, then the greatest angle measure in the triangle equals ....

(a) 60° (b) 75° (c) 90° (d) 120°

(2) In Δ ABC , $\frac{a}{a + b} =$ ....

(a) $\frac{\sin A}{\sin B}$ (b) $\frac{\sin A}{\sin C}$ (c) $\frac{\sin A}{\sin A + \sin B}$ (d) $\frac{\sin A}{\sin A + \sin C}$

(3) The diameter length of the circumcircle of Δ ABC in which a = 8 sin A cm. equals .... cm.

(a) 4 (b) 8 sin A (c) 8 (d) 5

(4) In Δ XYZ , $x = y$, $\frac{z^2 - 2x^2}{x^2} = 1$, then m (∠ Z) = ...

(a) 30° (b) 60° (c) 120° (d) 150°

(5) The area of Δ ABC is 24 cm.^{2} and the radius length of its circumcircle is 5 cm. , then sin A sin B sin (A + B) = ...

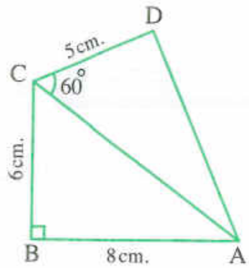
(a) $\frac{3}{25}$ (b) $\frac{6}{25}$ (c) $\frac{9}{25}$ (d) $\frac{12}{25}$

(6) In Δ ABC , if m (∠ A) = 60° , then $\left(1 + \frac{a}{c} + \frac{b}{c}\right)\left(1 + \frac{c}{b} - \frac{a}{b}\right) =$ ....

(a) zero (b) 1 (c) 2 (d) 3

Second question *4 marks*

In the opposite figure :



ABCD is a quadrilateral in which AB = 8 cm.

, BC = 6 cm. , m (∠ B) = 90°

, DC = 5 cm. and m (∠ ACD) = 60°

, then find the area of the circumcircle of the triangle ADC

Total mark

....

10

Quiz 3 till lesson 3 – unit 4

Answer the following questions :

First question *6 marks* **1 mark for each item**

Choose the correct answer from those given :

(1) The number of solutions of Δ ABC in which m ($\angle$ A) = 60° , a = 7 cm. , c = 9 cm. is ....

(a) one. (b) two. (c) zero. (d) three.

(2) If r is the radius length of the circumcircle of Δ ABC , then $\frac{a}{2 \sin A}$ = ...

(a) r (b) 2 r (c) $\frac{1}{2}$ r (d) r^2

(3) In Δ ABC , if a = c , then cos C = ...

(a) $\frac{2b}{c}$ (b) $\frac{c}{2b}$ (c) $\frac{b}{2a}$ (d) $\frac{c}{2a}$

(4) The greatest angle measure in the triangle whose side lengths are 6 cm. , 10 cm. and 14 cm. equals ....

(a) 120° (b) 150° (c) 135° (d) 60°

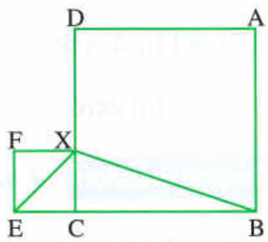
(5) If the area of the triangle ABC = 12 cm^2. , then $(b^2 + c^2 - a^2)$ tan A = ...

(a) 12 (b) 24 (c) 48 (d) 96

(6) In the opposite figure :

ABCD , XCEF are two squares.

If BC = 3 CE , then sin ($\angle$ BXE) = ...



(a) $\frac{1}{\sqrt{5}}$ (b) $\frac{2}{\sqrt{5}}$

(c) $\frac{-1}{\sqrt{5}}$ (d) $\frac{-2}{\sqrt{5}}$

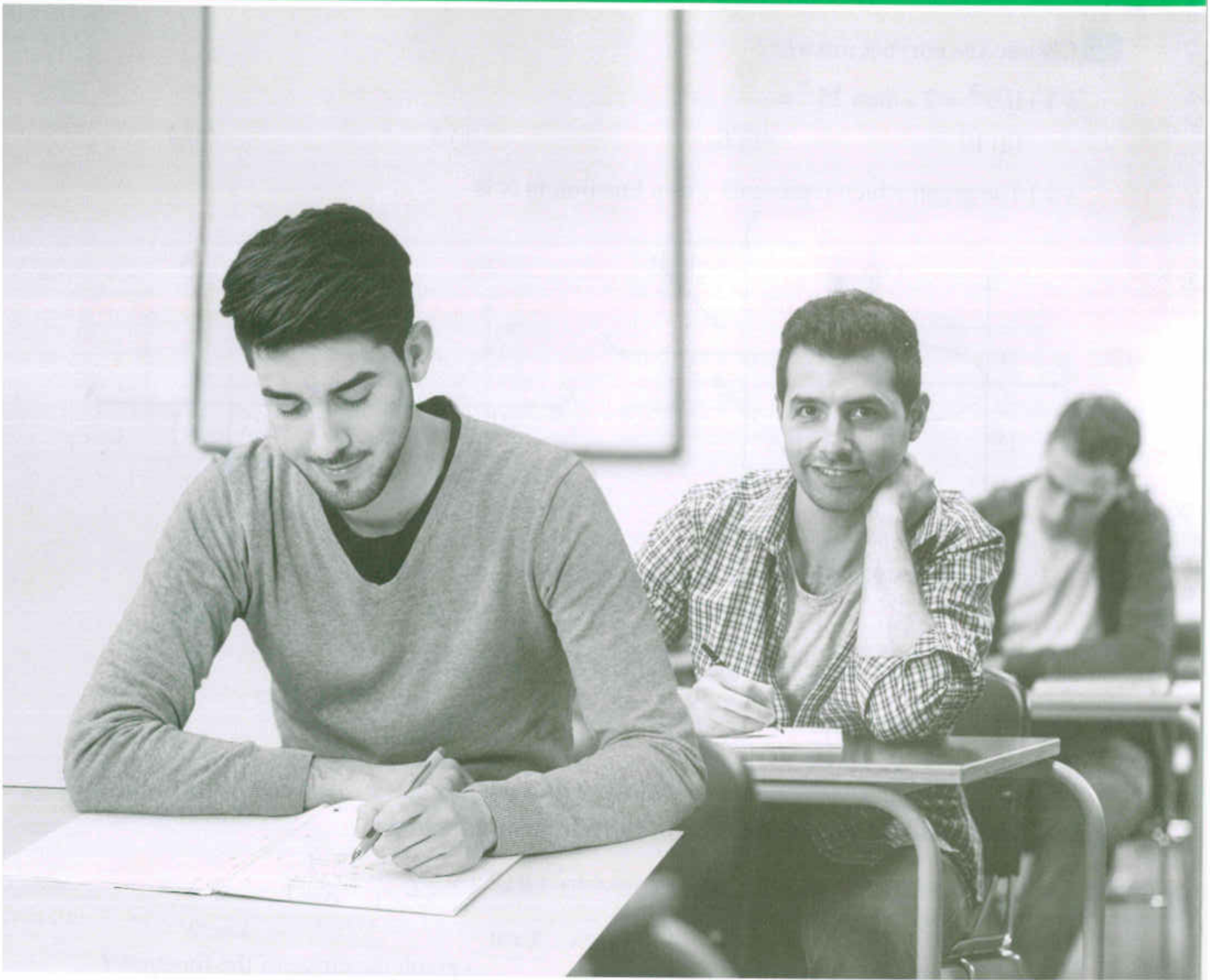
Second question *4 marks*

In Δ ABC , m ($\angle$ A) = 40° , a = 24 cm. , b = 30 cm.

How many triangles satisfy that ?

Find : c if possible in each case.

School book examinations



- **First :** School book examinations in algebra
- **Second :** School book examinations in calculus and trigonometry

First School book examinations in algebra

Model 1

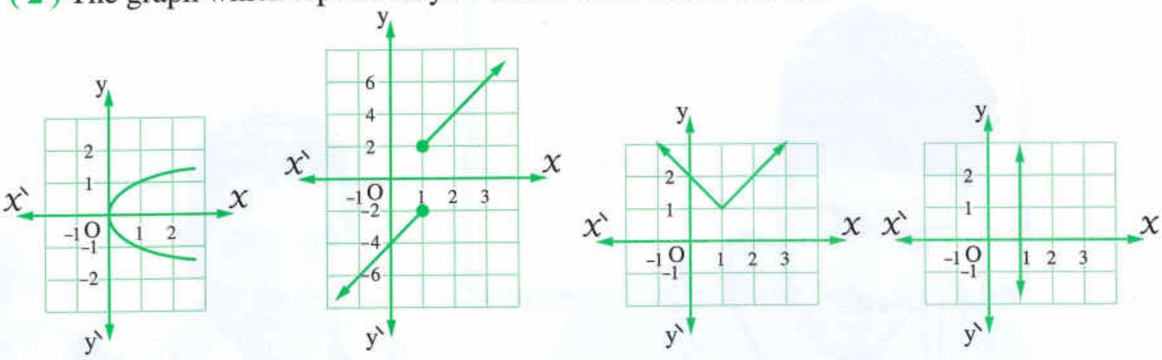
Answer the following questions :

1 Choose the correct answer :

(1) If $5^{x} = 2$, then $25^{x} = $ ....

(a) 10 (b) 6 (c) 5 (d) 4

(2) The graph which represents y as a function in x is ....



(a) (b) (c) (d)

(3) If the curve $y = \log_4 (1 - a x)$ passes through $\left(\frac{1}{4}, -\frac{1}{2}\right)$, then a = ...

(a) 2 (b) 3 (c) 4 (d) 8

(4) From the following functions , the one - to - one function is ....

(a) $f_1 : f_1 (x) = x + 2$

(b) $f_2 : f_2 (x) = x^2$

(c) $f_3 : f_3 (x) = |x|$

(d) $f_4 : f_4 (x) = 5$

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2 [a] Determine the domain of each of the following functions :

(1) $f(x) = \frac{x}{\sqrt[3]{1-x}}$

(2) $g(x) = \frac{x-1}{x^2-1} + \frac{1}{x+1}$

[b] If f is a function where $f(x) = \begin{cases} x^2 & , x > 0 \\ -2x & , x < 0 \end{cases}$, graph the curve of the function f and from the graph find the range of f

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3 [a] If $f_1 : \mathbb{R} \longrightarrow \mathbb{R}$ where $f_1(x) = 3x - 1$, $f_2 : [-2, 3] \longrightarrow \mathbb{R}$ where $f_2(x) = 3 - 2x$, then graph the function $(f_1 + f_2)$ showing its domain , then check its monotony.

[b] Find the inverse function of the function $y = x + 1$ and graph each of them on the same figure.

4 [a] **Find in $\mathbb{R}$ the solution set of each of the following equations :**

(1) $\log_4 x = 1 - \log_4 (x - 3)$ (2) $|x + 2| = |x - 3|$

[b] **Use the curve of the function f where $f(x) = x^2$ to graph each of the following functions :**

(1) $f_1 : f_1(x) = x^2 - 3$ (2) $f_2 : f_2(x) = x^2 + 3$

5 [a] **Find in $\mathbb{R}$ the solution set of the inequality :** $|3x - 2| \geq 7$

[b] **Find in $\mathbb{R}$ the solution set of the equation :** $x^{\frac{4}{3}} - 10\, x^{\frac{2}{3}} + 9 = 0$

Model 2

Answer the following questions :

1 **Choose the correct answer :**

(1) If $3^{x-2} = 2^{x-2}$, then $x =$ ....

(a) 3 (b) -2 (c) zero (d) 2

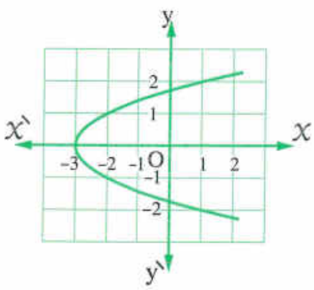
(2) If $y = \sqrt[3]{x}$ for all $x \geq 0$, then the inverse function of y is ....

(a) $y = \frac{1}{3} x^3$ (b) $y = x^3$ (c) $y = x^3 - 1$ (d) $y = x^{-\frac{1}{3}}$

(3) If f is an odd function on $[-x, x]$, then $f(-x) + f(x) =$ ....

(a) $2x$ (b) not defined (c) $-2x$ (d) zero

(4) The curve in the opposite figure is symmetric about the straight line whose equation is ....



(a) $x = 0$ (b) $y = 0$

(c) $y = -2$ (d) $x = 2$

2 [a] If $f(x) = a^x$, prove that the expression $\frac{1}{f(x) + 1} + \frac{1}{f(-x) + 1}$ has a constant value whatever the value of x

[b] Find the domain of the function f where $f(x) = \log_{x-1} x$

3 **[a] Use the curve of the function f where $f(x) = |x|$ to graph each of the following functions :**

(1) $f_1(x) = |x| + 1$ (2) $f_2(x) = 2 - |x|$

[b] Draw the graph of each of the following functions , then determine the domain and monotonicity of each :

(1) $f(x) = \sqrt{x^2 - 4x + 4}$ (2) $f(x) = |x^2 - 4x + 5| , x \in [0 , 4]$

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4 **[a] Determine whether each of the following functions is even , odd or otherwise :**

(1) $f_1(x) = x \cos x$ (2) $f_2(x) = \begin{cases} x^2 & \text{when } x \geq 0 \\ |x| & \text{when } x < 0 \end{cases}$

(3) $f_3(x) = x^2 |x| - 1$

[b] Find in $\mathbb{R}$ the solution set of each of the following :

(1) $|x| + x = 0$ (2) $|2x - 3| - |6 - 4x| > 0$

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5 [a] If $f(x) = x^2 - 1$, $g(x) = x + 1$, then graph the function $\frac{f}{g}(x)$ showing its domain , range and monotony.

[b] Without using calculator , find the value of : $\log 25 + \frac{\log 8 \times \log 16}{\log 64}$

Model 1

Answer the following questions :

1 **Choose the correct answer :**

(1) $\lim\limits_{x \to \infty} \dfrac{x^3+5}{x(2x^2+3)} = \dots\dots$

(a) $\frac{5}{8}$ (b) 1 (c) $\frac{1}{2}$ (d) $\frac{5}{3}$

(2) In the triangle ABC , if 4 sin A = 3 sin B = 6 sin C , then m (∠ C) = ...

(a) 89° (b) 29° (c) 57° (d) 82°

(3) If the function f where $f(x) = \begin{cases} \dfrac{x^2-1}{x-1} & , x \neq 1 \\ 2a & , x = 1 \end{cases}$ is continuous at $x = 1$

, then a = ...

(a) zero (b) − 2 (c) 2 (d) 1

(4) In the triangle XYZ , the expression $\dfrac{x^2+y^2-z^2}{2xy} = \dots\dots$

(a) cos X (b) cos Y (c) cos Z (d) sin Z

2 [a] **Find :** (1) $\lim\limits_{x \to 2} \dfrac{x^5-32}{x^2+3x-10}$ (2) $\lim\limits_{x \to 0} \dfrac{\sin 2x + 5 \sin 3x}{x}$

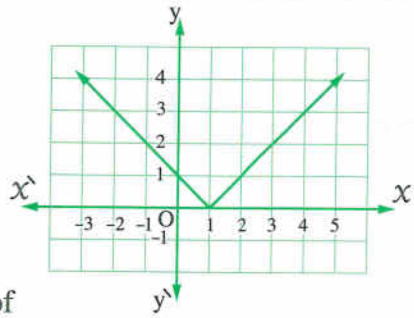
[b] Solve the acute-angled triangle ABC in which a = 21 cm. , b = 25 cm. and the length of the diameter of the circumcircle of the triangle ABC equals 28 cm.

3 [a] **From the opposite graph , find :**

(1) $\lim\limits_{x \to 1} f(x)$

(2) $\lim\limits_{x \to 2} f(x)$

(3) $f(1)$



[b] ABCD is a parallelogram in which m (∠ A) = 50° , m (∠ DBC) = 70° , BD = 8 cm. Find the perimeter of the parallelogram.

4 [a] ABC is a triangle in which a = 5 cm. , b = 7 cm. , m (∠ A) = 40° **Find :** m (∠ B)

[b] Find the value of a which makes the function f continuous at $x = 2$

where $f(x) = \begin{cases} x^2 - 1 & , \quad x \geq 2 \\ x - 2a & , \quad x < 2 \end{cases}$

5 [a] Discuss the existence of $\lim\limits_{x \to 0} f(x)$ where $f(x) = \begin{cases} \dfrac{\tan 2x}{\sin x} & , \quad x > 0 \\ \dfrac{5x+6}{x+3} & , \quad x < 0 \end{cases}$

[b] **Find :** $\lim\limits_{x \to 1} \dfrac{4 - \sqrt{x+15}}{1 - x^2}$

Model 2

Answer the following questions :

1 **Choose the correct answer :**

(1) $\lim\limits_{x \to 3} \dfrac{x-3}{x^2-9} = \dots\dots\dots\dots$

(a) 3 (b) $\frac{1}{9}$ (c) $\frac{1}{3}$ (d) $\frac{1}{6}$

(2) In the triangle ABC , cos A = ...

(a) $\dfrac{a^2+b^2-c^2}{2ab}$ (b) $\dfrac{a^2+c^2-b^2}{2ab}$ (c) $\dfrac{b^2+c^2-a^2}{2bc}$ (d) $\dfrac{c^2-a^2-b^2}{2ab}$

(3) ABC is a triangle in which $\dfrac{\sin A}{3} = \dfrac{2 \sin B}{5} = \dfrac{\sin C}{4}$, then a : b : c = ...

(a) 6 : 5 : 8 (b) 8 : 5 : 6 (c) 7 : 2 : 4 (d) 3 : 5 : 6

(4) $\lim\limits_{x \to \infty} \dfrac{\sqrt{x^2+3}}{2x+1} = \dots\dots\dots\dots$

(a) 1 (b) $\frac{3}{2}$ (c) $\frac{1}{2}$ (d) 3

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2 [a] If the function f where $f(x) = \begin{cases} \dfrac{x^2+2x-3}{x+3} & , \quad x \neq -3 \\ x + a & , \quad x = -3 \end{cases}$

is continuous at $x = -3$, find the value of a

[b] ABC is a triangle in which : $\frac{1}{3} \sin A = \frac{1}{4} \sin B = \frac{1}{5} \sin C$, find : m (∠ C)

and if the perimeter of the triangle = 24 cm. , find its surface area.

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3 [a] **Find :** $\lim\limits_{x \to 0} \dfrac{x^2 + \sin 3x}{5x \cos 2x}$

[b] Solve the triangle ABC in which a = 9 cm. , b = 15 cm. , m (∠ C) = 106°

4 [a] **Find :** $\lim\limits_{x \to -2} \dfrac{(x+3)^5 - 1}{x^2 - 4}$

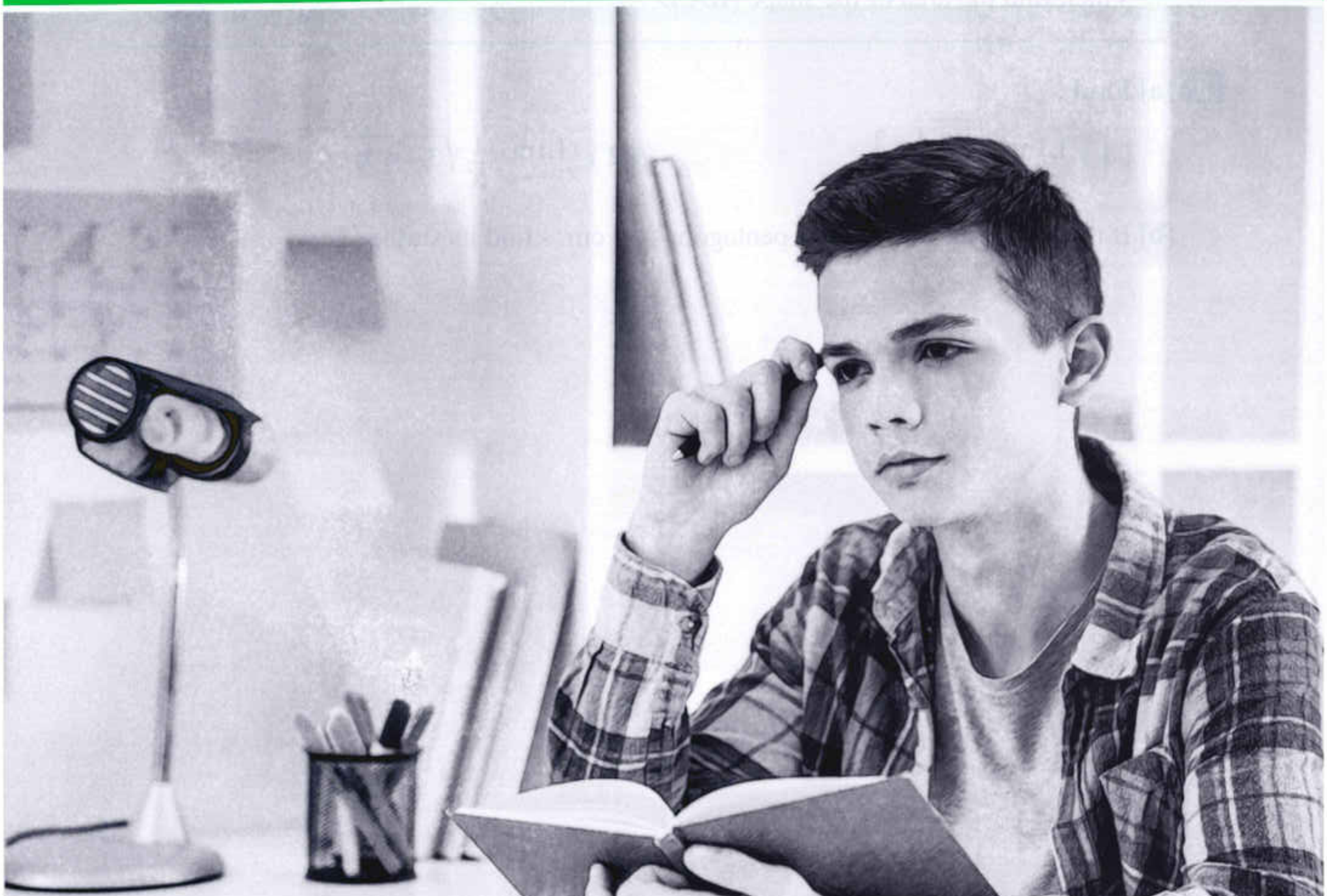
[b] ABCD is a quadrilateral in which AB = 27 cm. , BC = 12 cm. , CD = 8 cm. , DA = 12 cm. , AC = 18 cm. Prove that $\overrightarrow{AC}$ bisects ∠ BAD , then find the area of the shape ABCD

5 [a] **Find :**

(1) $\lim\limits_{x \to 0} \dfrac{\sqrt{x+4} - 2}{x^2 + x}$ (2) $\lim\limits_{x \to \infty} \dfrac{1}{x}\sqrt{3 + 4x^2}$

[b] If the perimeter of a regular pentagon is 30 cm. , find its surface area.

Final examinations



▶ **First :** **Examination models**

▶ **Second :** **Multiple choice examinations**

Scan the QR codes to solve interactive tests

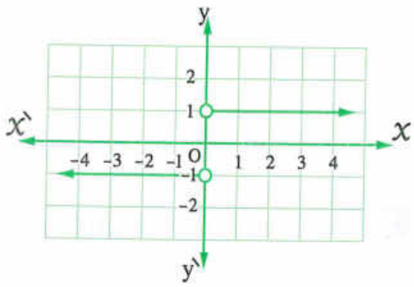
First — Examination models

Model 1

Interactive test 1

Answer the following questions :

1 The range of the given function in the opposite figure is ....



(a) $\{1\}$ (b) $\{1 , -1\}$

(c) $\{-1\}$ (d) $\mathbb{R}$

2 If the point $\left(x , \frac{4}{x}\right)$ is the point of intersection between the function f and its inverse function f^{-1} , then $x =$ ....

(a) 2 (b) 4 (c) ± 2 (d) ± 4

3 $\lim\limits_{x \to \infty} \frac{2x+3}{5x^2+4} =$ ....

(a) 2 (b) zero (c) $\frac{3}{4}$ (d) $\frac{2}{5}$

4 In Δ ABC , if $4 \sin A = 3 \sin B = 6 \sin C$, then $m(\angle C) \simeq$ ....

(a) 89° (b) 29° (c) 57° (d) 82°

5 In Δ ABC , $m(\angle C) = 96^\circ 23'$, a = 7 cm. , b = 9 cm.

Find : c , the area of Δ ABC to the nearest cm^2.

6 If $f(1) = 3$, $g(3) = 5$, then $(g \circ f)(1) =$ ....

(a) 3 (b) 5 (c) 15 (d) $\frac{3}{5}$

7 The solution set of the equation : $2^{2x} - 12 \times 2^{x} + 2^5 = 0$ in $\mathbb{R}$ is ....

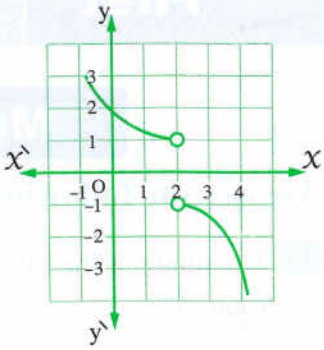
(a) $\{2 , 3\}$ (b) $\{2\}$ (c) $\{3\}$ (d) $\{4 , 8\}$

8 Use the curve of the function f where $f(x) = \frac{1}{x}$ to represent the function $g : g(x) = f(x-2) + 2$ and from the graph determine the range and discuss its monotony.

9 $\lim\limits_{x \to 0} \frac{(x+2)^5 - 32}{x} =$ ....

(a) 25 (b) 64 (c) 80 (d) 100

10 The opposite figure represents the curve of the function f, then $\lim_{x \to 2} |f(x)| = \text{.............}$



(a) 1 (b) -1

(c) 2 (d) does not exist.

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11 The number of possible solutions for Δ LMN given that m ($\angle$ L) = 40°, $\ell = 12$ cm. , m = 15 cm. is ....

(a) 1 (b) 2

(c) zero (d) infinite solutions.

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12 The sum of the roots of the equation : $25^x - 12 \times 5^x + 27 = 0$ in $\mathbb{R}$ equals ....

(a) $\log_5 12$ (b) 12 (c) $\log_5 27$ (d) 27

---

13 If $x = 5 + 2\sqrt{6}$, then $\log\left(x + \frac{1}{x}\right) = \text{.............}$

(a) 1 (b) $5 - 2\sqrt{6}$ (c) 10 (d) $5 + 2\sqrt{6}$

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14 ABC is an equilateral triangle, its side length = $5\sqrt{3}$ cm. , then the diameter length of its circumcircle equals .... cm.

(a) $5\sqrt{3}$ (b) $10\sqrt{3}$ (c) 10 (d) 5

---

15 If $f(x) = 2 + \sqrt{3 - x}$, find the domain and the range of f then find $f^{-1}(x)$ and determine the domain of f^{-1} and its range.

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16 If $f : \mathbb{R} \longrightarrow \mathbb{R}$, where $f(x) = (a + 1)x + b - 2$ and $f(x)$ maps each real number to itself, then (a , b) = ...

(a) (0 , 3) (b) (0 , -3) (c) (0 , 2) (d) (-1 , 2)

---

17 If $f(x) = \frac{x^2 - 16}{x - 4}$, $x \neq 4$, then the value of $f(4)$ which makes the function f continuous is ....

(a) undefined. (b) $\lim_{x \to 4} \frac{x^2 - 16}{x - 4}$ (c) zero (d) 16

18 The solution set of the equation : $\log_3 X \times \log_2 3 = 5$ in $\mathbb{R}$ is ....

(a) $\{32\}$ (b) $\{5\}$ (c) $\{3\}$ (d) $\{2\}$

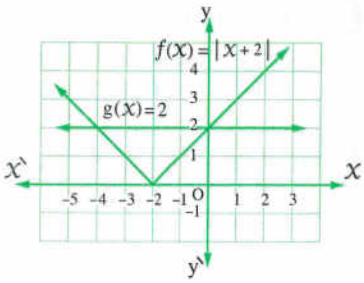
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19 In Δ ABC , m ($\angle$ A) : m ($\angle$ B) : m ($\angle$ C) = 3 : 5 : 4 , then $c^2 : a^2 =$ ....

(a) $\sqrt{6} : 2$ (b) $2 : 3$ (c) $4 : 3$ (d) $3 : 2$

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20 **In the opposite figure :**



The solution set of the inequality $f(X) < g(X)$ in $\mathbb{R}$ is ....

(a) $\{-4 , 0\}$ (b) $[-4 , 0]$

(c) $\mathbb{R} - [-4 , 0]$ (d) $]-4 , 0[$

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21 If f is a function where $f(X) = \begin{cases} X^2 - X + 4 & \text{at} \quad X < 2 \\ k & \text{at} \quad X = 2 \\ 5X - 4 & \text{at} \quad X > 2 \end{cases}$

[1] Discuss the existence of $\lim\limits_{X \to 2} f(X)$

[2] Find , if possible , the value of k which makes f continuous at $X = 2$

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22 The type of the function $f : f(X) = \dfrac{\sin X}{X}$ is ....

(a) even. (b) odd.

(c) neither odd nor even. (d) both odd and even.

---

23 $\lim\limits_{X \to 0} \dfrac{\sin \pi |X|}{4X} =$ ....

(a) $\dfrac{\pi}{4}$ (b) 1 (c) $\dfrac{1}{4}$ (d) does not exist.

---

24 If $X^{\frac{3}{2}} = 8$, then $X =$ ....

(a) 2 (b) 4 (c) 8 (d) 9

---

25 $\lim\limits_{X \to 2} \dfrac{X^3 - 7X + 6}{3X^2 - 8X + 4} =$ ....

(a) $\dfrac{5}{4}$ (b) $\dfrac{4}{5}$ (c) $\dfrac{1}{3}$ (d) $\dfrac{3}{2}$

---

26 If $\lim\limits_{X \to 1} \dfrac{2X + a}{X + 1} = 5$, then a = ...

(a) 2 (b) 5 (c) 8 (d) 10

27 In any triangle XYZ , $x^2 + y^2 - 2xy \cos Z =$ ....

(a) x^2 (b) y^2 (c) z^2 (d) z

---

28 The solution set in $\mathbb{R}$ of the inequality : $\sqrt{4x^2 - 12x + 9} \le 9$ equals ....

(a) $]-3 , 6]$ (b) $[-3 , 6]$ (c) $\mathbb{R} - [-3 , 6]$ (d) $\mathbb{R} -]-3 , 6[$

---

29 The solution set of the equation : $|3 - 2x| - 5x = 3$ in $\mathbb{R}$ is ....

(a) $\{0 , 3\}$ (b) $\{0 , -2\}$ (c) $\{0\}$ (d) $\varnothing$

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30 If $f(x - 1) = 2^{x-5}$, $f(x + 3) = \frac{1}{32}$, then $x =$ ....

(a) -4 (b) -2 (c) 4 (d) 6

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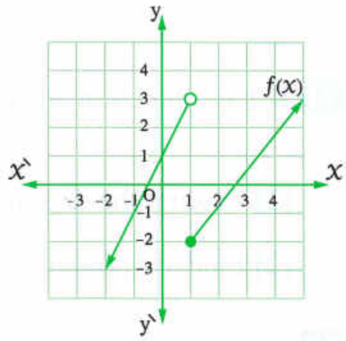
31 **In the opposite figure :**

The curve of the function $f(x)$

, then $\lim\limits_{x \to 0} f(x) + f(1^+) + f(1^-) =$ ....

(a) 2 (b) 3

(c) 4 (d) 6



---

32 The perimeter of Δ ABC = 33 cm. and $\sin A + \sin C = \frac{2}{3}$, $\sin B = \frac{1}{4}$

, then b = ...

(a) 6 (b) 9 (c) 12 (d) 15

Model 2

Interactive test 2



Answer the following questions :

1 If $f : \mathbb{N} \longrightarrow \mathbb{N}$ where $f(x) = 2x$,

$$n : \mathbb{N} \longrightarrow \mathbb{N} \text{ where } n(x) = \begin{cases} \frac{x}{2} & , \quad x \text{ is even} \\ \frac{x+1}{2} & , \quad x \text{ is odd} \end{cases}$$

, then $(f \circ n)(3) - (f \circ n)(8) =$ ....

(a) 4 (b) 8 (c) -4 (d) -5

2 $\lim\limits_{x \to \infty} \left(\frac{3}{5}\right)^{\frac{1}{x}} =$ ....

(a) 1 (b) -1 (c) $\frac{3}{5}$ (d) ∞

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3 In Δ ABC , D is the midpoint of $\overline{BC}$ **prove that :** $(AB)^2 + (AC)^2 = 2\,(AD)^2 + 2\,(BD)^2$ and if AB = 5 cm. , AC = 8 cm. , BC = 12 cm. , **find :** AD

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4 If $f : f(x) = \begin{cases} \frac{x^2 - 1}{x - 1} & , \quad x \neq 1 \\ 2\,a & , \quad x = 1 \end{cases}$ is continuous at $x = 1$, then a = ...

(a) zero (b) -2 (c) 2 (d) 1

---

5 In Δ ABC , $\frac{a}{\sin A} = 6$ cm. , then the radius length of its circumcircle = ... cm.

(a) 2 (b) 3 (c) 5 (d) 6

---

6 If f is an odd function and $x\,f(x) + x^3 f(-x) = 2$, then $f(2) =$ ....

(a) 3 (b) $\frac{1}{3}$ (c) $-\frac{1}{3}$ (d) -3

---

7 If $x = 5 + 2\sqrt{6}$, find in the simplest form the value of $\log\left(\frac{1}{x} + x\right)$ without using calculator.

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8 In Δ XYZ , $\frac{x^2 + y^2 - z^2}{2\,x\,y} =$ ....

(a) cos X (b) cos Y (c) cos Z (d) sin Z

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9 $\lim\limits_{x \to 1} \frac{x^2 - x^{-2}}{x - x^{-1}} =$ ....

(a) zero (b) 1 (c) 2 (d) -2

---

10 If $f(x) = 3^x$, then the solution set in $\mathbb{R}$ for $f(2\,x) - 28\,f(x) + f(3) =$ zero is ....

(a) $\{1 , 27\}$ (b) $\{27\}$ (c) $\{0 , 3\}$ (d) $\{3\}$

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11 If $\log a \in]0 , 1[$, then $a \in$ ....

(a) $]0 , 1[$ (b) $]1 , 2[$ (c) $]1 , 10[$ (d) $]1 , \infty[$

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12 The curve of the even function is symmetric about the straight line ....

(a) $y = x$ (b) $\overleftrightarrow{yy}$ (c) $\overleftrightarrow{xx}$ (d) $y = -x$

13 In Δ LMN , if $\frac{\sin L}{3} = \frac{2 \sin M}{3} = \frac{\sin N}{4}$, then $\ell : m : n =$ ....

(a) $6 : 8 : 3$ (b) $3 : 6 : 8$ (c) $8 : 3 : 6$ (d) $6 : 3 : 8$

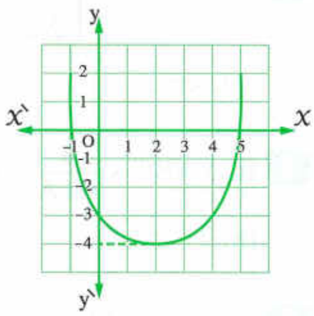
14 In Δ ABC , c = 7 cm. , $m(\angle A) = 70°$, $m(\angle B) = 40°$

, then b = ... cm. (to nearest one decimal)

(a) 6.3 (b) 8.4 (c) 3.6 (d) 4.8

15 The opposite figure represents the curve of the function f

, then $\lim\limits_{x \to 2} \frac{[f(x)]^2 + f(x) - 12}{f(x) + 4} =$ ....



(a) -7 (b) -4

(c) -1 (d) 3

16 Use the curve of the function $f : f(x) = \frac{1}{x}$ to graph the curve of the function $g : g(x) = \frac{1}{x - 2} + 3$ from the graph state the domain and range of g and the monotony and its type whether it is even , odd or otherwise. Is the function g one - to - one or not ?

17 The range of the function $f : f(x) = \frac{x - 2}{2 - x}$ equals ....

(a) $\mathbb{R}$ (b) $\mathbb{R} - \{2\}$ (c) $\mathbb{R} - \{-2\}$ (d) $\{-1\}$

18 If $y = a^{\log_a x}$, then ....

(a) $x + y = 0$ (b) $x = 2y$ (c) $x - y = 0$ (d) $x = \frac{1}{2} y$

19 $\log_3 5 \times \log_2 3 \times \log_5 16 =$ ....

(a) 30 (b) 15 (c) log 10000 (d) $\log_{30} 240$

20 The curve of the function $g : g(x) = x^2 + 4$ is the same as the curve of $f : f(x) = x^2$ by translation 4 units in the direction of ....

(a) $\overrightarrow{Ox}$ (b) $\overrightarrow{Ox'}$ (c) $\overrightarrow{Oy}$ (d) $\overrightarrow{Oy'}$

21 Discuss the existence of the limit of the function $f : f(x) = \begin{cases} x^2 + 1 & , \quad x < 3 \\ 3x + 1 & , \quad x \geq 3 \end{cases}$ at $x = 3$

22 The function f where $f(\mathcal{X}) = a^{\mathcal{X}}$ is decreasing on its domain if ....

(a) $a = 1$ (b) $a > 1$ (c) $0 < a < 1$ (d) $a = -1$

23 $\lim_{\mathcal{X} \to \frac{\pi}{2}} (2\mathcal{X} + \sin \mathcal{X}) =$ ....

(a) π (b) $\pi - 1$ (c) $1 - \pi$ (d) $\pi + 1$

24 $\lim_{\mathcal{X} \to \infty} \frac{\sqrt{\mathcal{X}^2}}{\mathcal{X}} =$ ....

(a) zero (b) 2 (c) -1 (d) 1

25 The solution set in $\mathbb{R}$ of the equation : $|\mathcal{X} - 7| = 2\mathcal{X} - 2$ equals ....

(a) $\{3, -5\}$ (b) $\{3\}$ (c) $\{-5\}$ (d) $\varnothing$

26 If $y = \sqrt[3]{\mathcal{X}}$ for every $\mathcal{X} \geq 0$, then its inverse function is $y =$ ....

(a) $\frac{1}{3}\mathcal{X}^3$ (b) $\mathcal{X}^3$ (c) $\mathcal{X}^3 - 1$ (d) $\mathcal{X}^{-\frac{1}{3}}$

27 If the perimeter of Δ ABC = 33 cm. , $\sin A + \sin C = \frac{2}{3}$, $\sin B = \frac{1}{4}$
, then AC = ... cm.

(a) 6 (b) 9 (c) 12 (d) 15

28 $\lim_{\mathcal{X} \to 0} \frac{\sin 2\mathcal{X} + 5 \tan 3\mathcal{X}}{\mathcal{X}} =$ ....

(a) 2 (b) 15 (c) 21 (d) 17

29 The solution set of the inequality $\sqrt{\mathcal{X}^2 - 4\mathcal{X} + 4} > 0$ in $\mathbb{R}$ is ....

(a) $\mathbb{R} - \{2\}$ (b) $\mathbb{R} - \{-2\}$ (c) $\mathbb{R}$ (d) $\varnothing$

30 If $n(\mathcal{X})$ is a function and $\lim_{\mathcal{X} \to 2} \frac{n(\mathcal{X}) - 8}{\mathcal{X} - 2} = 7$, then $\lim_{\mathcal{X} \to 2} \frac{2\mathcal{X}^2 - n(\mathcal{X})}{\mathcal{X} - 2} =$ ....

(a) 1 (b) 7 (c) 8 (d) 15

31 The number of possible solutions of Δ ABC where $m(\angle A) = 110°$, a = 7 cm.
, b = 4 cm. , then ....

(a) 1 (b) zero (c) infinite solutions. (d) 2

32 The solution set of the equation : $\log(\mathcal{X} + 2) + \log(\mathcal{X} - 2) = 1 - \log 2$ in $\mathbb{R}$ is ....

(a) $\{\log_5 125\}$ (b) $\{\log_2 16\}$ (c) $\{\log_3 9\}$ (d) $\{\log_{100} 100\}$

Model 3

Interactive test 3

Answer the following questions :

1 If $f(\mathcal{X}) = 7^{\mathcal{X}+1}$

, find the value of $\mathcal{X}$ that satisfies : $f(2\mathcal{X}-1) + f(\mathcal{X}-2) = 50$

2 If $\log 3 = \mathcal{X}$, $\log 5 = y$, then $\log 15 =$ ....

(a) $\mathcal{X}y$ (b) $\frac{\mathcal{X}}{y}$ (c) $\mathcal{X}+y$ (d) $\mathcal{X}-y$

3 $\lim\limits_{\mathcal{X}\to\infty} \frac{5+\mathcal{X}^{-2}}{1+3\mathcal{X}^{-2}} =$ ....

(a) $\frac{1}{3}$ (b) $\frac{5}{4}$ (c) $\frac{5}{3}$ (d) 5

4 If $9^{\mathcal{X}} = 2$, $27^{y} = 4$, then $\frac{\mathcal{X}-y}{\mathcal{X}+y} =$ ....

(a) $\frac{1}{7}$ (b) $-\frac{1}{7}$ (c) $\frac{3}{4}$ (d) $\frac{-4}{3}$

5 If $f(\mathcal{X}) = \log_a(2\mathcal{X}+4)$, $f^{-1}(5) = 14$, then $a =$ ....

(a) 1 (b) 2 (c) 3 (d) 4

6 $\lim\limits_{\mathcal{X}\to 16} \frac{\sqrt[4]{\mathcal{X}}-1}{\mathcal{X}-16} =$ ....

(a) zero (b) $\frac{1}{2}$ (c) 1 (d) does not exist.

7 $\lim\limits_{\mathcal{X}\to 0} \frac{\mathcal{X}(\cos\mathcal{X}+\cos 3\mathcal{X}+\cos 5\mathcal{X})}{\sin\mathcal{X}} =$ ....

(a) 1 (b) 3 (c) 9 (d) 15

8 $\log 25 + \frac{\log 8 \times \log 16}{\log 64} =$ ....

(a) 2 (b) $\log 2$ (c) 3 (d) 1

9 In the parallelogram ABCD , $m(\angle A) = 50°$, $m(\angle DBC) = 70°$, BD = 8 cm.

Find the perimeter of the parallelogram to the nearest cm.

10 $\lim\limits_{\mathcal{X}\to 2} \frac{\mathcal{X}-2}{\mathcal{X}+1} =$ ....

(a) zero (b) 1 (c) 2 (d) ∞

11 $\lim\limits_{x \to 0} \frac{1 - \sec x}{\cos x - 1} = \dots\dots$

(a) 2 (b) 1 (c) zero (d) -1

---

12 $\lim\limits_{x \to 5} \frac{x^2 - 8x + 15}{x^2 - 10x + 25} = \dots\dots$

(a) does not exist. (b) zero (c) 2 (d) 3

---

13 The included area between the curves of the two functions $f : f(x) = |x + 3| - 2$, g : g (x) = zero is .... square units.

(a) 2 (b) 3 (c) 4 (d) 5

---

14 If $\log_3 y = x$, then the exponential form is ....

(a) $y = x^3$ (b) $x = y^3$ (c) $x = 3^y$ (d) $y = 3^x$

---

15 If f is an odd function on $[-x , x]$, then $f(-x) + f(x) = \dots\dots$

(a) $2x$ (b) undefined. (c) $-2x$ (d) zero

---

16 In Δ ABC , if 2 sin A = 3 sin B = 4 sin C , then a : b : c = ...

(a) 2 : 3 : 4 (b) 4 : 3 : 2 (c) 3 : 4 : 6 (d) 6 : 4 : 3

---

17 $\lim\limits_{x \to 2} \frac{x^5 - 32}{x^2 + 3x - 10} = \dots\dots$

(a) 80 (b) $\frac{80}{7}$ (c) $\frac{16}{7}$ (d) 16

---

18 The radius length of the circumcircle of the triangle ABC in which m ($\angle$ A) = 30° , a = 10 cm. equals ....

(a) 10 cm. (b) 20 cm. (c) 5 cm. (d) 40 cm.

---

19 The function $f : f(x) = \begin{cases} x^2 & , x > 2 \\ -x^2 & , x \le 2 \end{cases}$ is decreasing on the interval ....

(a) $]0 , 2[$ (b) $]-\infty , 0[$ (c) $\mathbb{R} - [0 , 2[$ (d) $]0 , \infty[$

---

20 The domain of the function $f : f(x) = \log_{1-x} x$ is ....

(a) $x > 0$ (b) $x < 1$ (c) $0 < x < 1$ (d) $0 \le x \le 1$

21 If $f(x) = \sqrt{x-2}$, $g(x) = \sqrt{5-x}$, then the domain of $(f \circ g) =$ ....

(a) $]-\infty , 0]$ (b) $]-\infty , 1]$ (c) $[1 , \infty[$ (d) $[0 , \infty[$

---

22 Graph the function $f : f(x) = \begin{cases} |x| & , \quad x \le 0 \\ x^2 & , \quad x > 0 \end{cases}$, from the graph state the range of the function and discuss its montony.

---

23 The solution set of the equation : $|x + 2| + x = -2$ in $\mathbb{R}$ is ....

(a) $\varnothing$ (b) $\mathbb{R}$ (c) $]-\infty , -2[$ (d) $]-\infty , -2]$

---

24 The measure of the greatest angle in the triangle whose side lengths are 3 cm. , 5 cm. , 7 cm. equals ....

(a) 150° (b) 120° (c) 60° (d) 30°

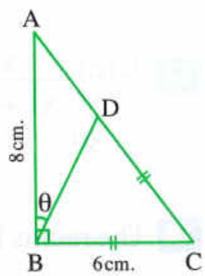
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25 An acute angled triangle ABC in which a = 5 cm. , b = 7 cm. , m (∠ A) = 40° , then the area of Δ ABC ≃ .... cm.²

(a) 16 (b) 17 (c) 18 (d) 7

---

26 **In the opposite figure :**



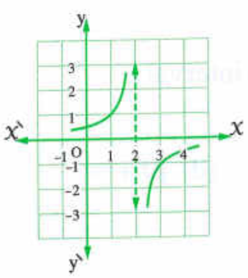
If CD = CB = 6 cm. , then tan θ = ...

(a) $\frac{3}{4}$ (b) $\frac{4}{3}$

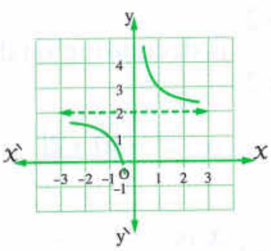
(c) $\frac{1}{2}$ (d) 2

---

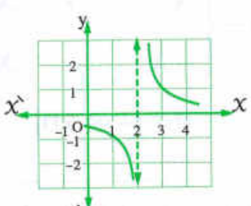
27 If $f(x) = \frac{1}{x-2}$, then the graph that represents the function f is ....



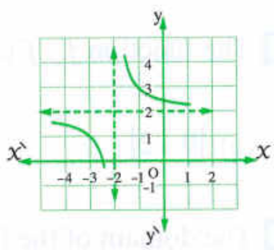
(a)



(b)



(c)



(d)

28 If $f : f(x) = \begin{cases} \frac{(x+3)^5 - 243}{x} & , \quad x \neq 0 \\ k & , \quad x = 0 \end{cases}$

is continuous at x = zero , find the value of k.

---

29 $\lim\limits_{x \to \infty} \frac{\sqrt{9x^2+4} + 5x}{4x+3} = \dots\dots\dots$

(a) ∞ (b) 5 (c) 3 (d) 2

---

30 The solution set of the inequality :

$|2x - 6| + |3 - x| > 12$ is ....

(a) $]-1 , 7[$ (b) $\mathbb{R} - [-3 , 9]$ (c) $\mathbb{R} - [-1 , 7]$ (d) $]-3 , 9[$

---

31 If $2 \log y + 4 \log x - 3 \log x y = 2 (1 - \log 2)$ and $x = k y$, then k = ...

(a) 4 (b) 5 (c) 16 (d) 25

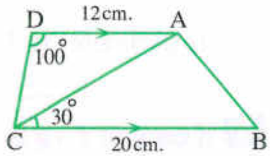
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32 **In the opposite figure :**

$\overline{AD} // \overline{BC}$, m ($\angle$ ACB) = 30° , BC = 20 cm.

, m ($\angle$ ADC) = 100° , AD = 12 cm.

, then the area of Δ ABC $\simeq$ .... cm².



(a) 60 (b) 77 (c) 104 (d) 120

Model 4

Interactive test 4

Answer the following questions :

1 $\lim\limits_{x \to 0} 5x \csc 2x = \dots\dots\dots$

(a) $\frac{5}{2}$ (b) 10 (c) $\frac{2}{5}$ (d) zero

---

2 In Δ ABC , $\frac{b^2 + c^2 - a^2}{2bc} = \dots\dots\dots$

(a) cos A (b) cos B (c) cos C (d) sin A

---

3 The solution set in $\mathbb{R}$ of the inequality : $|x - 1| \geq 3$ equals ....

(a) $[-2 , 4]$ (b) $]-2 , 4[$ (c) $\mathbb{R} -]-2 , 4[$ (d) $\mathbb{R} - [-2 , 4]$

4 $\lim\limits_{x \to -1} \frac{x^2 + x}{x^3 + 1} =$ ....

(a) zero (b) $-\frac{1}{3}$ (c) -1 (d) does not exist.

5 The radius length of the circumcircle of Δ XYZ in which X = 20 sin X cm. equals .... cm.

(a) 5 (b) 10 (c) 20 (d) 40

6 If $2^x = 3^y = 6$, which of the following is true ?

(a) $y - x = xy$ (b) $x - y = xy$ (c) $xy = \frac{x}{y}$ (d) $x + y = xy$

7 The solution set of the equation : $\log 5x = -1$ in $\mathbb{R}$ is ....

(a) $\left\{\frac{1}{10}\right\}$ (b) $\left\{\frac{1}{50}\right\}$ (c) $\{1\}$ (d) $\{50\}$

8 Find the measure of the smallest angle in Δ ABC in which a = 8 cm. , b = 7 cm. and its perimeter is 21 cm.

9 If each of f , g is an inverse function of the other where $f(x) = 2x + a$, $g(x) = bx + 3$, what is the value of each of a , b ?

10 $\lim\limits_{h \to 0} \frac{(2 - 3h)^7 - 128}{4h} =$ ....

(a) -336 (b) 336 (c) 192 (d) -192

11 If the curve of the function $f : f(x) = \log_4 (1 - ax)$ passes through the point $\left(\frac{1}{8}, \frac{-1}{2}\right)$, then a = ...

(a) 3 (b) 2 (c) 4 (d) 8

12 $|f(x)| =$ ....

(a) $\begin{cases} f(x) & , x \geq 0 \\ -f(x) & , x < 0 \end{cases}$ (b) $\begin{cases} f(x) & , x < 0 \\ -f(x) & , x \geq 0 \end{cases}$

(c) $\begin{cases} f(x) & , f(x) \geq 0 \\ -f(x) & , f(x) < 0 \end{cases}$ (d) $\begin{cases} f(x) & , f(x) < 0 \\ -f(x) & , f(x) \geq 0 \end{cases}$

13 If the function $f : f(X) = \begin{cases} aX - 3 & \text{at } X < -1 \\ 3X - 2 & \text{at } X > -1 \end{cases}$ has a limit at $X = -1$, find the value of a

14 $\lim\limits_{X \to \infty} \log 5^{\frac{1}{X}} = $ ....

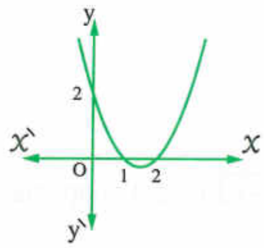
(a) 5 (b) 1 (c) $\frac{1}{5}$ (d) zero

15 All the following relations represent function y in terms of X except ....

(a) $y = 3X + 1$ (b) $y = X^2 - 4$ (c) $X = y^2 - 2$ (d) $y = \sin X$

16 Graph the function $f : f(X) = \begin{cases} -X^3 & , \quad X < 0 \\ X & , \quad X \geq 0 \end{cases}$, from the graph find the range and its type whether it is odd , even or otherwise and discuss its monotony.

17 The opposite figure represents the curve of the function f , then $\lim\limits_{X \to 0} \frac{f(X)}{f(X) + 2} = $ ....

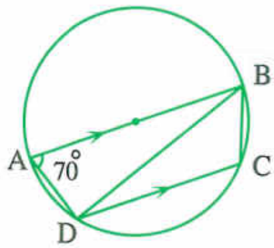


(a) zero (b) 1

(c) $\frac{1}{2}$ (d) 2

18 **In the opposite figure :**

If BC = 10 cm. , then the perimeter of Δ BDC $\simeq$ .... cm.



(a) 60 (b) 62

(c) 64 (d) 67

19 If $f(X) = 2^X$, then the value of X which satisfies : $f(X + 1) - f(X - 1) = 24$ equals ....

(a) 16 (b) 4 (c) 8 (d) 2

20 If $3^{X-2} = 2^{X-2}$, then $X = $ ....

(a) 3 (b) -2 (c) zero (d) 2

21 The domain of the function $f : f(X) = \frac{1}{|X| - 3}$ is ....

(a) $\{3 , -3\}$ (b) $[-3 , 3]$ (c) $\mathbb{R} - [-3 , 3]$ (d) $\mathbb{R} - \{-3 , 3\}$

22 The vertex of the curve of the function $f : f(X) = (2 - X)^2 + 3$ is ....

(a) $(2 , 3)$ (b) $(2 , -3)$ (c) $(-2 , 3)$ (d) $(-2 , -3)$

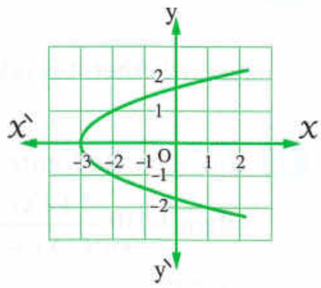
23 In ΔABC , m ($\angle$ A) : m ($\angle$ B) : m ($\angle$ C) = 3 : 4 : 3 If a = 5 cm. , then the circumference of the circle passing through the vertices of $\Delta ABC \simeq$ .... cm.

(a) 17 (b) 18 (c) 19 (d) 15

24 $\lim\limits_{X \to 0} \dfrac{3^{2X} - 1}{3^{X+2} - 9} =$ ....

(a) $\frac{1}{9}$ (c) $\frac{2}{9}$ (c) $\frac{9}{2}$ (d) $\frac{1}{6}$

25 The curve represented in the opposite figure is symmetric about the straight line whose equation is ....



(a) $X = 0$ (b) $y = 0$

(c) $y = -2$ (d) $X = 2$

26 If $\angle$ A supplements $\angle$ C , then cos A + cos C = ...

(a) 1 (b) zero (c) $\frac{1}{2}$ (d) -1

27 If the function $f : f(X) = \begin{cases} \sin 9X \cot X & , \quad X \in \left] \frac{-\pi}{2} , \frac{\pi}{2} \right[- \{0\} \\ k^2 & , \quad X = 0 \end{cases}$

is continuous at $X = 0$, then k = ...

(a) 3 (b) ± 3 (c) 9 (d) $\pm \frac{1}{3}$

28 log (cos θ) + log (sec θ) = ... where $\theta \in \left[0 , \frac{\pi}{2}\right[$

(a) 1 (b) zero (c) 2 (d) -1

29 If $f(X) = |X - 2| + 4$, then the solution set of the equation $f(X + 2) = 6$ is ....

(a) $\{0 , 4\}$ (b) $\{2 , -2\}$ (c) $\{2 , 4\}$ (d) $\{-2 , -4\}$

30 $\lim\limits_{X \to 0} \dfrac{2X \cos 8X + 2 \sin 5X}{\tan 2X} =$ ....

(a) 13 (b) 10 (c) 9 (d) 6

31 If the function $f : f(x)$ is one - to - one function , $f(2k+3) = f(k-1)$, then k = ...

(a) – 1 (b) – 2 (c) – 3 (d) – 4

32 ABC is a triangle in which : BC = 14 cm. , m (∠ B) = 60° , the area of the triangle equals $42\sqrt{3}$ cm². , then AB = ... cm.

(a) 14 (b) 12 (c) 7 (d) 6

Model 5

Interactive test 5

Answer the following questions :

1 $\lim\limits_{x \to 1} \dfrac{x^{6\frac{1}{2}} - x^{\frac{1}{2}}}{x^{3\frac{1}{2}} - x^{\frac{1}{2}}} =$ ....

(a) $\frac{13}{7}$ (b) 1 (c) 2 (d) x

2 If $5^{x+1} = 7^{x+1}$, then $3^{x+1} =$ ....

(a) zero (b) 3 (c) 2 (d) 1

3 If $x < 1$, then $|3 - x| - |x - 4| =$ ....

(a) – 1 (b) 1 (c) $2x - 7$ (d) $7 - 2x$

4 The solution set in ℝ of the equation : $|2x - 4| = |x + 1|$ equals ....

(a) {0 , 5} (b) {1 , 5} (c) {2 , 5} (d) {5}

5 The domain of the function $f : f(x) = \dfrac{\sqrt{x-2}}{x-3}$ is ....

(a) ℝ (b) {3} (c) [2 , ∞[(d) [2 , ∞[– {3}

6 If the function $f : f(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3} & , \quad x \neq 3 \\ 2a & , \quad x = 3 \end{cases}$ is continuous at $x = 3$

, then a = ...

(a) 2 (b) $\frac{3}{2}$ (c) – 3 (d) 3

7 In Δ DEF , if m (∠ D) = 30° , e = $15\sqrt{3}$ cm. , m (∠ E) = 60°

, then d = ... cm.

(a) 30 (b) 45 (c) 15 (d) 60

8 **Showing steps , find the solution set of the equation :**

$3^{2x-1} - 4 \times 3^{x} + 9 = 0$, where x is a real number.

---

9 $\lim\limits_{x \to \infty} (3 + 5x^2 + 3x) = \cdots\cdots$

(a) does not exist. (b) 5 (c) ∞ (d) 11

---

10 If $f(x) = 4x - 5$, $g(x) = 3^{x}$, then $(f \circ g)(2) = \cdots\cdots$

(a) 3 (b) 9 (c) 27 (d) 31

---

11 If $f(a) = 2^{a}$, then $\log_2 f(a) = \cdots\cdots$

(a) 2 (b) $f(a)$ (c) a (d) $\frac{1}{2a}$

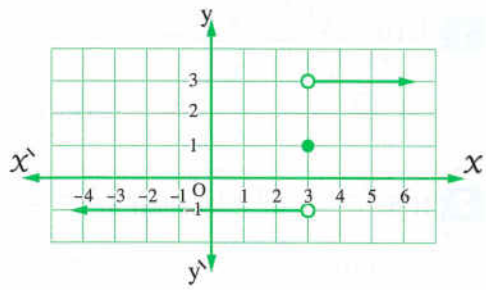
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12 **From the opposite figure :**

$f(3^{+}) + f(3) = \cdots\cdots$

(a) 2 (b) 0

(c) 4 (d) 6



---

13 $\lim\limits_{x \to 0} \left(\frac{\sin 2x}{x} - \cos 5x\right) = \cdots\cdots$

(a) zero (b) 1 (c) 2 (d) 3

---

14 From the following functions , the even function is $f : f(x) = \cdots\cdots$

(a) $\sin x$ (b) $\sin 30°$ (c) $x \cos x$ (d) $x^2 + \tan x$

---

15 In Δ XYZ , $2xz \times \cdots\cdots = x^2 + z^2 - y^2$

(a) cos X (b) cos Z (c) cos Y (d) sin Y

---

16 ABCD is a quadrilateral in which , AB = 22 cm. , m ($\angle$ ADB) = 65°

, m ($\angle$ DBA) = 50° , BC = 25 cm. , DC = 18 cm. **Find :** m ($\angle$ CBD)

---

17 If $\lim\limits_{x \to -1} \frac{x^2 + kx + m}{x^2 - 1} = 3$, then $k + m = \cdots\cdots$

(a) -4 (b) -5 (c) -8 (d) -9

---

18 The range of the function $f : f(x) = \frac{x^2 - 1}{x - 1}$ is $\cdots\cdots$

(a) $\mathbb{R}$ (b) $\mathbb{R} - \{0\}$ (c) $\mathbb{R} - \{1\}$ (d) $\mathbb{R} - \{2\}$

19 If $\frac{a+b}{13} = \frac{b+c}{11} = \frac{c+a}{12}$, then cos A = ...

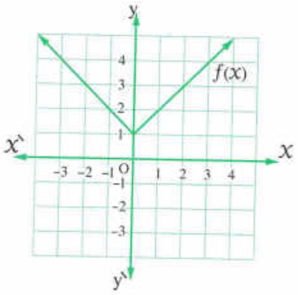
(a) $\frac{1}{5}$ (b) $\frac{5}{7}$ (c) $\frac{19}{35}$ (d) $\frac{4}{11}$

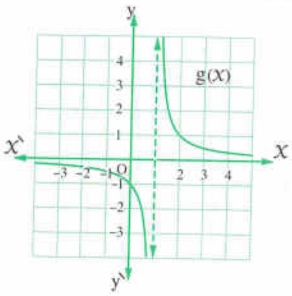
20 The number of possible solutions for the triangle ABC where : m ($\angle$ A) = 47° , a = 4 cm. , b = 6 cm. equals ....

(a) 1 (b) 0 (c) 2 (d) infinite solutions.

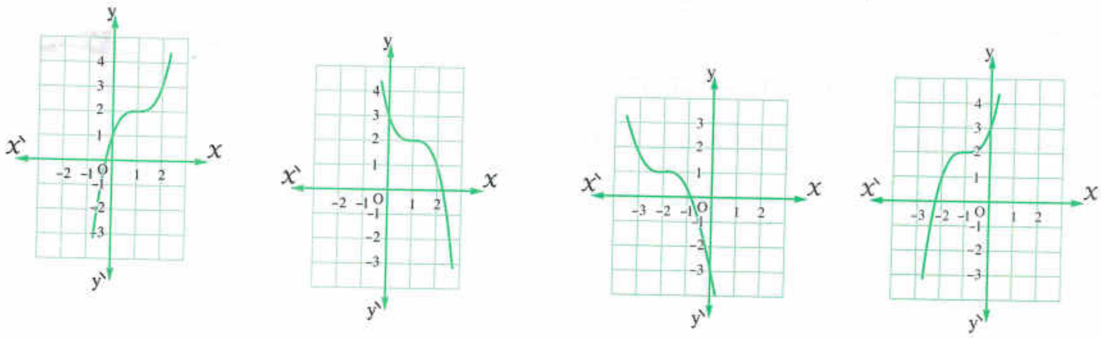
21 **In the opposite figure :**

Find ($f \circ$ g) and represent it graphically , state the domain and if it is even , odd or otherwise and discuss its symmetry and whether it is one - to - one or not.





22 If $f(x) = 2 - (x-1)^3$, then the graph that represents the function f is ....



(a) (b) (c) (d)

23 $\lim_{x \to 1} (2x - 5) =$ ....

(a) 2 (b) – 3 (c) 7 (d) zero

24 Solution set of the equation : $\log_x (x+6) = 2$ in $\mathbb{R}$ is ....

(a) {3 , – 2} (b) {3} (c) {3 , 1} (d) {6 , 1}

25 The inverse function of the function $f : f(x) = 8x^3 - 1$ is $f^{-1}(x) =$ ....

(a) $\sqrt[3]{x^3 - \frac{1}{8}}$ (b) $\frac{\sqrt[3]{x+1}}{2}$ (c) $8\sqrt[3]{x-1}$ (d) $\frac{\sqrt[3]{x-1}}{2}$

26 In Δ LMN , m ($\angle$ L) = 30° , MN = 7 cm. , then the diameter length of the circle passing through its vertices equals ....

(a) 7 cm. (b) 3.5 cm. (c) 14 cm. (d) $\frac{14}{\sqrt{2}}$ cm.

27 The solution set of the equation : $2^{x^2} = 16$ in $\mathbb{R}$ is ....

(a) $\{2\}$ (b) $\{-2\}$ (c) $\{2 , -2\}$ (d) $\{4 , -4\}$

28 **Discuss the continuity of the function at $x = 0$:**

$$f : f(x) = \begin{cases} \frac{x + \sin x}{3x - \sin 2x} & , \quad x < 0 \\ 3x - 1 & , \quad x \geq 0 \end{cases}$$

29 The function $f : f(x) = 5 - x^2 - 4x$ is increasing when $x \in$ ....

(a) $]2 , \infty[$ (b) $]-2 , \infty[$ (c) $]-\infty , 2[$ (d) $]-\infty , -2[$

30 $\lim\limits_{4x \to \pi} \frac{\cos 2x}{4x - \pi} =$ ....

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) $\frac{1}{4}$ (d) $-\frac{1}{4}$

31 In Δ ABC : 3 sin A = 4 sin B = 2 sin C

, then cosine the smallest angle in Δ ABC = ...

(a) $\frac{11}{24}$ (b) $\frac{43}{48}$ (c) $\frac{29}{36}$ (d) $\frac{11}{36}$

32 The solution set of the inequality : $\sqrt{x^2 - 6x + 9} + 2 \leq 9$ is ....

(a) $\mathbb{R} -]-4 , 10[$ (b) $\mathbb{R} -]-8 , 10[$ (c) $[-4 , 10]$ (d) $[-8 , 10]$

Model 6

Interactive test 6

Answer the following questions :

1 If $\lim\limits_{x \to 4} \frac{x^2 + 7x + b}{x^2 - 6x + 8} = \frac{15}{2}$, then b = ...

(a) – 44 (b) 7 (c) – 8 (d) 8

2 If $\log_b x + \log_b 3 = \log_b 27 - 1$, which of the following represents x in terms of b ?

(a) $x = 9$ b (b) $x = \frac{1}{9}$ b (c) $x = \frac{9}{b}$ (d) $x = \frac{1}{9b}$

3 If $\log_a (x+2) - \log_a (x-1) = \log_a 4$, then $x =$ ....

(a) -2 (b) 2 (c) 1 (d) -1

4 **Determine which of the functions defined by the following rules is even , odd or neither even nor odd :**

[1] $f_1(x) = x \cos x$ **[2]** $f_2(x) = \begin{cases} x^2 & , \quad x \geq 0 \\ |x| & , \quad x < 0 \end{cases}$

5 $\lim\limits_{x \to 0} \dfrac{x^2 - 1}{x} =$ ....

(a) zero (b) 1 (c) does not exist. (d) -1

6 If $x^{\frac{3}{2}} = 64$, then $x =$ ....

(a) 512 (b) 16 (c) 4 (d) 2

7 The area of the circle passing through the vertices of the equilateral triangle ABC whose side length is 9 cm. equals .... cm^2.

(a) 9π (b) $9\sqrt{3}\pi$ (c) 27π (d) 81π

8 If $f(x) = 3x - 1$, $h(x) = x^2$, then $(h \circ f)(-2) =$ ....

(a) -7 (b) 11 (c) -49 (d) 49

9 $\lim\limits_{h \to 0} \dfrac{(x+h)^7 - x^7}{h} =$ ....

(a) x^7 (b) $7x^6$ (c) zero (d) 1

10 In Δ ABC , $a^2 + b^2 - c^2 =$ ....

(a) cos A (b) a b cos C (c) cos C (d) 2 a b cos C

11 The curve $g(x) = |x + 3|$ is the same as the curve $f(x) = |x|$ by translation 3 units in the direction of ....

(a) $\overrightarrow{Ox}$ (b) $\overleftarrow{Ox}$ (c) $\overrightarrow{Oy}$ (d) $\overleftarrow{Oy}$

12 The solution set of the inequality :

$|3 - 2x| \leq 1$ in $\mathbb{R}$ is ....

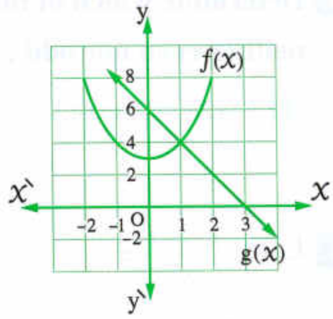
(a) $[1 , 2]$ (b) $]1 , 2[$ (c) $\mathbb{R} -]1 , 2[$ (d) $\mathbb{R} - [1 , 2]$

13 If $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ where $f(x) = \frac{1}{x^2 + 1}$

Find : $f^{-1}(x)$ and state the domain of f^{-1} and its range.

---

14 The opposite figure represents the two curves $f(x)$, $g(x)$, then $(g \circ f)(1) =$ ....



(a) -8 (b) -2 (c) 4 (d) 5

---

15 The point of symmetry of the function $f : f(x) = \frac{2x - 1}{x}$ is ....

(a) $(1 , 1)$ (b) $(2 , 1)$ (c) $(1 , 2)$ (d) $(0 , 2)$

---

16 $\lim\limits_{x \to 0} \frac{\sin 2x^2 + \tan^2 2x}{3x^2} =$ ....

(a) $\frac{8}{3}$ (b) $\frac{4}{3}$ (c) 2 (d) 3

---

17 If $\log(2x + y) = \frac{1}{2}(\log x + \log y) + \log 3$ where $x \neq y$ and $y = kx$, then $k =$ ....

(a) 2 (b) 4 (c) $\frac{1}{4}$ (d) -4

---

18 $\lim\limits_{x \to \frac{\pi}{2}} (2x - \cos x) =$ ....

(a) zero (b) 2 (c) π (d) $\frac{\pi}{2}$

---

19 The even continuous function at the point (a , b) is also continuous at the point ....

(a) (a , b) (b) $(-a , b)$ (c) $(a , -b)$ (d) $(-a , -b)$

---

20 The range of the function $f : f(x) = \begin{cases} 0 & , \quad x \leq 0 \\ 1 & , \quad x > 0 \end{cases}$ is ....

(a) $\{1\}$ (b) $\{0\}$ (c) $\mathbb{R}$ (d) $\{0 , 1\}$

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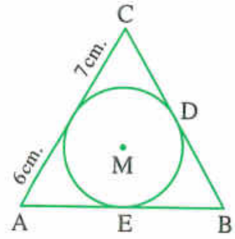
21 The radius length of the circumcircle of the triangle XYZ in which :

$x = 3$ cm. , $y = 5$ cm. , $z = 7$ cm. equals .... cm.

(a) $\frac{7}{3}\sqrt{3}$ (b) $2\sqrt{3}$ (c) 4.4 (d) $\frac{14}{3}\sqrt{3}$

22 **In the opposite figure :**

If the perimeter of Δ ABC = 42 cm. and the circle M is the inscribed circle in it , then m ($\angle$ A) = ...



(a) $53° \grave{7}$ (b) $67° \grave{23}$

(c) $36° \grave{53}$ (d) $22° \grave{37}$

23 If $5^{x-3} = 4^{3-x}$, then x = ...

(a) $\frac{5}{4}$ (b) 3 (c) $\frac{4}{5}$ (d) zero

24 Solve the triangle ABC approximate the results to the nearest 1 decimal place where m ($\angle$ A) = 40° , m ($\angle$ B) = 30° , b = 10 cm.

25 The numerical value of the expression $\frac{\log 64}{\log 8}$ equals ....

(a) 2 (b) 8 (c) 80 (d) 72

26 In Δ DEF , m ($\angle$ D) = 80° , m ($\angle$ E) = 60° , if f = 12 cm. , then d = ... cm.

(a) $\frac{12 \sin 80°}{\sin 40°}$ (b) $\frac{12 \sin 80°}{\sin 60°}$ (c) $\frac{12 \sin 40°}{\sin 80°}$ (d) $\frac{12 \cos 80°}{\cos 40°}$

27 $\lim\limits_{x \to \infty} \frac{(2x+1)^{40}(4x-1)^5}{(2x+3)^{45}}$ = ...

(a) 16 (b) 32 (c) 64 (d) 8

28 If the function $f : f(x) = \begin{cases} \frac{x^2+2x-3}{x+3} & , x \neq -3 \\ x + a & , x = -3 \end{cases}$ is continuous at $x = -3$, then find a

29 The point of intersection of the curve : $f(x) = 4 - \frac{2}{x-1}$ with the y-axis is ....

(a) (0 , 2) (b) (1 , 4) (c) (0 , 6) (d) (0 , – 2)

30 $\lim\limits_{x \to 3} \frac{5x^2 - 45}{\sqrt{5x+1} - 4}$ = ...

(a) 5 (b) 6 (c) 8 (d) 48

31 The solution set of the equation : $3^{x+1} + 3^{3-x} = 30$ in $\mathbb{R}$ is ....

(a) $\{1 , 9\}$ (b) $\{\log_3 27 , \log 1\}$

(c) $\{\log_5 25 , \log 1\}$ (d) $\{\log_2 25 , \log_2 5\}$

32 In Δ ABC , m ($\angle$ B) = 3 m ($\angle$ C) = 84° , AC = 16 cm.

, then the perimeter of Δ ABC = ... cm. (to nearest one decimal)

(a) 81.5 (b) 62.2 (c) 41.3 (d) 38.5

Model 7

Interactive test 7

Answer the following questions :

1 If $f(x) = (10)^{2x}$, $g(x) = \log\left(\sqrt{x}\right)$, then $(g \circ f)(x) =$ ....

(a) 10^{2x} (b) $\log\sqrt{x}$ (c) $10^{2x}\log\sqrt{x}$ (d) x

2 In Δ ABC , $\frac{a}{a+b} = \frac{\sin A}{\dots\dots\dots}$

(a) sin B (b) sin C (c) sin A + sin B (d) sin A + sin C

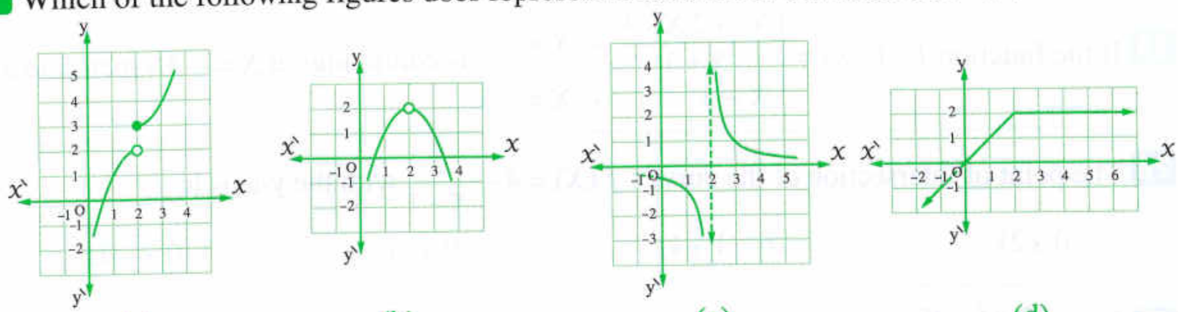
3 In Δ ABC , if sin A = 2 sin C , BC = 6 cm. , then AB = ... cm.

(a) 2 (b) 3 (c) 4 (d) 6

4 **Find in $\mathbb{R}$ the solution set of the equation :**

$x^{\frac{4}{3}} - 10\, x^{\frac{2}{3}} + 9 = 0$

5 Which of the following figures does represent a continuous function at $x = 2$?



(a) (b) (c) (d)

6 The domain of the function $f : f(x) = \sqrt{x+2} - \sqrt{5-x}$ is ....

(a) $[-2 , \infty[$ (b) $[-2 , 5[$ (c) $[-2 , 5]$ (d) $[3 , \infty[$

7 In Δ ABC , $b^2 + c^2 - a^2 = 2\,b\,c \times$ ....

(a) sin (90° – B) (b) sin (90° – A) (c) cos B (d) cos (90° – B)

8 The number of possible solutions of the triangle XYZ in which :

m (∠ X) = 30° , $\mathcal{X}$ = 6 cm. , y = 9 cm. equals ....

(a) 1 (b) 2 (c) 0 (d) infinite solutions.

---

9 $\lim\limits_{x \to 1} \frac{4-\sqrt{x+15}}{1-x^2} =$ ....

(a) $\frac{1}{16}$ (b) 16 (c) $\frac{1}{4}$ (d) 4

---

10 If the radius length of the circle passing through the vertices of Δ ABC equals 6 cm.

, then : $\frac{2\,a}{\sin A} =$ .... cm.

(a) 12 (b) 6 (c) 18 (d) 24

---

11 The solution set of the equation : $(\log_5 y)^2 - 7 \log_5 y + 12 = 0$ in $\mathbb{R}$ is ....

(a) $\{25 , 125\}$ (b) $\{25 , 625\}$ (c) $\{\frac{1}{25} , 625\}$ (d) $\{125 , 625\}$

---

12 If $f(x) = \frac{x-5}{|x-5|}$, then $[f(5^+)]^2 + [f(5^-)]^2 =$ ....

(a) 5 (b) zero (c) 4 (d) 2

---

13 $\log_2 \frac{3}{25} + 5 \log_2 5 + \log_2 27 - \log_2 \frac{125}{12} - \log_2 243 =$ ....

(a) 4 (b) $\log_3 9$ (c) log 25 (d) $\log \frac{1}{100}$

---

14 If $\lim\limits_{x \to 1} \frac{x^2-k^2}{x+2} = -1$, then k = ...

(a) 2 (b) – 2 (c) 4 (d) ± 2

---

15 If $f : f(x) = x^3 + 1$, which of the following statements is not true ?

(a) f is one - to - one (b) f is an odd function.

(c) f is increasing in its domain.

(d) The curve of the function f intersects the x-axis at $x = -1$

---

16 In Δ XYZ , $\mathcal{X}$ = 5 cm. , y = 7 cm. , m (∠ Z) = 65°

Calculate : z , then find the perimeter of Δ XYZ

---

17 If $\log_2 x = 3$, then $\log_x 2 =$ ....

(a) 2 (b) $\frac{1}{3}$ (c) 8 (d) 9

18 If f^{-1} is the inverse function of the function f, then ....

(a) the domain of f^{-1} = the domain of f
(b) the domain of f^{-1} = the range of f
(c) the range of f^{-1} = the range of f
(d) the range of f^{-1} = the domain of f^{-1}

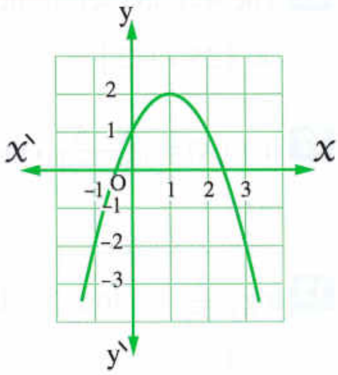
19 $\lim\limits_{y \to 2} \frac{y^5 - 32}{y - 2} =$ ....

(a) $31\ y^4$ (b) 32×2^4 (c) 64 (d) 5×2^4

20 Discuss the existence of limit of f where :

$$f(x) = \begin{cases} \frac{\tan 2x}{\sin x} & \text{at } \frac{-\pi}{4} < x < 0 \\ \frac{5x + 6}{x + 3} & \text{at } x > 0 \end{cases}$$ at x tends to zero

21 The rule of the function represented in the opposite figure is $f(x) =$ ....



(a) $(x - 2)^2 + 1$
(b) $-(x - 2)^2 + 1$
(c) $-(x - 1)^2 + 2$
(d) $(-x + 1)^2 + 2$

22 The solution set of the equation : $|2x - 1| = 5$ in $\mathbb{R}$ is ....

(a) $\{3\}$ (b) $\{-2\}$ (c) $\varnothing$ (d) $\{3, -2\}$

23 If $f(x) = \begin{cases} x - 4 & , \quad x \geq 4 \\ g(x) & , \quad x < 4 \end{cases}$ is symmetric about the straight line $x = 4$

, then the function g is ....

(a) an increasing function.
(b) a decreasing function.
(c) an even function.
(d) a constant function.

24 If $f(x) = x^2 - 3$, $g(x) = \sqrt{x - 2}$, find $(f \circ g)(x)$ in the simplest form and state its domain, then find $(f \circ g)(3)$

25 $\lim\limits_{x \to 0} \frac{3x + 2x^{-1}}{x + 4x^{-1}} =$ ....

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) 2 (d) 4

26 If $a \in]0 , 9]$, then $\log_3 a \in$ ....

(a) $]-\infty , 2]$ (b) $]2 , 81]$ (c) $[2 , \infty[$ (d) $]-\infty , 0]$

27 $\lim_{x \to \infty} \left(\frac{1}{x-2} + 1\right) =$ ....

(a) 2 (b) 1 (c) zero (d) ∞

28 The solution set in $\mathbb{R}$ of the equation : $\sqrt{x^2 - 6x + 9} + 2x = 9$ equals ....

(a) $\{4 , 6\}$ (b) $\{6\}$ (c) $\{4\}$ (d) $\varnothing$

29 The solution set in $\mathbb{R}$ of the inequality $|x - 3| \leq 2$ is ....

(a) $]1 , 5[$ (b) $[1 , 5]$ (c) $\mathbb{R} -]1 , 5[$ (d) $\mathbb{R} - [1 , 5]$

30 If the function $f : f(x) = \begin{cases} \frac{x^2 - 4}{x - 2} & , x \neq 2 \\ K & , x = 2 \end{cases}$ is continuous at $x = 2$, then K = ...

(a) zero (b) 8 (c) 2 (d) 4

31 If $b^{x} - 2b^{-x} = 1$ where $b > 1$, then $x =$ ....

(a) 2 (b) $\log 2$ (c) $\log_2 b$ (d) $\log_b 2$

32 If the area of Δ ABC is 24 cm^2, the radius length of it circumcircle is 5 cm.

, then $\sin A \sin B \sin (A + B) =$ ....

(a) $\frac{3}{25}$ (b) $\frac{6}{25}$ (c) $\frac{9}{25}$ (d) $\frac{12}{25}$

Model 8

Interactive test 8

Answer the following questions :

1 $\log_a x \div \log_{ab} x =$ ....

(a) $1 - \log_a b$ (b) $1 + \log_a b$ (c) $1 - \log_b a$ (d) $1 + \log_b a$

2 If $f(x) = \begin{cases} 3x^2 + ax - 2 & , x > 3 \\ 2x + b & , x < 3 \end{cases}$ and $\lim_{x \to 3} f(x) = 16$, then a + b = ...

(a) 4 (b) 10 (c) -13 (d) 7

3 The diameter length of the circle inscribed in an equilateral triangle whose side length is $4\sqrt{3}$ cm. equals .... cm.

(a) $2\sqrt{3}$ (b) $4\sqrt{3}$ (c) 4 (d) 8

4 The value of : $\log_3 54 - \log_3 \frac{8}{15} + \log_3 \frac{4}{5} =$ ....

(a) log 3 (b) 3 (c) 27 (d) 4

5 If $y = f(x)$ is a real function , then its image by translation 2 units right is $g(x) =$ ....

(a) $f(x-2)$ (b) $f(x+2)$ (c) $f(x)+2$ (d) $f(x)-2$

6 The number of possible solutions of Δ ABC where $m(\angle A) = 60°$, b = 3 cm. , a = 5 cm. is ....

(a) 1 (b) 2

(c) no solution. (d) an infinite number of triangles.

7 $\lim\limits_{x \to \text{zero}} \frac{x^2 + x}{x} =$ ....

(a) zero (b) 1 (c) 2 (d) 3

8 If $f(x) = 5^x$ **, find the solution set in $\mathbb{R}$ of the equation :** $f(x) + f(x-1) = 150$

9 In Δ ABC , cos (A + B) = ...

(a) $\frac{a^2 + b^2 - c^2}{2ab}$ (b) $\frac{a^2 + c^2 - b^2}{2ac}$ (c) $\frac{b^2 + c^2 - a^2}{2bc}$ (d) $\frac{c^2 - a^2 - b^2}{2ab}$

10 If $1 < x < 2$, then $\sqrt{x^2 - 2x + 1} + \sqrt{x^2 - 4x + 4} =$ ....

(a) $2x - 3$ (b) $2x - 1$ (c) 1 (d) 3

11 $\lim\limits_{x \to 3} \frac{x^2 - 7x + 12}{x - 3} =$ ....

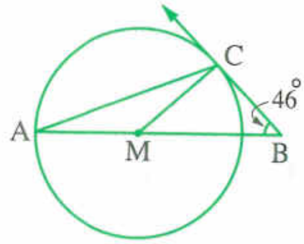
(a) 1 (b) – 1 (c) – 2 (d) 7

12 The domain of the function $f : f(x) = \sqrt{\frac{x-5}{x+4}} =$ ....

(a) $[5, \infty[$ (b) $[-4, 5]$ (c) $\mathbb{R} - [-4, 5[$ (d) $\mathbb{R} -]-4, 5]$

13 **In the opposite figure :** If AC = 20 cm.

, then the perimeter of Δ ACM = ... cm.



(a) 41.5 (b) 43.5

(c) 45 (d) 47.5

14 $3^{\log_3 4} + \log_5 25 =$ ....

(a) 6 (b) 4 (c) 2 (d) 1

15 The domain of the function $f : f(x) = \sqrt{9 - x}$ is ....

(a) $\mathbb{R}$ (b) $\mathbb{R} - \{9\}$ (c) $]-\infty , 9]$ (d) $[9 , \infty[$

16 $\lim_{x \to 5} \frac{x^2 - 5x}{\sqrt{x + 4} - 3} =$ ....

(a) 30 (b) 6 (c) 5 (d) 25

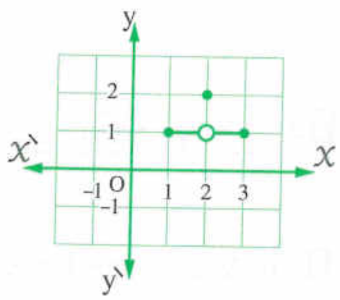
17 In Δ ABC , c = 19 cm. , m ($\angle$ A) = 112° , m ($\angle$ B) = 33° , find the area of Δ ABC to the nearest cm^2 and the radius length of its circumcircle to the nearest cm.

18 The domain of the function $f : f(x) = \log_{1-x} x$ is ....

(a) $x > 0$ (b) $x < 1$ (c) $0 < x < 1$ (d) $0 \le x \le 1$

19 The opposite figure represents the curve of the function f

, which of the following statements is true ?



(a) f is continuous on the interval $[1 , 3]$

(b) f is continuous on the interval $]1 , 3[$

(c) $\lim_{x \to a} f(x)$ exists where $a \in [1 , 3]$

(d) $\lim_{x \to a} f(x)$ exists where $a \in]1 , 3[$

20 $\lim_{x \to \infty} \left(\frac{3x^2 + 2x + 1}{x^2 - 3x + 2}\right)^4 =$ ....

(a) 3 (b) 9 (c) 27 (d) 81

21 If $f : \mathbb{R} \longrightarrow \mathbb{R}$ where $f(x) = 4x - 2$, h : $[-2 , 3] \longrightarrow \mathbb{R}$ where h $(x) = 4 - 3x$ graph the function $(f + h)$, determine its domain and range and discuss its monotony.

22 Which of the following does not equal $\left(\sqrt[5]{x^4}\right)$?

(a) $\left(\sqrt[5]{x}\right)^4$ (b) $\sqrt[4]{x^5}$ (c) $x^{\frac{4}{5}}$ (d) $\left(x^{\frac{1}{5}}\right)^4$

23 If the function f is even in $[c , d]$, then c + d = ...

(a) 2 c (b) 2 d (c) c – d (d) zero

24 Find the value of a that makes the function f continuous at a where

$$f(x) = \begin{cases} 2 - x^2 & , \quad x \le a \\ x & , \quad x > a \end{cases}$$

25 If $f(x) = (x - 5)(x + 5)$, $g(x) = x - 5$, then $\frac{f}{g}(5) =$ ....

(a) 10 (b) 1 (c) $\frac{f}{g}(-5)$ (d) undefined.

26 $\lim\limits_{x \to 0} \frac{\sin\left(2\sqrt[3]{x}\right)}{\sqrt[3]{x}} =$ ....

(a) 8 (b) – 2 (c) $\frac{2}{3}$ (d) 2

27 If $\left(\frac{1}{2}\right)^{a^2 - a - 2} = 1$, where a > zero , then a = ...

(a) 1 (b) – 3 (c) 2 (d) 3

28 Δ ABC in which m (∠ C) = 116° , c = 12 cm. , a = 10 cm.

, then b = ... cm. (to nearest one decimal)

(a) 2.6 (b) 3.6 (c) 4.6 (d) 5.6

29 $\lim\limits_{x \to 0} \frac{1 - \cos x + \tan 5x}{1 - \cos x - \tan x} =$ ....

(a) – 5 (b) 5 (c) zero (d) undefined.

30 In Δ ABC , $\cos B = \frac{c}{2a}$, then Δ ABC is ....

(a) an equilateral triangle. (b) an isosceles triangle.

(c) a scalene triangle. (d) otherwise.

31 The solution set of the inequality $\frac{1}{|2x - 3|} > 2$ in ℝ is ....

(a) $\left[\frac{5}{4} , \frac{7}{4}\right] - \left\{\frac{3}{2}\right\}$ (b) $\left]\frac{5}{4} , \frac{7}{4}\right[$

(c) $\left]\frac{5}{4} , \frac{7}{4}\right[- \left\{\frac{3}{2}\right\}$ (d) $\left[\frac{5}{4} , \frac{7}{4}\right]$

32 The one-to-one function from the following functions defined by the rules ....

(a) $f_1(x) = x + 5$ (b) $f_2(x) = x^2$ (c) $f_3(x) = |x - 2|$ (d) $f_4(x) = -3$

Model 9

Interactive test 9

Answer the following questions :

1 The solution set of the equation : $\log_{(x+3)} 125 = 3$ in $\mathbb{R}$ is ....

(a) $\{5\}$ (b) $\{3\}$ (c) $\varnothing$ (d) $\{2\}$

2 Δ LMN in which m ($\angle$ L) = 30° , m = 9 cm. has two solutions when ℓ = ... cm.

(a) 6 (b) 10 (c) 11 (d) 2

3 The opposite figure represents the curve of the function f

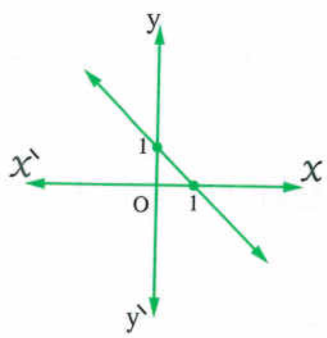
, then $\lim\limits_{x \to 2} |f(x)| =$ ....

(a) -1

(b) zero

(c) 1

(d) does not exist.



4 If $f(x) = 3 + \sqrt{x - 1}$, find its inverse function.

5 If $f : \mathbb{R} \longrightarrow \mathbb{R}$ where $f(x + 1) - f(x) = x - 1$, then $f(10) - f(9) =$ ....

(a) 1 (b) 9 (c) 8 (d) 18

6 $\lim\limits_{x \to 0} \frac{2x}{\sin 3x} =$ ....

(a) $\frac{2}{3}$ (b) $\frac{3}{2}$ (c) 6 (d) does not exist.

7 The image of the curve $y = |x| - 5$ by translation 3 units in the direction of $\overrightarrow{Ox}$ and 5 units in the direction of $\overrightarrow{Oy}$ is ....

(a) $y = |x - 3| + 5$ (b) $y = |x - 3|$ (c) $y = |x - 3| - 10$ (d) $y = |x + 3|$

8 If the function $f : f(x) = \begin{cases} x^2 + a x - 2 & , \quad x > 2 \\ 4 & , \quad x = 2 \\ 5a + b x & , \quad x < 2 \end{cases}$ is continuous at $x = 2$

, find the value of each of a , b

9 $\lim\limits_{x \to \infty} \frac{\sqrt{x+5} - \sqrt{5}}{\sqrt{x} - \sqrt{5}} =$ ....

(a) 1 (b) -1 (c) ∞ (d) $-\infty$

10 If $y = \sqrt[5]{x}$, then its inverse function is y = ...

(a) $\frac{1}{5} x^5$ (b) x^5 (c) $x^5 - 1$ (d) $5 x^5$

11 In Δ ABC , c (a cos B + b cos A) = ...

(a) a^2 (b) b^2 (c) c^2 (d) $2 c^2$

12 ABCD is a parallelogram in which : AB = 9 cm. , BC = 13 cm. , AC = 20 cm.

, then the length of $\overline{BD}$ = ... cm.

(a) 5 (b) 10 (c) 205 (d) 4

13 If $f(x) = x - 1$, $g(x) = \sqrt{x}$, then the domain of $(g \circ f)$ is ....

(a) $\mathbb{R}$ (b) $]-\infty , 1[$ (c) $[1 , \infty[$ (d) $\mathbb{R} - \{1\}$

14 $\lim\limits_{x \to 4} \frac{x^3 \sqrt{x} - 128}{x - 4} =$ ....

(a) 112 (b) 96 (c) 84 (d) 72

15 If $f(x) = \frac{\sqrt{x^2 - 2x + 1}}{x - 1}$, then the range of the function f is ....

(a) $\{1\}$ (b) $\mathbb{R}$ (c) $[-1 , 1[$ (d) $\{-1 , 1\}$

16 Graph the function $f : f(x) = \sqrt{x^2 - 4x + 4}$ and determine its range and discuss its monotony.

17 The solution set of the following equation in $\mathbb{R}$: $\log_2 x - \frac{3}{\log_2 x} = 2$ equals ....

(a) $\{-1 , 3\}$ (b) $\{8 , \frac{1}{2}\}$ (c) $\{8 , 2\}$ (d) $\{\frac{1}{8} , 2\}$

18 $\lim\limits_{h \to 0} \frac{(x+h)^9 - x^9}{h} =$ ....

(a) x^9 (b) $9 x^8$ (c) zero (d) does not exist.

19 If $a > b > c > 1$, then $\log_c \log_b \log_a a^{b^c} = \dots\dots\dots$

(a) zero (b) 1 (c) 2 (d) a b c

---

20 If ABC is a triangle in which $a = 4$ cm. , $b = 4\sqrt{3}$ cm. , $c = 8$ cm. , then sine of its smallest angle equals ....

(a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) 1 (d) zero

---

21 Solve the acute-angled triangle ABC in which $a = 21$ cm. , $b = 25$ cm. and the diameter length of its circumcircle is 28 cm.

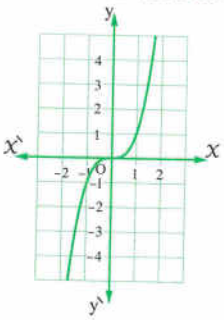
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22 If $x = 5 + 2\sqrt{6}$, then $\log\left(\frac{1}{x} + x\right) = \dots\dots\dots$

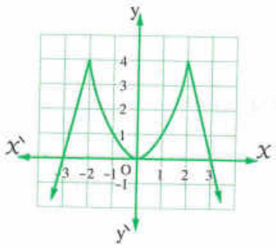
(a) 1 (b) $5 - 2\sqrt{6}$ (c) 10 (d) $5 + 2\sqrt{6}$

---

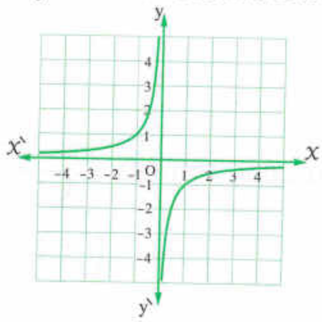
23 Which of the functions represented graphically as follows is neither even nor odd ?



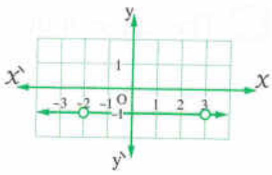
(a)



(b)



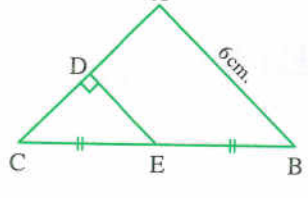
(c)



(d)

---

24 **In the opposite figure :**



If $\tan(\angle DEC) = \frac{3}{4}$, then the radius length of the circumcircle of $\Delta ABC = \dots\dots\dots$ cm.

(a) 9 (b) 5.7

(c) $4\frac{3}{4}$ (d) 3.75

---

25 If $f(x) = x^2 + |x|$ where x is a real number , then the solution set in $\mathbb{R}$ of the equation : $f(x) = 2$ equals ....

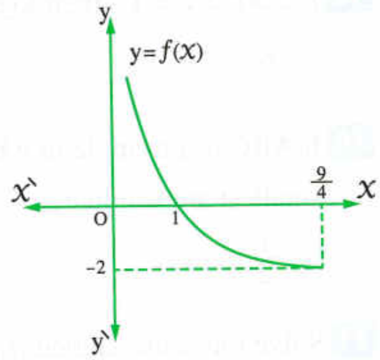
(a) $\{-2\}$ (b) $\{-2, -1, 1\}$ (c) $\{1, -1\}$ (d) $\{1\}$

---

26 $\lim\limits_{x \to 0} \sqrt{4 - x^2} = \dots\dots\dots$

(a) -1 (b) zero (c) 2 (d) does not exist.

27 If the opposite figure represents the curve of the function $f : f(x) = \log_a x$, then $\log \frac{16}{81} = \dots\dots$



(a) -2

(b) 1

(c) 2

(d) 4

---

28 $\lim\limits_{x \to 0} \frac{1 - \cos x + \sin x}{1 - \cos x - \sin x} = \dots\dots$

(a) 1 (b) -1 (c) 0 (d) ∞

---

29 If the perimeter of $\Delta ABC = 33$ cm., $\sin A + \sin C = \frac{2}{3}$, $\sin B = \frac{1}{4}$, then $AC = \dots\dots$ cm.

(a) 6 (b) 9 (c) 12 (d) 15

---

30 The range of the function $f : f(x) = \frac{x+1}{x+2}$ equals $\dots\dots$

(a) $\mathbb{R}$ (b) $\mathbb{R} - \{-2\}$ (c) $\mathbb{R} - \{1\}$ (d) $\mathbb{R}^+$

---

31 If $f(x) = a x + b$, $f^{-1}(9) = 3$, $f^{-1}(5) = 2$, then $a + b = \dots\dots$

(a) -1 (b) 1 (c) 7 (d) -7

---

32 $\lim\limits_{x \to -1} \frac{2|x+1|}{x+1} = \dots\dots$

(a) 2 (b) -2 (c) 0 (d) not exist.

Model 10

Interactive test 10

Answer the following questions :

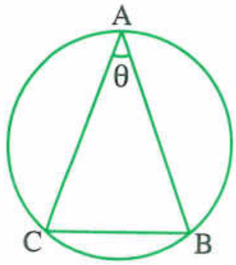
1 $\log \tan 1° + \log \tan 2° + \log \tan 3° + \cdots + \log \tan 88° + \log \tan 89° = \dots\dots$

(a) zero (b) 1 (c) 10 (d) 89

2 **In the opposite figure :**

ABC is a triangle inscribed in a circle whose radius length is 4 cm. , m (∠ BAC) = θ^{rad}

, then $\lim\limits_{\theta^{rad} \to 0} \frac{BC}{\theta^{rad}} =$ ....



(a) 2 (b) 4

(c) 6 (d) 8

3 If the ratio among the measures of the angles of a triangle is 8 : 3 : 1 , then the ratio between the longest two sides in the triangle is ....

(a) $\sqrt{3} : 2$ (b) $\sqrt{6} : 2$ (c) 8 : 3 (d) 8 : 5

4 $\lim\limits_{x \to -3} \frac{\sqrt{x+7}-2}{x+3} =$ ....

(a) 2 (b) 4 (c) $\frac{1}{4}$ (d) $\frac{1}{2}$

5 If $3^a = 4^b$, then $9^{\frac{a}{b}} + 16^{\frac{b}{a}} =$ ....

(a) 7 (b) 12 (c) 20 (d) 25

6 If $\lim\limits_{x \to \infty} \frac{3k|x|}{4x+3} = 6$, then k = ...

(a) 6 (b) $\frac{3}{4}$ (c) 8 (d) 3

7 If $f(x) = x^3$, then the image of the curve of f by reflection in x-axis and translation 3 units in the direction of $\overrightarrow{Ox}$ and two units in the direction of $\overrightarrow{Oy}$ is ....

(a) $-(x-3)^3 - 2$ (b) $-(x+3)^3 + 2$

(c) $-(x+3)^3 - 2$ (d) $-[(x+3)^3 + 2]$

8 If the function $f : f(x) = \begin{cases} x^2 + ax - 2 & , \quad x > 2 \\ 4 & , \quad x = 2 \\ 5a + bx & , \quad x < 2 \end{cases}$ is continuous at $x = 2$

, find the value of each of a , b

9 The perimeter of Δ ABC is 70 cm. , a = 26 cm. , m (∠ A) = 60° , find its area.

10 If $f(x) = x + 1$, $g(x) = \frac{x^2 - 1}{x - 1}$, then $\lim\limits_{x \to 1} (g \circ f)(x) =$ ....

(a) 1 (b) 2 (c) − 2 (d) 3

11 If $\log_2 3 \times \log_3 4 \times \log_4 5 \times \cdots \times \log_n (n+1) = 10$, then n = ...

(a) 9 (b) 10 (c) 11 (d) 1023

---

12 The domain of the function $f : f(X) = \sqrt{\sqrt{X^2} - 1}$ is ....

(a) $]-1 , 1[$ (b) $[-1 , 1]$ (c) $\mathbb{R} -]-1 , 1[$ (d) $\mathbb{R} - \{-1 , 1\}$

---

13 In Δ ABC , m ($\angle$ A) = 112° , m ($\angle$ B) = 33° , c = 19 cm.

, then the diameter length of its circumcircle $\simeq$ .... cm.

(a) 16 (b) 17 (c) 32 (d) 33

---

14 If $2^X = 20$, $n < X < n + 1$, n is an integer , then n = ...

(a) 4 (b) 5 (c) 6 (d) 10

---

15 In Δ XYZ , $y^2 + z^2 - X^2 = 2\, y\, z \times$ ....

(a) cos X (b) sin Z (c) cos Z (d) sin X

---

16 If $f(X) = \begin{cases} 3X - 1 & , \quad X \neq 2 \\ 6 & , \quad X = 2 \end{cases}$, then $\lim\limits_{X \to 2} f(X) =$ ....

(a) – 5 (b) 5 (c) 6 (d) does not exist.

---

17 If $f(X) = \log_2 (X + a)$ and $f^{-1}(2) = -3$, then a = ...

(a) – 7 (b) 7 (c) 3 (d) 1

---

18 The exponential function whose base is a , is increasing if ....

(a) a > 0 (b) a > 1 (c) 0 < a < 1 (d) a = 1

---

19 $\lim\limits_{X \to \infty} (4 - 3X - X^3) =$ ....

(a) ∞ (b) does not exist. (c) – 1 (d) $-\infty$

---

20 If f is an odd function , a $\in$ the domain of f , then $f(a) + f(-a) =$ ....

(a) $2 f(a)$ (b) $2 f(-a)$ (c) zero (d) $f(a)$

---

21 **Find algebraically in $\mathbb{R}$ the solution set of the equation :** $|X - 3| = |9 - 2X|$

22 If f is an odd function , then $\frac{2 f(3) + 7 f(-3)}{10 f(-3)} =$ ....

(a) 3 (b) -3 (c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

23 If $f(X) = \sqrt{X + 3}$, $g(X) = \sqrt{6 - X}$, then $(f \circ g)(5) =$ ....

(a) undefined. (b) zero (c) 5 (d) 2

24 The range of the function $f : f(X) = \begin{cases} 2X + 3 & , X > 3 \\ 9 & , X < 3 \end{cases}$ is ....

(a) $\{3\}$ (b) $\mathbb{R}$ (c) $]9 , \infty[$ (d) $[9 , \infty[$

25 In Δ ABC , if m ($\angle$ B) = 60° , m ($\angle$ C) = 30° , c = 4 cm. , then b = ... cm.

(a) 4 (b) 8 (c) $2\sqrt{3}$ (d) $4\sqrt{3}$

26 $\lim\limits_{X \to 0} \frac{1 - \tan X}{\sin X - \cos X} =$ ....

(a) 1 (b) -1 (c) zero (d) does not exist.

27 If the area of Δ ABC is "X" and the radius length of its circumcircle is "r" , then $\frac{4 r X}{a b c} =$ ....

(a) $\frac{a}{\sin A}$ (b) cos A (c) 1 (d) r

28 If $f(X) = 7^{X + 1}$

, find the value of X which satisfies : $f(2X - 1) + f(X - 2) = 50$

29 If $\lim\limits_{X \to a^+} f(X) = \ell$, $\lim\limits_{X \to a^-} f(X) = m$ and the function is continuous at $X = a$, then $\ell^2 + m^2 - 2 \ell m =$ ....

(a) 1 (b) 3 (c) zero (d) 6

30 If a = sin B , b = sin C , c = sin A , then the circumference of the circumcircle of triangle ABC equals ....

(a) 1 (b) 2π (c) $\frac{1}{2}\pi$ (d) π

31 The solution set of the inequality : $\sqrt{9X^2 - 12X + 4} + 2|4 - 6X| \geq 20$ is ....

(a) $\mathbb{R} -]\frac{-2}{3} , 2[$ (b) $]\frac{-2}{3} , 2[$ (c) $\mathbb{R} - [\frac{-2}{3} , 2]$ (d) $[\frac{-2}{3} , 2]$

32 If $f(X) = X^3 + 3X^2 + 3X + 1$, then $f^{-1}(X) =$ ....

(a) $(X + 1)^3$ (b) $\sqrt[3]{X} - 1$ (c) $\sqrt[3]{X} + 1$ (d) $X^3 - 1$

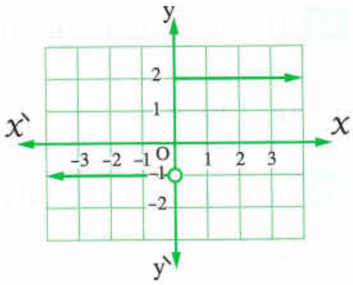
Second Multiple choice examinations

Model 1

Answer the following questions :

Choose the correct answer from the given ones :

1 **In the opposite figure :**



The range of the function is ....

(a) $\{-1, 2\}$ (b) $\{2\}$

(c) $\{-3, 3\}$ (d) $\{-1\}$

2 If $\log_2 x = \log_4 9$, then $x =$ ....

(a) 3 (b) 4 (c) 9 (d) 12

3 $\lim\limits_{x \to 0} \dfrac{\sqrt{1-\cos^2 x}}{x} =$ ....

(a) 0 (b) 1 (c) -1 (d) not exist.

4 ABC is a triangle in which AB = 3 cm. , BC = 5 cm. and AC = 7 cm. , then m ($\angle$ B) = ...

(a) 30° (b) 60° (c) 120° (d) 150°

5 $\log_{xy}\left(\frac{1}{x}\right) + \log_{xy}\left(\frac{1}{y}\right) =$ ....

(a) 1 (b) -1 (c) xy (d) $-xy$

6 $\lim\limits_{x \to 16} \dfrac{\sqrt[4]{x^7} - 128}{x - 16} =$ ....

(a) 12 (b) 14 (c) 16 (d) 20

7 $f(x) = \begin{cases} \dfrac{2x + \sin x}{\tan 2x + 4x} & , \quad x > 0 \\ 5x + \frac{1}{2} & , \quad x < 0 \end{cases}$, then $\lim\limits_{x \to 0} f(x) =$ ....

(a) $\frac{5}{3}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) otherwise.

8 XYZ is a triangle , in which $x = 20$ cm. , $z = 16$ cm. and $\cos Y = 0.4$, then $y =$ .... cm.

(a) 20 (b) 16 (c) 25 (d) 15

9 The curve of the function $r : r(x) = |x + 3|$ is the same as the curve of the function $f : f(x) = |x|$ with displacement 3 units in the direction of ....

(a) $\overrightarrow{OX}$ (b) $\overrightarrow{OX'}$ (c) $\overrightarrow{OY}$ (d) $\overrightarrow{OY'}$

---

10 If $f(x) = 3^x$, then $f^{-1}(x) =$ ....

(a) $\log x^3$ (b) $\log_3 x$ (c) x^{-3} (d) 3^x

---

11 The function $f : f(x) = \sqrt{x + 3}$, then its domain equals ....

(a) $]-3, \infty[$ (b) $]-\infty, 3[$ (c) $[3, \infty[$ (d) $[-3, \infty[$

---

12 $\lim_{x \to \frac{\pi}{4}} \frac{\tan x}{x} =$ ....

(a) 1 (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{4}{\pi}$

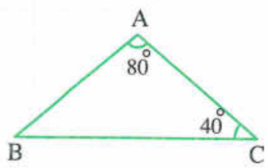
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13 $\lim_{n \to \infty} \frac{\sqrt[n]{3}}{9} =$ ....

(a) $\frac{1}{3}$ (b) ∞ (c) $\frac{1}{9}$ (d) zero

---

14 **In the opposite figure :**



If the perimeter of the triangle ABC equals 20 cm.

, then the length of the radius of the circle which passes through its vertices $\simeq$ .... cm.

(a) 2 (b) 4 (c) 6 (d) 8

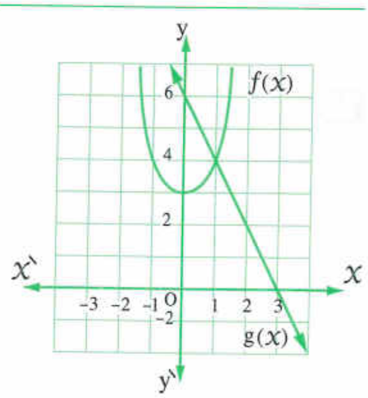
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15 The solution set of the equation $\log_x 3 = -2$ in $\mathbb{R}$ is ....

(a) $\{\frac{1}{9}\}$ (b) $\{9\}$ (c) $\{\sqrt{3}\}$ (d) $\{\frac{1}{\sqrt{3}}\}$

---

16 The opposite figure represents two curves $f(x)$ and $g(x)$, then $(g \circ f)(1) =$ .... cm.



(a) -2 (b) 2

(c) 4 (d) 5

17 $\lim\limits_{x \to 0} \frac{1 - \cos x}{5x} =$ ....

(a) 0 (b) 5 (c) $\frac{1}{5}$ (d) otherwise.

18 In the triangle XYZ , $y^2 + z^2 - x^2 = 2yz$ ....

(a) $\cos x$ (b) $\sin z$ (c) $\cos z$ (d) $\sin x$

19 The type of the function $f : f(x) = \frac{x - \tan x}{|x|}$ is ....

(a) odd function. (b) even function.

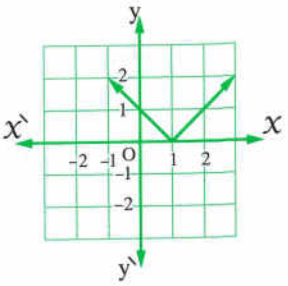
(c) neither even nor odd function. (d) one - to - one

20 If $|x| < a$, then $x \in$ ....

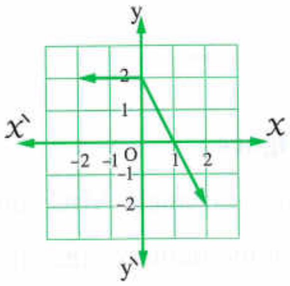
(a) $\mathbb{R} - [-a , a]$ (b) $[-a , a]$ (c) $\mathbb{R} -]-a , a[$ (d) $]-a , a[$

21 Which of the following functions is even function ?

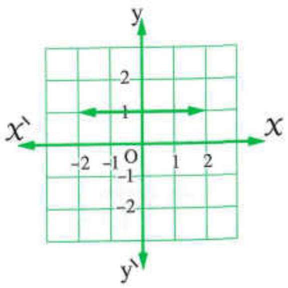
(a)



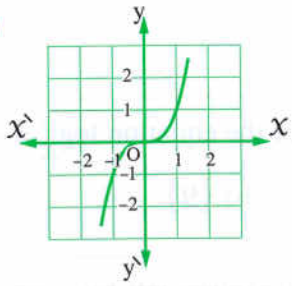
(b)



(c)



(d)



22 The solution set of the equation in $\mathbb{R} : 3^{x+1} + 3^{x} = 36$ is ....

(a) $\{0\}$ (b) $\{3\}$ (c) $\{2\}$ (d) $\{0 , 1\}$

23 The function $f : f(x) = \begin{cases} 3 & , \quad x > 0 \\ -3 & , \quad x < 0 \end{cases}$ is symmetric about the point ....

(a) (0 , 0) (b) (3 , 0) (c) (– 3 , 0) (d) (– 3 , 3)

24 The function $f : f(x) = \begin{cases} a - x^2 & , \quad x \leq -2 \\ x & , \quad x > -2 \end{cases}$ is continuous at $x = -2$, then a = ...

(a) – 2 (b) – 4 (c) 2 (d) 4

25 ABC is a triangle , then $\frac{5\,c}{\sin C}$ = ... r

(a) 5 (b) 10 (c) 15 (d) 20

26 The axis of symmetry of the function $f : f(x) = x^2 - 1$ is the straight line ....

(a) $x = 1$ (b) $x = 0$ (c) $y = 1$ (d) $y = 0$

27 ABC is a triangle , where cos (A + B) = ...

(a) cos A + cos B (b) sin A + sin B (c) cos C (d) – cos C

28 $\lim\limits_{x \to \infty} \frac{x^2 - 5x}{2x + 3x^2}$ = ...

(a) 3 (b) 6 (c) 9 (d) $\frac{1}{3}$

Model 2

Answer the following questions :

Choose the correct answer from the given ones :

1 The vertex of curve of the function $f : f(x) = (2 - x)^2 + 3$ is ....

(a) (2 , 3) (b) (2 , – 3) (c) (– 2 , 3) (d) (– 2 , – 3)

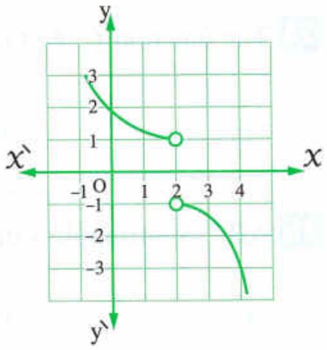
2 If ABC is a triangle in which a = 4 cm. , b = $4\sqrt{3}$ cm. , c = 8 cm. , then sine of its smallest angle = ...

(a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) 1 (d) zero

3 If $5^{x+1} = 7^{x+1}$, then 3^{x+1} = ...

(a) zero (b) 3 (c) 2 (d) 1

4 The opposite figure represents the curve of the function f, then $\lim\limits_{x \to 2} |f(x)| = \cdots\cdots$



(a) 1 (b) – 1
(c) 2 (d) does not exist.

---

5 The curve of the function g : g $(x) = x^2 + 4$ is the same as the curve $f(x) = x^2$ by translation 4 units in direction of $\cdots\cdots$

(a) $\overrightarrow{Ox}$ (b) $\overrightarrow{Ox`}$ (c) $\overrightarrow{Oy}$ (d) $\overrightarrow{Oy`}$

---

6 In Δ LMN : $\frac{\sin L}{3} = \frac{2 \sin M}{3} = \frac{\sin N}{4}$, then $\ell : m : n = \cdots\cdots$

(a) 6 : 8 : 3 (b) 3 : 6 : 8 (c) 8 : 3 : 6 (d) 6 : 3 : 8

---

7 If $y = \sqrt[5]{x}$, then its inverse function is y = $\cdots\cdots$

(a) $\frac{1}{5} x^5$ (b) x^5 (c) $x^5 - 1$ (d) $5 x^5$

---

8 $\lim\limits_{x \to 0} \frac{(x+2)^5 - 32}{x} = \cdots\cdots$

(a) 25 (b) 64 (c) 80 (d) 100

---

9 If $f(x) = x - 1$, g $(x) = \sqrt{x}$, then the domain of (g ∘ f) is $\cdots\cdots$

(a) $\mathbb{R}$ (b) $]-\infty , 1[$ (c) $[1 , \infty[$ (d) $\mathbb{R} - \{1\}$

---

10 In Δ LMN, m ($\angle$ L) = 30°, MN = 7 cm., then the diameter length of the circle passes through its vertices equals $\cdots\cdots$

(a) 7 cm. (b) 3.5 cm. (c) 14 cm. (d) $\frac{14}{\sqrt{2}}$ cm.

---

11 Solution set in $\mathbb{R}$ of the equation : $\log_x (x + 6) = 2$ is $\cdots\cdots$

(a) $\{3 , -2\}$ (b) $\{3\}$ (c) $\{3 , 1\}$ (d) $\{6 , 1\}$

---

12 From the following functions, the even function is $f : f(x) = \cdots\cdots$

(a) $\sin x$ (b) sin 30° (c) $x \cos x$ (d) $x^2 + \tan x$

---

13 $\lim\limits_{x \to 3} \frac{x^2 - 7x + 12}{x - 3} = \cdots\cdots$

(a) 1 (b) – 1 (c) – 2 (d) 7

14 If the curve of the function $f : f(X) = \log_4 (1 - a X)$ passes through the point $\left(\frac{1}{8}, \frac{-1}{2}\right)$, then a = ...

(a) 3 (b) 2 (c) 4 (d) 8

15 $\lim\limits_{X \to \infty} \frac{(2X+1)(4X-1)^2}{(2X+3)^3} =$ ....

(a) 4 (b) 32 (c) 1 (d) 8

16 The solution set of the inequality : $|X - 1| < -2$ in $\mathbb{R}$ is ....

(a) $]-1, 3[$ (b) $\mathbb{R} - [-1, 3]$ (c) $]-2, 2[$ (d) $\varnothing$

17 The domain of the function $f : f(X) = \frac{\sqrt{X-2}}{X-3}$ is ....

(a) $\mathbb{R}$ (b) $\{3\}$ (c) $[2, \infty[$ (d) $[2, \infty[- \{3\}$

18 In Δ ABC : $\frac{b^2 + c^2 - a^2}{2bc} =$ ....

(a) cos A (b) cos B (c) cos C (d) sin A

19 The solution set of the equation : $\log_3 X \times \log_2 3 = 5$ in $\mathbb{R}$ is ....

(a) $\{32\}$ (b) $\{5\}$ (c) $\{3\}$ (d) $\{2\}$

20 $\lim\limits_{X \to 0} 5X \csc 2X =$ ....

(a) $\frac{5}{2}$ (b) 10 (c) $\frac{2}{5}$ (d) zero

21 The exponential function whose base a is increasing if ....

(a) $a > 0$ (b) $a > 1$ (c) $0 < a < 1$ (d) $a = 1$

22 If $X^{\frac{3}{2}} = 8$, then $X =$ ....

(a) 2 (b) 4 (c) 8 (d) 9

23 If the function $f : f(X) = \begin{cases} \sin 9X \cot 16X & , X \neq 0 \\ k^2 & , X = 0 \end{cases}$ is continuous at $X = 0$, then k = ...

(a) $\frac{9}{16}$ (b) $\frac{3}{4}$ (c) $\pm\frac{3}{4}$ (d) $\pm\frac{4}{3}$

24 All the following relations represent function y in terms of X except ....

(a) $y = 3X + 1$ (b) $y = X^2 - 4$ (c) $X = y^2 - 2$ (d) $y = \sin X$

25 If $\frac{a+b}{13} = \frac{b+c}{11} = \frac{c+a}{12}$, then cos A = ...

(a) $\frac{1}{5}$ (b) $\frac{5}{7}$ (c) $\frac{19}{35}$ (d) $\frac{4}{11}$

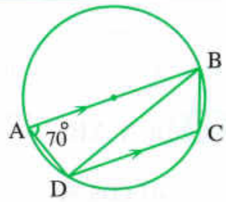
26 $\lim\limits_{x \to 1} : \frac{x^{6\frac{1}{2}} - x^{\frac{1}{2}}}{x^{3\frac{1}{2}} - x^{\frac{1}{2}}} =$ ....

(a) $\frac{13}{7}$ (b) 1 (c) 2 (d) x

27 If $f(x) = \begin{cases} 3x^2 + ax - 2 & , x > 3 \\ 2x + b & , x < 3 \end{cases}$ and $\lim\limits_{x \to 3} f(x) = 16$, then a + b = ...

(a) 4 (b) 10 (c) – 13 (d) 7

28 **In the opposite figure :**



If BC = 10 cm. , then perimeter of Δ BDC ≃ .... cm.

(a) 60 (b) 62

(c) 64 (d) 67

Model 3

Answer the following questions :

Choose the correct answer from the given ones :

1 If $f(x) = x^3$, then its image by reflection in x-axis and translation 3 units in direction by $\overrightarrow{OX}$ is ....

(a) $g(x) = -(x-3)^3$ (b) $g(x) = (x-3)^3$

(c) $g(x) = -(x+3)^3$ (d) $g(x) = (x+3)^3$

2 $\lim\limits_{x \to \infty} \frac{7x^{-2} + 5x^{-1} - 1}{4x^{-2} + 3x^{-1} + 2} =$ ....

(a) $\frac{7}{4}$ (b) $\frac{5}{3}$ (c) $-\frac{1}{2}$ (d) ∞

3 In Δ HED , cos (D + H) = ...

(a) cos D + cos H (b) cos E (c) – sin (90° + E) (d) cos (90° + E)

4 Image of point (– 4 , 2) by reflection in straight line y = x is ....

(a) (4 , 2) (b) (2 , – 4) (c) (4 , – 2) (d) (– 4 , – 2)

5 $\lim\limits_{x \to 4} \dfrac{x^2 + 16k}{x - 4}$ exists when K = ...

(a) -2 (b) -1 (c) 1 (d) 2

6 If $5 \times 3^{x-2} = 45$, then 7^{2-x} = ...

(a) $\frac{1}{49}$ (b) $\frac{1}{7}$ (c) 7 (d) 49

7 $\lim\limits_{x \to 0} \left(\cos 3x - \dfrac{\tan 4x}{x}\right)$ = ...

(a) -4 (b) -3 (c) -1 (d) 4

8 Range of the function : $f(x) = x^2 - 6x$ is ....

(a) $[6, \infty[$ (b) $[3, \infty[$ (c) $[0, \infty[$ (d) $[-9, \infty[$

9 If $x > 1$, $y > 1$ and $\log_{xy} \frac{1}{x} + \log_{xy} \frac{1}{y}$ = ...

(a) -1 (b) $\frac{x}{y}$ (c) $\frac{y}{x}$ (d) 1

10 If $f : \mathbb{N} \longrightarrow \mathbb{R}$, $f(x) = x^2$, then function is ....

(a) even. (b) odd.
(c) decrease on its domain. (d) one-to-one.

11 $\lim\limits_{x \to 3} \dfrac{|x - 6| - 3}{x^2 - 9}$ = ...

(a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{-1}{6}$ (d) no limit.

12 Δ XYZ in which m ($\angle$ X) = 75° , m ($\angle$ Y) = 45° , then z : y = ...

(a) $\sqrt{6} : 3$ (b) $\sqrt{6} : 2$ (c) $\left(1 + \sqrt{3}\right) : 2$ (d) $\left(\sqrt{6} + 3\sqrt{2}\right) : 6$

13 Range of the function $f : f(x) = \dfrac{|x - 2|}{x - 2}$ is ....

(a) $]2, \infty[$ (b) $]-\infty, 2[$ (c) $\mathbb{R} - \{2\}$ (d) $\{-1, 1\}$

14 $f(x) = \begin{cases} 3x - 2 & , \quad x > -2 \\ kx - 6 & , \quad x < -2 \end{cases}$, $\lim\limits_{x \to -2} f(x)$ exists , then k = ...

(a) -2 (b) -1 (c) 1 (d) 2

15 S.S. of the inequality : $|2x - 1| \leq 5$ is ....

(a) $[-5, 5]$ (b) $[-4, 6]$ (c) $[-2, 3]$ (d) $\mathbb{R} -]-2, 3[$

16 $\lim\limits_{x \to 0} \frac{2x^2 + \tan 3x}{5x + \sin 7x} = \ldots\ldots$

(a) $\frac{5}{12}$ (b) $\frac{1}{4}$ (c) $\frac{7}{12}$ (d) $\frac{3}{7}$

17 Δ ABC in which a = 23 cm. , b = 15 cm. and its perimeter = 70 cm. , then measure of the biggest angle in the triangle equals ……

(a) $77° 43'$ (b) $113° 2'$ (c) $131° 2'$ (d) $150°$

18 If f is increasing function on the interval $]1, \infty[$, then $g(x) = f(x + 2)$ is increasing on ……

(a) $]-1, \infty[$ (b) $]1, \infty[$ (c) $]2, \infty[$ (d) $]3, \infty[$

19 If $f(x) = 3^{x+1}$, then $\frac{f(x+2)}{f(x-2)} + \frac{f(2x+1)}{f(2x-1)} = \ldots\ldots$

(a) 27 (b) 81 (c) 90 (d) 243

20 $\lim\limits_{x \to \infty} \frac{2x^2 + 3x - 1}{4x - 5x^2 + 2} = \ldots\ldots$

(a) $\frac{-3}{5}$ (b) $-\frac{2}{5}$ (c) $\frac{1}{2}$ (d) $\frac{3}{4}$

21 Δ LMN in which $m(\angle M) = 60°$, $\ell = 20$ cm. has two solutions when m = …… cm.

(a) 21 (b) 15 (c) $10\sqrt{3}$ (d) 18

22 If $f(x) = \log_c (7x + 1)$ and $f^{-1}(3) = 1$, then c = ……

(a) $\frac{1}{3}$ (b) $\frac{1}{2}$ (c) 1 (d) 2

23 $f(x) = \frac{6 \cos x}{2x - \pi}$ to be continuous at $x = \frac{\pi}{2}$, then $f\left(\frac{\pi}{2}\right) = \ldots\ldots$

(a) -3 (b) 3 (c) $\frac{\pi}{2}$ (d) π

24 $\lim\limits_{x \to 1} \frac{\sqrt[5]{x} + \sqrt[3]{x} - 2}{x^2 - 1} = \ldots\ldots$

(a) $\frac{4}{15}$ (b) $\frac{8}{15}$ (c) $\frac{6}{15}$ (d) no limit.

25 If radius length of the circumcircle of Δ XYZ equals 4 cm. , then $\frac{y\,z}{\sin Y \sin Z} = \ldots\ldots$

(a) 8 (b) 16 (c) 32 (d) 64

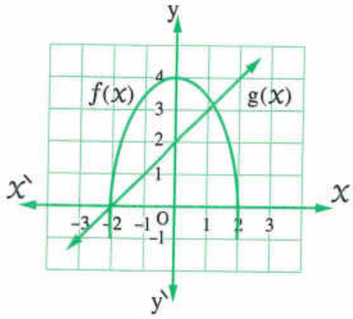
26 If $4^x = 3$, $8^y = 9$, then $\frac{x + y}{x - y} = \ldots\ldots$

(a) -7 (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) 7

27 If A (4 , 3) , B (– 2 , 5) , C (– 1 , – 2) , then cos (∠ ABC) = ...

(a) $\frac{\sqrt{5}}{5}$ (b) $\frac{3}{5}$ (c) $\frac{4}{5}$ (d) $-\frac{4}{5}$

28 **In the opposite figure :**



$(f \circ g)(0)$ ....

(a) – 12 (b) zero

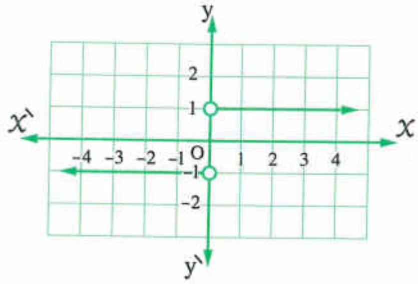
(c) 2 (d) 4

Model 4

Answer the following questions :

Choose the correct answer from the given ones :

1 The range of the given function in the opposite figure is ....



(a) {1} (b) {1 , – 1}

(c) {– 1} (d) ℝ

2 The type of the function $f : f(x) = \frac{\sin x}{x}$ is ....

(a) even (b) odd.

(c) neither odd nor even. (d) both odd and even.

3 If log 3 = x , log 5 = y , then log 15 = ...

(a) x y (b) $\frac{x}{y}$ (c) x + y (d) x – y

4 If $3^{x-2} = 2^{x-2}$, then x = ...

(a) 3 (b) – 2 (c) zero (d) 2

5 In Δ ABC : $b^2 + c^2 - a^2 = 2\,b\,c \times$ ....

(a) sin (90° – B) (b) sin (90° – A) (c) cos B (d) cos (90° – B)

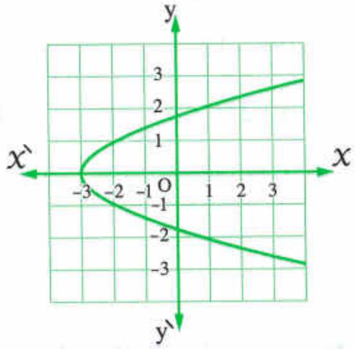
6 The function f where $f(x) = a^x$ is decreasing on its domain if ....

(a) a = 1 (b) a > 1 (c) 0 < a < 1 (d) a = – 1

7 In Δ ABC : $\frac{a}{\sin A} = 6$ cm. , then the radius length of its circumcircle equals .... cm.

(a) 2 (b) 3 (c) 5 (d) 6

8 The curve represented in the opposite figure is symmetric about the straight line whose equation is ....



(a) $x = 0$ (b) $y = 0$

(c) $y = -2$ (d) $x = 2$

9 If $f : f(x) = \begin{cases} \frac{x^2 - 1}{x - 1} & , x \neq 1 \\ 2a & , x = 1 \end{cases}$ is continuous at $x = 1$, then a = ...

(a) zero (b) – 2 (c) 2 (d) 1

10 $\lim_{x \to 0} \frac{x^2 - 1}{x} =$ ....

(a) zero (b) 1 (c) does not exist. (d) – 1

11 The area of the circle passes through the vertices of an equilateral triangle ABC whose side length 9 cm. equals .... cm^2

(a) 9π (b) $9\sqrt{3}\pi$ (c) 27π (d) 81π

12 If $\lim_{x \to 1} \frac{x^2 - k^2}{x + 2} = -1$, then k = ...

(a) 2 (b) – 2 (c) 4 (d) ± 2

13 $\lim_{x \to \infty} \frac{2x + 3}{5x^2 + 4} =$ ....

(a) 2 (b) zero (c) $\frac{3}{4}$ (d) $\frac{2}{5}$

14 $\lim_{x \to 2} \frac{x^5 - 32}{x^2 + 3x - 10} =$ ....

(a) 87 (b) $\frac{80}{7}$ (c) 78 (d) $\frac{7}{80}$

15 $\lim_{x \to 0} \frac{\sin x}{4x} =$ ....

(a) $\frac{\pi}{4}$ (b) 1 (c) $\frac{1}{4}$ (d) does not exist.

16 If $y = \sqrt[5]{x}$, then its inverse function is y = ...

(a) $\frac{1}{5} x^5$ (b) x^5 (c) $x^5 - 1$ (d) $5 x^5$

17 The exponential function whose base a is increasing if ....

(a) $a > 0$ (b) $a > 1$ (c) $0 < a < 1$ (d) $a = 1$

18 If $f(x) = \begin{cases} x - 4 & , \quad x \geq 4 \\ g(x) & , \quad x < 4 \end{cases}$ is symmetric about the straight line $x = 4$

, then the function g is ....

(a) an increasing function. (b) a decreasing function.

(c) an even function. (d) a constant function.

19 In Δ ABC , 2 sin A = 3 sin B = 4 sin c , then a : b : c = ...

(a) 2 : 3 : 4 (b) 4 : 3 : 2 (c) 3 : 4 : 6 (d) 6 : 4 : 3

20 If f is an odd function , $a \in$ the domain of f , then $f(a) + f(-a) =$ ....

(a) $2 f(a)$ (b) $2 f(-a)$ (c) zero (d) $f(a)$

21 If the ratio among the measures of angles in a triangle is 8 : 3 : 1 , then the ratio between the longest two sides in the triangle equals ....

(a) $\sqrt{3} : 2$ (b) $\sqrt{6} : 2$ (c) 8 : 3 (d) 8 : 5

22 If $f(x) = 4x - 5$, $g(x) = 3^x$, then $(f \circ g)(2) =$ ....

(a) 3 (b) 9 (c) 27 (d) 31

23 All the following relations represent function y in terms of x except ....

(a) $y = 3x + 1$ (b) $y = x^2 - 4$ (c) $x = y^2 - 2$ (d) $y = \sin x$

24 In Δ DEF , if m (∠ D) = 30° , $e = 15\sqrt{3}$ cm. , m (∠ E) = 60° , then d = ... cm.

(a) 30 (b) 45 (c) 15 (d) 60

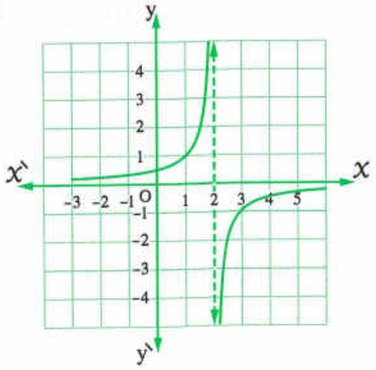
25 If f^{-1} is the inverse function of the function f , then ....

(a) the domain of f^{-1} = the domain of f (b) the domain of f^{-1} = the range of f

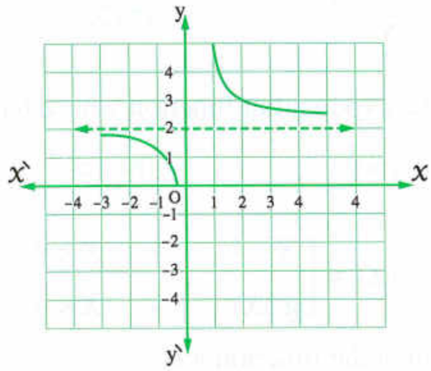
(c) the range of f^{-1} = the range of f (d) the range of f^{-1} = the domain of f^{-1}

26 If $f(x) = \frac{1}{x-2}$, then the graph that represents the function f is ....

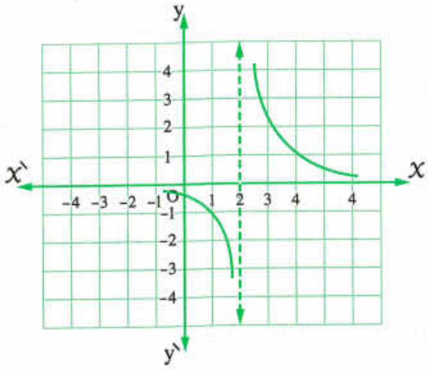
(a)



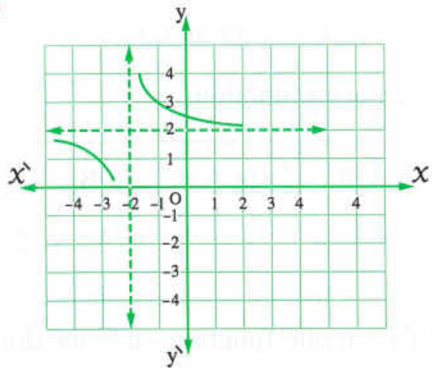
(b)



(c)



(d)



27 $\lim_{x \to \frac{\pi}{2}} (2x - \cos x) =$ ....

(a) zero (b) 2 (c) π (d) $\frac{\pi}{2}$

28 The even continuous function at the point (a , b) is also continuous at the point ....

(a) (a , b) (b) (– a , b) (c) (a , – b) (d) (– a , – b)

Model 5

Answer the following questions :

Choose the correct answer from the given ones :

1 The domain of the function $f : f(x) = \frac{\sqrt{x-2}}{x-3}$ is ....

(a) $\mathbb{R}$ (b) $\{3\}$ (c) $[2, \infty]$ (d) $[2, \infty[- \{3\}$

2 The diameter length of the circumcircle of the equilateral triangle ABC whose side length $5\sqrt{3}$ cm. is ....

(a) $5\sqrt{3}$ (b) $10\sqrt{3}$ (c) 10 (d) 5

3 If $\lim\limits_{x \to \infty} \frac{a x + 6}{2 x + 7} = 4$, $a \in \mathbb{R}$, then a = ...

(a) 2 (b) 4 (c) 6 (d) 8

4 The curve $f(x) = x^2 + 4$ is the same curve of $g(x) = x^2$ by translation 4 units in direction of ....

(a) $\overrightarrow{Ox}$ (b) $\overrightarrow{Ox'}$ (c) $\overrightarrow{Oy}$ (d) $\overrightarrow{Oy'}$

5 Measure of greatest angle in triangle whose side lengths are 3 cm. , 5 cm. , 7 cm. is ....

(a) 150° (b) 120° (c) 60° (d) 30°

6 If $f(x) = 4x - 5$, $g(x) = 3^x$, then $(f \circ g)(2) =$ ....

(a) 3 (b) 9 (c) 27 (d) 31

7 If $f(x) = 7x$, then $f^{-1}(x) =$ ....

(a) $7x$ (b) $\frac{x}{7}$ (c) $\frac{7}{x}$ (d) $7 - x$

8 **In the opposite figure :**

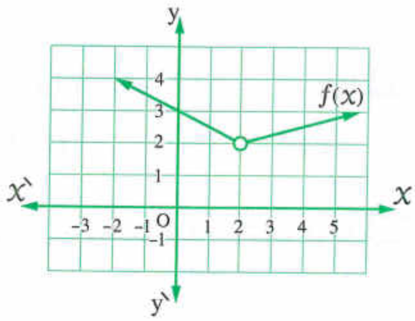
$\lim\limits_{x \to 2} f(x) =$ ....

(a) 1

(b) – 1

(c) dose not exist.

(d) 2



9 $\lim\limits_{x \to 3} \frac{x^2 - 7x + 12}{x - 3} =$ ....

(a) 1 (b) – 1 (c) 7 (d) – 2

10 Solution set of inequality $\sqrt{x^2 - 4x + 4} > 0$ in $\mathbb{R}$ is ....

(a) $\mathbb{R} - \{2\}$ (b) $\mathbb{R} - \{-2\}$ (c) $\mathbb{R}$ (d) $\varnothing$

11 Number of possible solutions of Δ ABC where m (∠ A) = 60° , b = 3 cm. , a = 5 cm. is ....

(a) 1 (b) 2

(c) no solution. (d) infinite number of triangles.

12 $\lim\limits_{x \to 0} \frac{\sqrt{x+1} - 1}{x} =$ ....

(a) zero (b) $\sqrt{2}$ (c) $\frac{1}{2}$ (d) does not exist.

13 The curve of the even function is symmetric about the straight line ....

(a) $y = x$ (b) $\overleftrightarrow{yy`}$ (c) $\overleftrightarrow{xx`}$ (d) $y = -x$

---

14 If $x^3 = 8$, then $x =$ ....

(a) 2 (b) 4 (c) 8 (d) 9

---

15 In Δ DEH , if m ($\angle$ D) = 30° , e = $15\sqrt{3}$ cm. , m ($\angle$ E) = 60° , then d = ... cm.

(a) 30 (b) 45 (c) 15 (d) 60

---

16 $\lim\limits_{x \to 0} \frac{(x+2)^5 - 32}{x} =$ ....

(a) 25 (b) 64 (c) 80 (d) 100

---

17 If $\log_3 y = x$, then the equivalent exponential form is ....

(a) $y = x^3$ (b) $x = y^3$ (c) $x = 3^y$ (d) $y = 3^x$

---

18 If $f(x) = 4^{x-1}$, then $f(x+1) =$ ....

(a) 4^x (b) 4^{x+1} (c) 4^{x+2} (d) 2^x

---

19 In Δ ABC , b = 2 cm. , c = 2.5 cm. , cos A = $\frac{2}{5}$, then Δ ABC will be .... triangle.

(a) right-angled (b) an isosceles (c) equilateral (d) scalen

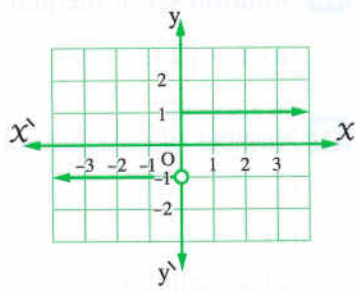
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20 If $\log_x (x + 6) = 2$, then $x =$ ....

(a) $\{3 , -2\}$ (b) $\{3\}$ (c) $\{3 , 1\}$ (d) $\{6 , 1\}$

---

21 Range of the function which represented in opposite figure is ....



(a) $\{1\}$ (b) $\{1 , -1\}$

(c) $\{-1\}$ (d) $\mathbb{R}$

---

22 In Δ XYZ , $\frac{2X}{\sin X} =$ .... r "where r is the radius length of the circumcircle of Δ XYZ"

(a) 1 (b) 2 (c) 4 (d) 8

23 If $f(x) = \begin{cases} \frac{\sin^2 2x}{x^2} & , \quad x < 0 \\ 2a + 3\cos x & , \quad x > 0 \end{cases}$ and $\lim_{x \to 0} f(x)$ is exist, then a = ...

(a) $\frac{1}{2}$ (b) zero (c) 2 (d) $-\frac{1}{2}$

---

24 $\lim_{x \to 0} 5x \csc 2x =$ ....

(a) $\frac{5}{2}$ (b) 10 (c) $\frac{2}{5}$ (d) zero

---

25 If $\log x + \log 5 = 2$, then $x =$ ....

(a) 3 (b) 8 (c) 17 (d) 20

---

26 If $\lim_{x \to 1} \frac{x^2 + kx + m}{2x - 1} = 3$, then k + m = ...

(a) – 4 (b) – 5 (c) – 8 (d) 2

---

27 The function which is one - to - one from the functions defined by the following rules is ....

(a) $f_1(x) = \cos x$ (b) $f_2(x) = x^2$

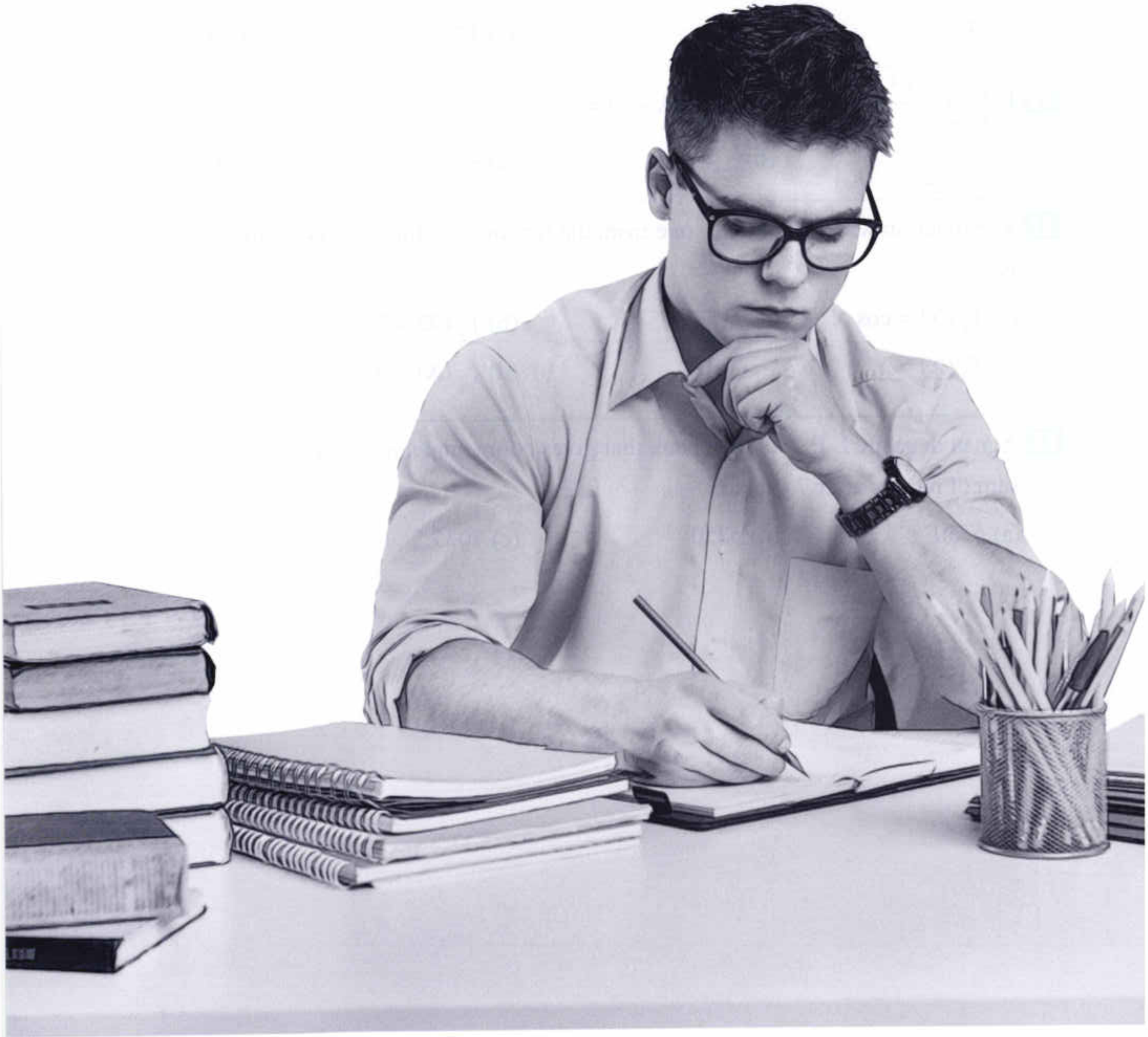
(c) $f_3(x) = x^3$ (d) $f_4(x) = x^4 + x^2$

---

28 A man deposite L.E 5000 in a bank that gives compound annually interest 5 % , then the sum of money after 7 years is .... L.E.

(a) 6750 (b) 5350 (c) 7035.5 (d) 8500

Guide answers



Guide answers of accumulative quizzes in algebra

Accumulative quiz 1

1 (1) c (2) d (3) c (4) a

2
(1) increasing on $]0 , 2[$, constant on $]2 , \infty[$
(2) increasing on $]-1 , 0[$, decreasing on $]0 , 4[$
(3) decreasing on $]-\infty , 1[$, increasing on $]1 , \infty[$

Accumulative quiz 2

1 (1) c (2) c (3) c
(4) b (5) d (6) c

2
(1) $(f \circ g)(X) = \frac{1}{X+3}$, domain of $(f \circ g) = \mathbb{R} - \{-3\}$
(2) $(g \circ f)(X) = \frac{1}{X} + 3$, domain of $(g \circ f) = \mathbb{R} - \{0\}$

Accumulative quiz 3

1 (1) b (2) c (3) d
(4) b (5) c (6) a

2
$(f_1 + f_2)(X) = X^5 + \sin X$ and it is odd.

Accumulative quiz 4

1
Represent by yourself , the range $= [0 , \infty[$
, the function is neither odd nor even
, decreasing on $]-\infty , 0[$, increasing on $]0 , \infty[$

2
The domain $= \mathbb{R} - \{2\}$, prove by yourself.

3
Represent by yourself , the domain $= \mathbb{R} - \{-1\}$
, the range $= \mathbb{R} - \{-2\}$, increasing on $\mathbb{R} - \{-1\}$

4
Represent by yourself , the range $=]1 , 3] \cup \{-1\}$
increasing on $]2 , 4[$, constant on $]-2 , 2[$

Accumulative quiz 5

1 (1) c (2) a (3) c
(4) a (5) c (6) b

2
Represent by yourself , the range $= [0 , \infty[$
, the function is even , decreasing on $]-\infty , -2[$
, $]0 , 2[$, increasing on $]-2 , 0[,]2 , \infty[$

Accumulative quiz 6

1 (1) b (2) c (3) c
(4) c (5) d (6) d

2
(1) The S.S. $= \{4\}$
(2) The S.S. $= [\frac{5}{4} , \frac{7}{4}] - \{\frac{3}{2}\}$

Accumulative quiz 7

1 (1) b (2) a (3) b
(4) d (5) a (6) a

2
(1) The S.S. $= \{1 , -1 , 27 , -27\}$
(2) The S.S. $= \{6\}$

Accumulative quiz 8

1 (1) d (2) c (3) c
(4) b (5) d (6) b

2
$f(t) = 60(1 + 0.25)^t$, approximately 229 bees

Accumulative quiz 9

1 (1) a (2) a (3) b
(4) d (5) b (6) c

2
(1) The domain $= \mathbb{R} - \{-1\}$, the range $= \mathbb{R} - \{2\}$
(2) $f^{-1}(X) = \frac{-X+3}{X-2}$, the domain of $f^{-1} = \mathbb{R} - \{2\}$
and the range of $f^{-1} = \mathbb{R} - \{-1\}$

Accumulative quiz 10

1 (1) c (2) c (3) c
(4) c (5) b (6) a

2
(1) The S.S. $= \{8\}$
(2) The S.S. $= \{-1 , 5\}$

Accumulative quiz 11

1 (1) c (2) a (3) c
(4) b (5) a (6) d

2 Prove by yourself.

Guide answers of accumulative quizzes in calculus

Accumulative quiz 1

1 (1) 1 (2) 4 (3) undefined (4) 3

2 (1) d (2) d (3) c

Accumulative quiz 2

1 (1) a (2) d (3) c (4) c

2 (1) $\frac{1}{4}$ (2) $\frac{7}{2}$ (3) 33 (4) 1

Accumulative quiz 3

1 (1) a (2) d (3) b
(4) a (5) a (6) d

2 (1) 64 (2) 5

Accumulative quiz 4

1 (1) d (2) b (3) b (4) d

2 (1) 2 (2) $\frac{1}{4}$ (3) ∞ (4) 7

Accumulative quiz 5

1 (1) c (2) c (3) c
(4) c (5) c (6) b

2 (1) 2 (2) $\frac{1}{4}$

Accumulative quiz 6

1 (1) c (2) b (3) a
(4) d (5) a (6) c

2 $\lim_{x \to 3} f(x) = -1$

Accumulative quiz 7

1 (1) d (2) a (3) c
(4) c (5) c (6) a

2 The function f is continuous at $x = 1$

Guide answers of accumulative quizzes in trigonometry

Accumulative quiz 1

1
(1) d (2) c (3) c (4) b

2 b = 18.04 cm. , r = 16.56 cm.

Accumulative quiz 2

1
(1) c (2) c (3) c
(4) c (5) d (6) d

2
The area of the circle that passes through the vertices of Δ ADC = 25 π cm^2.

Accumulative quiz 3

1
(1) c (2) a (3) c
(4) a (5) c (6) b

2
There are two solutions , c = 37.3 cm. or 8.7 cm.

Answers of school book examinations in algebra

Model 1

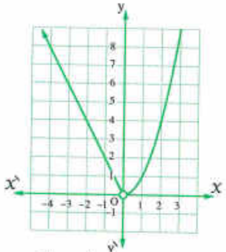
1

(1) (d) (2) (c) (3) (a) (4) (a)

2

(a) (1) $]-\infty , 1[$ (2) $\mathbb{R} - \{-1 , 1\}$

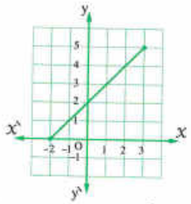
(b)



The range $=]0 , \infty[$

3

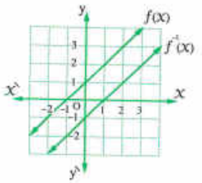
(a) $(f_1 + f_2)(x) = x + 2$, its domain $= [-2 , 3]$



The function is increasing on $]-2 , 3[$

(b) $\because y = x + 1$ $\therefore x = y + 1$

$\therefore y = x - 1$ $\therefore f^{-1}(x) = x - 1$



4

(a) (1) $\because \log_4 x = \log_4 4 - \log_4 (x-3)$

$\therefore \log_4 x = \log_4 \frac{4}{x-3}$ $\therefore x = \frac{4}{x-3}$

$\therefore x^2 - 3x - 4 = 0$

$\therefore (x+1)(x-4) = 0$

$\therefore x = -1$ (refused) or $x = 4$

$\therefore$ The S.S. $= \{4\}$

(2) Squaring both sides

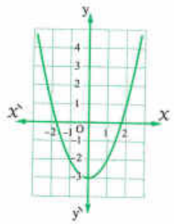
$\therefore x^2 + 4x + 4 = x^2 - 6x + 9$

$\therefore 10x = 5$

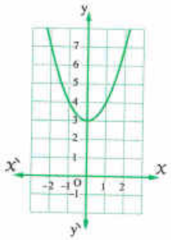
$\therefore x = \frac{1}{2}$ (satisfies)

$\therefore$ The S.S. $= \left\{\frac{1}{2}\right\}$

(b) (1)



(2)



5

(a) $3x - 2 \geq 7$, then $x \geq 3$

or $3x - 2 \leq -7$, then $x \leq \frac{-5}{3}$

$\therefore$ The S.S. $= \mathbb{R} -]\frac{-5}{3} , 3[$

(b) $\therefore (x^{\frac{2}{3}} - 1)(x^{\frac{2}{3}} - 9) = 0$

$\therefore x^{\frac{2}{3}} = 1$

$\therefore x = \pm 1$ or $x^{\frac{2}{3}} = 9$

$\therefore x = \pm (3^2)^{\frac{3}{2}} = \pm 27$

$\therefore$ The S.S. $= \{1 , -1 , 27 , -27\}$

Model 2

1

(1) (d) (2) (b) (3) (d) (4) (b)

2

(a) The expression $= \frac{1}{a^x + 1} + \frac{1}{a^{-x} + 1} \times \frac{a^x}{a^x}$

$= \frac{1}{a^x + 1} + \frac{a^x}{1 + a^x} = \frac{1 + a^x}{1 + a^x} = 1$

$\therefore$ The value of the expression is constant (= 1) whatever the value of x

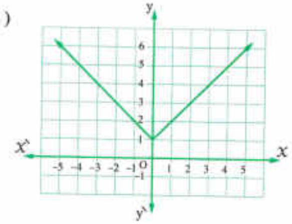
(b) The function is defined for every x

satisfying $\begin{cases} x > 0 \\ x - 1 > 0 \\ x - 1 \neq 1 \end{cases}$ *i.e.* $\begin{cases} x > 0 \\ x > 1 \\ x \neq 2 \end{cases}$

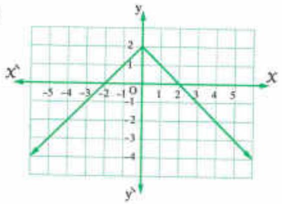
i.e. The domain $=]1 , \infty[- \{2\}$

3

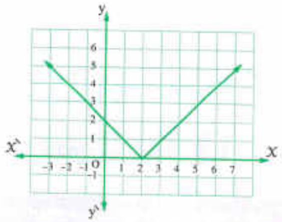
(a) (1)



(2)



(b) (1) $f(x) = \sqrt{x^2 - 4x + 4} = \sqrt{(x-2)^2} = |x - 2|$

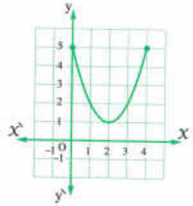


* The domain $= \mathbb{R}$

* The function is decreasing on $]-\infty , 2[$ and increasing on $]2 , \infty[$

(2) $f(x) = |x^2 - 4x + 5| = |(x-2)^2 + 1|$

$= (x-2)^2 + 1$



* The domain $= [0 , 4]$

* The function is decreasing on $]0 , 2[$ and increasing on $]2 , 4[$

4

(a) (1) $\because f_1(-x) = -x \cos(-x) = -x \cos x$

$= -f_1(x)$

$\therefore f_1$ is odd.

(2) $\because f_2(-x) = \begin{cases} (-x)^2 & , -x \geq 0 \\ |-x| & , -x < 0 \end{cases}$

$= \begin{cases} x^2 & , x \leq 0 \\ |x| & , x > 0 \end{cases}$

$\neq f_2(x) \neq -f_2(x)$

$\therefore f_2$ is neither even nor odd.

(3) $\because f_3(-x) = (-x)^2 |-x| - 1 = x^2 |x| - 1$

$= f_3(x)$

$\therefore f_3$ is even.

(b) (1) $|x| + x = 0$

When $x \geq 0$: $\therefore x + x = 0$

$\therefore 2x = 0$ $\therefore x = 0 \in [0 , \infty[$

When $x < 0$: $\therefore -x + x = 0$

$\therefore 0 = 0$ and this is satisfying for any value of $x < 0$ $\therefore$ The S.S. $=]-\infty , 0]$

(2) $\because |2x - 3| - 2|2x - 3| > 0$

$\therefore -|2x - 3| > 0$ $\therefore |2x - 3| < 0$

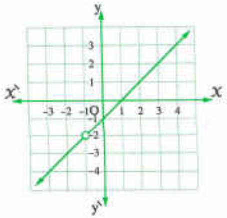
and this is impossible $\therefore$ The S.S. $= \varnothing$

5

(a) $\because$ The domain of $f = \mathbb{R}$, the domain of $g = \mathbb{R}$

$\therefore$ The domain of $\left(\frac{f}{g}\right) = \mathbb{R} - \{-1\}$

, $\because \left(\frac{f}{g}\right)(X) = \frac{X^2 - 1}{X + 1} = \frac{(X+1)(X-1)}{X+1} = X - 1$



* The range of $\left(\frac{f}{g}\right) = \mathbb{R} - \{-2\}$

* $\frac{f}{g}$ is increasing on each of $]-\infty, -1[$ and $]-1, \infty[$

(b) $\log 25 + \frac{\log 8 \times \log 16}{\log 64}$

$= \log 25 + \frac{3 \log 2 \times 4 \log 2}{6 \log 2}$

$= \log 25 + 2 \log 2 = \log 25 + \log 4$

$= \log (25 \times 4) = \log 100 = 2$

Answers of school book examinations in calculus and trigonometry

Model 1

1

(1) (c) (2) (b) (3) (d) (4) (c)

2

(a) (1) $\lim_{X \to 2} \frac{X^5 - 2^5}{(X+5)(X-2)}$

$= \lim_{X \to 2} \frac{1}{X+5} \times \lim_{X \to 2} \frac{X^5 - 2^5}{X - 2}$

$= \frac{1}{7} \times 5 \times 2^4 = \frac{80}{7}$

(2) $\lim_{X \to 0} \frac{\sin 2X}{X} + \lim_{X \to 0} \frac{5 \sin 3X}{X}$

$= 2 + 5 \times 3 = 17$

(b) $\because \frac{21}{\sin A} = \frac{25}{\sin B} = \frac{c}{\sin C} = 28$

$\therefore m(\angle A) \simeq 48° 35'$, $m(\angle B) \simeq 63° 14'$

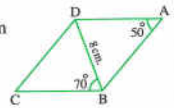
$\therefore m(\angle C) \simeq 68° 11'$

$\therefore c = 28 \sin 68° 11' \simeq 26$ cm.

3

(a) (1) 0 **(2)** 1 **(3)** 0

(b) $\because$ ABCD is a parallelogram



$\therefore m(\angle C) = 50°$

In Δ BDC :

$m(\angle BDC) = 180° - (50° + 70°) = 60°$

$\therefore \frac{8}{\sin 50°} = \frac{BC}{\sin 60°} = \frac{DC}{\sin 70°}$

$\therefore BC \simeq 9$ cm. , $DC \simeq 9.8$ cm.

$\therefore$ The perimeter of the parallelogram

$= 2(BC + CD) \simeq 38$ cm.

4

(a) $\because \frac{5}{\sin 40°} = \frac{7}{\sin B}$ $\therefore \sin B = \frac{7 \sin 40°}{5}$

$\therefore m(\angle B) \simeq 64° 9'$ or $m(\angle B) \simeq 115° 51'$

(b) $\because f(2) = 2^2 - 1 = 3$

, $f(2^-) = \lim_{X \to 2^-} (X - 2a) = 2 - 2a$

, $\because f$ is continuous at $X = 2$

$\therefore 3 = 2 - 2a$ $\therefore a = -\frac{1}{2}$

5

(a) $\because f(0^-) = \lim_{X \to 0^-} \frac{5X + 6}{X + 3} = 2$

, $f(0^+) = \lim_{X \to 0^+} \frac{\tan 2X}{\sin X}$

$= \lim_{X \to 0^+} \left(\frac{\tan 2X}{X} \times \frac{X}{\sin X}\right) = 2$

$\therefore f(0^-) = f(0^+) = 2$ $\therefore \lim_{X \to 0} f(X) = 2$

(b) $\lim_{X \to 1} \frac{\left(4 - \sqrt{X+15}\right)\left(4 + \sqrt{X+15}\right)}{(1-X)(1+X)\left(4 + \sqrt{X+15}\right)}$

$= \lim_{X \to 1} \frac{(16 - X - 15)}{(1-X)(1+X)\left(4 + \sqrt{X+15}\right)}$

$= \lim_{X \to 1} \frac{1}{(1+X)\left(4 + \sqrt{X+15}\right)} = \frac{1}{16}$

Model 2

1

(1) (d) (2) (c) (3) (a) (4) (c)

2

(a) $f(-3) = -3 + a$

, $\lim_{X \to -3} f(X) = \lim_{X \to -3} \frac{(X+3)(X-1)}{X+3} = -4$

, $\because f$ is continuous at $X = -3$

$\therefore -3 + a = -4$ $\therefore a = -1$

(b) $\because a : b : c = 3 : 4 : 5$

Let $a = 3k$, $b = 4k$, $c = 5k$

$\therefore \cos C = \frac{(3k)^2 + (4k)^2 - (5k)^2}{2 \times 3k \times 4k} =$ zero

$\therefore m(\angle C) = 90°$

$\because 3k + 4k + 5k = 24$ $\therefore k = 2$

$\therefore a = 6$ cm. , $b = 8$ cm. , $c = 10$ cm.

$\therefore$ The area of $\Delta ABC = \frac{1}{2} \times 6 \times 8 = 24$ cm.^{2}

3

(a) $\lim_{X \to 0} \frac{X + \frac{\sin 3X}{X}}{5 \cos 2X} = \frac{0 + 3}{5 \times 1} = \frac{3}{5}$

(b) $\because c^2 = (9)^2 + (15)^2 - 2 \times 9 \times 15 \cos 106°$

$\therefore c \simeq 19.5$ cm. $\therefore \cos A = \frac{(15)^2 + (19.5)^2 - (9)^2}{2 \times 15 \times 19.5}$

$\therefore m(\angle A) \simeq 26° 21'$

$\therefore m(\angle B) = 180° - (26° 21' + 106°) = 47° 39'$

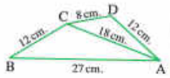
4

(a) $\lim\limits_{X \to -2} \dfrac{(X+3)^5 - 1}{(X-2)(X+2)}$

$= \lim\limits_{X \to -2} \dfrac{1}{X-2} \times \lim\limits_{(X+3) \to 1} \dfrac{(X+3)^5 - 1^5}{(X+3)-1}$

$= -\frac{1}{4} \times 5 \times (1)^4 = -\frac{5}{4}$

(b) In Δ ADC :



$\cos(\angle DAC)$

$= \dfrac{(12)^2 + (18)^2 - (8)^2}{2 \times 12 \times 18} = \dfrac{101}{108}$

$\therefore m(\angle DAC) \simeq 20° 45'$

, in Δ CAB :

$\cos(\angle CAB) = \dfrac{(27)^2 + (18)^2 - (12)^2}{2 \times 27 \times 18} = \dfrac{101}{108}$

$\therefore m(\angle DAC) = m(\angle CAB)$

$\therefore \overrightarrow{AC}$ bisects ∠ BAD

∴ The area of the figure ABCD

= the area of Δ ADC + the area of Δ ACB

$= \frac{1}{2} \times 12 \times 18 \times \sin 20° 45' + \frac{1}{2} \times 18 \times 27 \times \sin 20° 45' \simeq 124 \text{ cm}^2$

5

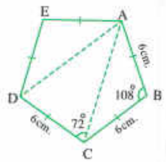
(a) (1) $\lim\limits_{X \to 0} \dfrac{(\sqrt{X+4}-2)(\sqrt{X+4}+2)}{X(X+1)(\sqrt{X+4}+2)}$

$= \lim\limits_{X \to 0} \dfrac{X+4-4}{X(X+1)(\sqrt{X+4}+2)}$

$= \lim\limits_{X \to 0} \dfrac{1}{(X+1)(\sqrt{X+4}+2)} = \dfrac{1}{4}$

(2) $\lim\limits_{X \to \infty} \sqrt{\dfrac{3}{X^2} + 4} = \sqrt{4} = 2$

(b) In Δ ABC :



$\because (AC)^2 = 6^2 + 6^2 - 2 \times 6 \times 6 \cos 108°$

$\therefore AC \simeq 9.7$ cm.

∴ The area of ABCDE

= 2 the area of Δ ABC + the area of Δ ACD

$= 2 \times \frac{1}{2} \times 6 \times 6 \times \sin 108° + \frac{1}{2} \times 9.7 \times 6 \times \sin 72° \simeq 62 \text{ cm}^2$

Answers of final examination models

Model 1

1 ⓑ **2** ⓒ **3** ⓑ **4** ⓑ

5

$c^2 = a^2 + b^2 - 2ab\cos c$

$= 49 + 81 - 2 \times 7 \times 9 \times \cos 96° 23'$ $\therefore c \simeq 12$ cm.

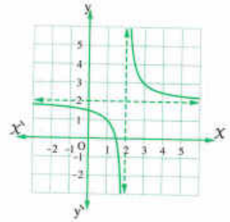
, the area of $\Delta ABC = \frac{1}{2} ab \sin C$

$= \frac{1}{2} \times 7 \times 9 \times \sin 96° 23'$

$\simeq 31 \text{ cm}^2$

6 ⓑ **7** ⓐ

8



$g(X) = \dfrac{1}{X-2} + 2$

The range of g is $\mathbb{R} - \{2\}$, g is decreasing on $]-\infty, 2[$, $]2, \infty[$

9 ⓒ **10** ⓐ **11** ⓑ

12 ⓒ **13** ⓐ **14** ⓒ

15

Putting $3 - X \geq 0$ $\therefore X \leq 3$

∴ The domain of f is $]-\infty, 3]$

, the range of f is $[2, \infty[$

, putting $y = 2 + \sqrt{3-X}$, $X \leq 3$, $y \geq 2$

, by exchanging the two variables

$\therefore X = 2 + \sqrt{3-y}$, $y \leq 3$, $X \geq 2$

$\therefore \sqrt{3-y} = X - 2$ $\therefore 3 - y = X^2 - 4X + 4$

$\therefore y = -X^2 + 4X - 1$ $\therefore f^{-1}(X) = -X^2 + 4X - 1$

, domain of $f^{-1} = [2, \infty[$, range of $f^{-1} =]-\infty, 3]$

16 ⓒ **17** ⓑ **18** ⓐ **19** ⓓ **20** ⓓ

21

$f(2^+) = \lim\limits_{X \to 2^+} (5X - 4) = 6$

, $f(2^-) = \lim\limits_{X \to 2^-} (X^2 - X + 4) = 6$

$\therefore f(2^+) = f(2^-) = 6$ $\therefore \lim\limits_{X \to 2} f(X) = 6$

, $\because f(2) = k$ $\therefore k = 6$

so that the function becomes continuous at $X = 2$

22 ⓐ **23** ⓓ **24** ⓑ **25** ⓐ **26** ⓒ

27 ⓒ **28** ⓑ **29** ⓒ **30** ⓐ **31** ⓐ

32 ⓑ

Model 2

1 ⓒ **2** ⓐ

3

$(AB)^2 = (AD)^2 + (BD)^2 - 2 \times AD \times BD \times \cos\theta$ (1)

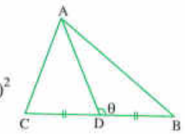
, $(AC)^2 = (AD)^2 + (CD)^2 - 2 \times AD \times CD \times \cos(180° - \theta)$

, $\because BD = CD$

, $\cos(180° - \theta) = -\cos\theta$

$\therefore (AC)^2 = (AD)^2 + (BD)^2 + 2 \times AD \times BD \times \cos\theta$ (2)

By adding (1) , (2) :



$\therefore (AB)^2 + (AC)^2$

$= 2(AD)^2 + 2(BD)^2$

$\therefore (5)^2 + (8)^2 = 2(AD)^2 + 2(6)^2$

$\therefore AD = \dfrac{\sqrt{34}}{2} \simeq 2.9$ cm.

4 ⓓ **5** ⓑ **6** ⓒ

7

$\dfrac{1}{X} = \dfrac{1}{5+2\sqrt{6}} \times \dfrac{5-2\sqrt{6}}{5-2\sqrt{6}} = \dfrac{5-2\sqrt{6}}{25-24} = 5 - 2\sqrt{6}$

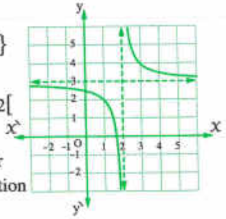
$\therefore X + \dfrac{1}{X} = 5 + 2\sqrt{6} + 5 - 2\sqrt{6} = 10$

$\therefore \log\left(X + \dfrac{1}{X}\right) = \log 10 = 1$

8 ⓒ **9** ⓒ **10** ⓒ **11** ⓒ **12** ⓑ

13 ⓓ **14** ⓓ **15** ⓐ

16



- Domain of g = ℝ − {2}
- Range = ℝ − {3}
- Decreasing on]− ∞ , 2[,]2 , ∞[
- The function is neither even nor odd the function is one - to - one.

17 ⓓ 18 ⓒ 19 ⓒ 20 ⓒ

21

$f(3^+) = \lim_{x \to 3^+} (3x + 1) = 10$

, $f(3^-) = \lim_{x \to 3^-} (x^2 + 1) = 10$

, ∵ $f(3^+) = f(3^-) = 10$

∴ $\lim_{x \to 3} f(x)$ is exist and equals 10

22 ⓒ 23 ⓓ 24 ⓓ 25 ⓑ 26 ⓑ

27 ⓑ 28 ⓓ 29 ⓐ 30 ⓐ 31 ⓐ

32 ⓐ

Model 3

1

∵ $f(2x - 1) + f(x - 2) = 50$

∴ $7^{2x-1+1} + 7^{x-2+1} = 50$

∴ $7^{2x} + 7^{x-1} = 50$ ∴ $7 \times 7^{2x} + 7^x - 350 = 0$

∴ $(7^x - 7)(7 \times 7^x + 50) = 0$

∴ $7^x = 7$ ∴ $x = 1$ or $7^{x+1} = -50$ (refused)

2 ⓒ 3 ⓓ 4 ⓑ 5 ⓑ 6 ⓓ

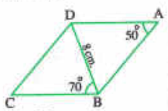
7 ⓑ 8 ⓐ

9

m (∠ ADB) = 70°

, m (∠ ABD) = 180° − (50° + 70°) = 60°

∴ $\frac{8}{\sin 50°} = \frac{AB}{\sin 70°} = \frac{AD}{\sin 60°}$



∴ AB ≈ 9.8 cm.

AD ≈ 9 cm.

∴ The perimeter of paralleogram ≈ 38 cm.

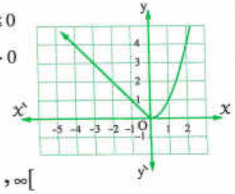
10 ⓐ 11 ⓑ 12 ⓐ 13 ⓒ 14 ⓓ

15 ⓓ 16 ⓓ 17 ⓑ 18 ⓐ 19 ⓐ

20 ⓒ 21 ⓑ

22

$f(x) = \begin{cases} -x & , x \le 0 \\ x^2 & , x > 0 \end{cases}$



- The range is [0 , ∞[
- f is decreasing on]− ∞ , 0[and increasing on]0 , ∞[

23 ⓓ 24 ⓑ 25 ⓑ 26 ⓒ 27 ⓒ

28

∵ f is continuous at $x = 0$

∴ $f(0) = \lim_{x \to 0} f(x)$ ∴ $k = \lim_{x \to 0} \frac{(x+3)^5 - 3^5}{x}$

∴ $k = 5 \times 3^4 = 405$

29 ⓓ 30 ⓒ 31 ⓓ 32 ⓑ

Model 4

1 ⓐ 2 ⓐ 3 ⓒ 4 ⓑ 5 ⓑ

6 ⓓ 7 ⓑ

8

c = 21 − (8 + 7) = 6 cm.

∵ c is the length of the smallest side.

∴ m (∠ C) is the measure of the smallest angle.

∴ $\cos C = \frac{64 + 49 - 36}{2 \times 8 \times 7}$

∴ m (∠ C) = 46° 34′ 3″

9

∵ $f(x) = 2x + a$

, by putting y = 2x + a

, exchanging the two variables.

∴ $x = 2y + a$ ∴ $y = \frac{1}{2}x - \frac{1}{2}a$

∴ $f^{-1}(x) = \frac{1}{2}x - \frac{1}{2}a$

, ∵ g is the inverse function of f

∴ $g(x) = f^{-1}(x)$ ∴ $bx + 3 = \frac{1}{2}x - \frac{1}{2}a$

by comparing the coeffecients.

∴ $b = \frac{1}{2}$, a = − 6

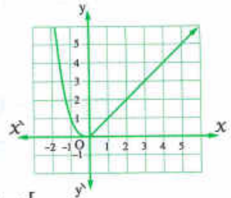
10 ⓐ 11 ⓒ 12 ⓒ

13

∵ $f(-1^-) = f(-1^+)$ ∴ − a − 3 = − 3 − 2 ∴ a = 2

14 ⓓ 15 ⓒ

16



- The range = [0 , ∞[
- The function is neither even nor odd
- f is decreasing on]− ∞ , 0[and increasing on]0 , ∞[

17 ⓒ 18 ⓐ 19 ⓑ 20 ⓓ 21 ⓓ

22 ⓐ 23 ⓒ 24 ⓑ 25 ⓑ 26 ⓑ

27 ⓑ 28 ⓑ 29 ⓑ 30 ⓓ 31 ⓓ

32 ⓑ

Model 5

1 ⓒ 2 ⓓ 3 ⓐ 4 ⓑ 5 ⓓ

6 ⓓ 7 ⓒ

8

∵ $3^{2x-1} - 4 \times 3^x + 9 = 0$ (Multiplying by 3)

∴ $3^{2x} - 12 \times 3^x + 27 = 0$

∴ $(3^x - 3)(3^x - 9) = 0$

∴ $3^x = 3$ ∴ $x = 1$

or $3^x = 9 = 3^2$ ∴ $x = 2$

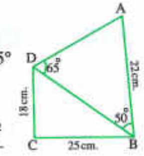
∴ The S.S. = {1 , 2}

9 ⓒ 10 ⓓ 11 ⓒ 12 ⓒ 13 ⓑ

14 ⓑ 15 ⓒ

16

In Δ ABD :



m (∠ A) = 180° − (50° + 65°) = 65°

∴ BD = AB = 22 cm.

, in Δ BDC :

$\cos(\angle CBD) = \frac{(22)^2 + (25)^2 - (18)^2}{2 \times 22 \times 25}$

∴ m (∠ CBD) = 44° 28′ 6″

17 ⓓ 18 ⓓ 19 ⓐ 20 ⓑ

21

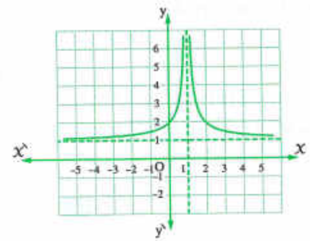
$f(x) = |x| + 1$, $g(x) = \frac{1}{x - 1}$

∴ $(f \circ g)(x) = f[g(x)] = f\left(\frac{1}{x-1}\right) = \left|\frac{1}{x-1}\right| + 1$

∴ D_1 = domain of g = ℝ − {1} , D_2 = ℝ

∴ Domain of $(f \circ g) = D_1 \cap D_2$ = ℝ − {1}

- Graphical representation of the function $(f \circ g)$



- The domain = ℝ − {1} • f neither even nor odd.
- The curve symmetrical about $x = 1$
- f is not one - to - one.

22 ⓑ 23 ⓑ 24 ⓑ 25 ⓑ 26 ⓒ

27 ⓒ

28

$f(0^-) = \lim_{x \to 0^-} \frac{x + \sin x}{3x - \sin 2x} = \lim_{x \to 0^-} \frac{x\left(1 + \frac{\sin x}{x}\right)}{x\left(3 - \frac{\sin 2x}{x}\right)} = \frac{1 + 1}{3 - 2} = 2$

, $f(0^+) = \lim_{x \to 0^+} (3x - 1) = -1$

∵ $f(0^-) \ne f(0^+)$ ∴ f is discontinuous at $x = 0$

29 ⓓ 30 ⓑ 31 ⓑ 32 ⓒ

Model 6

1 ⓐ 2 ⓒ 3 ⓑ

4

(1) $f_1(-X) = (-X)\cos(-X) = -X\cos X = -f_1(X)$

$\therefore f_1$ is odd function.

(2) $f_2(-X) = \begin{cases} (-X)^2 & , -X \ge 0 \\ |-X| & , -X < 0 \end{cases}$

$= \begin{cases} X^2 & , X \le 0 \\ |X| & , X > 0 \end{cases} \ne \pm f_2(X)$

$\therefore f_2$ is neither even nor odd.

5 ⓒ 6 ⓑ 7 ⓒ 8 ⓓ 9 ⓑ

10 ⓓ 11 ⓑ 12 ⓐ

13

Putting : $y = \frac{1}{X^2+1}$

, where $X \in \mathbb{R}^+$, $y \in]0 , 1[$ by exchanging the two variables

$\therefore X = \frac{1}{y^2+1}$ where $y \in \mathbb{R}^+$, $X \in]0 , 1[$

$\therefore y^2 + 1 = \frac{1}{X}$ $\therefore y^2 = \frac{1}{X} - 1$, $y = \sqrt{\frac{1}{X} - 1}$

$\therefore f^{-1}(X) = \sqrt{\frac{1}{X} - 1}$

, domain of $f^{-1} =]0 , 1[$

, range of f^{-1} = domain of $f = \mathbb{R}^+$

14 ⓑ 15 ⓓ 16 ⓒ 17 ⓑ 18 ⓒ

19 ⓑ 20 ⓓ 21 ⓐ 22 ⓑ 23 ⓑ

24

$m(\angle C) = 180° - (40° + 30°) = 110°$

$, \because \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$\therefore \frac{a}{\sin 40°} = \frac{10}{\sin 30°} = \frac{c}{\sin 110°}$

$\therefore a \simeq 12.9$ cm. , $c = 18.8$ cm.

25 ⓐ 26 ⓐ 27 ⓑ

28

$\because f$ is continuous at $X = -3$

$\therefore \lim_{X \to -3} \frac{X^2 + 2X - 3}{X + 3} = -3 + a$

$\therefore \lim_{X \to -3} \frac{(X+3)(X-1)}{(X+3)} = -3 + a$

$\therefore -4 = -3 + a$ $\therefore a = -1$

29 ⓒ 30 ⓓ 31 ⓒ 32 ⓓ

Model 7

1 ⓓ 2 ⓒ 3 ⓑ

4

$\because X^{\frac{4}{3}} - 10X^{\frac{2}{3}} + 9 = 0$ $\therefore (X^{\frac{2}{3}} - 9)(X^{\frac{2}{3}} - 1) = 0$

$\therefore X^{\frac{2}{3}} = 9$ $\therefore X = \pm 27$

or $X^{\frac{2}{3}} = 1$ $\therefore X = \pm 1$

$\therefore$ The S.S. = $\{1 , -1 , 27 , -27\}$

5 ⓓ 6 ⓒ 7 ⓑ 8 ⓑ 9 ⓐ

10 ⓓ 11 ⓓ 12 ⓓ 13 ⓑ 14 ⓓ

15 ⓑ

16

$z^2 = 25 + 49 - 2 \times 5 \times 7 \cos 65°$

$\therefore z \simeq 6.7$ cm. $\therefore$ perimeter of $\Delta XYZ \simeq 18.7$ cm.

17 ⓑ 18 ⓑ 19 ⓓ

20

$f(0^+) = \lim_{X \to 0^+} \frac{5X + 6}{X + 3} = 2$, $f(0^-) = \lim_{X \to 0^-} \frac{\tan 2X}{\sin X}$

(dividing both numerator and denominator by X)

$= \lim_{X \to 0^-} \frac{\frac{\tan 2X}{X}}{\frac{\sin X}{X}} = 2$

$\therefore f(0^+) = f(0^-) = 2$ $\therefore \lim_{X \to 0} f(X) = 2$

21 ⓒ 22 ⓓ 23 ⓑ

24

D_1 = domain of $g = [2 , \infty[$

$, (f \circ g)(X) = f[g(X)] = f(\sqrt{X-2})$

$= (\sqrt{X-2})^2 - 3$

$= X - 2 - 3$

$= X - 5$

$\therefore D_2$ = (values of X which make $\sqrt{X-2}$ in domain of f) $= \mathbb{R}$

$\therefore$ The domain of $(f \circ g) = [2 , \infty[$

$\because (f \circ g)(3) = 3 - 5 = -2$

25 ⓑ 26 ⓐ 27 ⓑ 28 ⓒ 29 ⓑ

30 ⓓ 31 ⓓ 32 ⓓ

Model 8

1 ⓑ 2 ⓓ 3 ⓒ 4 ⓓ 5 ⓐ

6 ⓐ 7 ⓑ

8

$5^X + 5^{X-1} = 150$ $\therefore 5^{X-1}(5 + 1) = 150$

$\therefore 5^{X-1} = 25 = 5^2$ $\therefore X = 3$

9 ⓓ 10 ⓒ 11 ⓑ 12 ⓒ 13 ⓐ

14 ⓐ 15 ⓒ 16 ⓐ

17

$m(\angle C) = 180° - (112° + 33°) = 35°$

$\therefore \frac{a}{\sin 112°} = \frac{b}{\sin 33°} = \frac{19}{\sin 35°} = 2r$

$\therefore r \simeq 17$ cm. , $b = 18$ cm.

$\therefore$ area of $\Delta ABC = \frac{1}{2} b c \sin A$

$= \frac{1}{2} \times 18 \times 19 \times \sin 112°$

$\simeq 159$ cm.^{2}

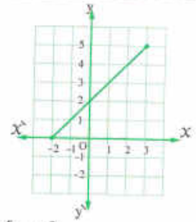
18 ⓒ 19 ⓓ 20 ⓓ

21

$(f + h)(X) = f(X) + h(X) = 4X - 2 + 4 - 3X = X + 2$

• Domain of $(f + h)$ is $\mathbb{R} \cap [-2 , 3] = [-2 , 3]$

X	-2	0	3
$(f+h)(X)$	0	2	5



• The range = $[0 , 5]$

• The function is increasing on its domain.

22 ⓑ 23 ⓓ

24

$\because f$ is continuous at a

$\therefore 2 - a^2 = a$ $\therefore a^2 + a - 2 = 0$

$\therefore (a + 2)(a - 1) = 0$ $\therefore a = -2$ or $a = 1$

25 ⓓ 26 ⓓ 27 ⓒ 28 ⓑ 29 ⓐ

30 ⓑ 31 ⓒ 32 ⓐ

Model 9

1 ⓓ 2 ⓐ 3 ⓒ

4

Put $y = 3 + \sqrt{X - 1}$ where $X \ge 1$, $y \ge 3$

(by exchanging the two variables)

$\therefore X = 3 + \sqrt{y - 1}$

Where $y \ge 1$, $X \ge 3$ $\therefore \sqrt{y - 1} = X - 3$

$y - 1 = (X - 3)^2$ $\therefore y = (X - 3)^2 + 1$

$\therefore f^{-1}(X) = (X - 3)^2 + 1$ where $X \ge 3$

5 ⓒ 6 ⓐ 7 ⓑ

8

$\because f$ is continuous at $X = 2$

$\therefore f(2^+) = f(2^-) = f(2) = 4$

$\therefore \lim_{X \to 2^+} (X^2 + aX - 2) = 4$

$\therefore 4 + 2a - 2 = 4$ $\therefore a = 1$

$, \lim_{X \to 2^-} (5a + bX) = 4 \therefore 5a + 2b = 4$

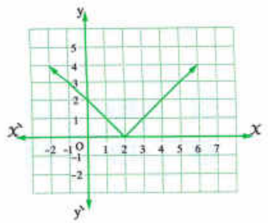
$\therefore 5 + 2b = 4$ $\therefore b = \frac{-1}{2}$

9 ⓐ 10 ⓑ 11 ⓒ 12 ⓑ 13 ⓒ

14 ⓐ 15 ⓓ

16

$f(x) = \sqrt{x^2 - 4x + 4} = \sqrt{(x-2)^2} = |x - 2|$



- The range $= [0, \infty[$
- f is decreasing on $]-\infty, 2[$ and increasing on $]2, \infty[$

17 ⓑ 18 ⓑ 19 ⓑ 20 ⓐ

21

$\because \frac{21}{\sin A} = \frac{25}{\sin B} = \frac{c}{\sin C} = 28$

$\therefore \sin A = \frac{3}{4}$

$, \because$ the triangle is acute angled triangle.

$\therefore m(\angle A) = 48° 35' 25''$

$, \sin B = \frac{25}{28}$

$, \because$ the triangle is acute angled triangle.

$\therefore m(\angle B) = 63° 14' 4''$, $m(\angle C) = 68° 10' 31''$

$\therefore c = 28 \sin(68° 10' 31'') = 26$ cm.

22 ⓐ 23 ⓓ 24 ⓓ 25 ⓒ 26 ⓒ

27 ⓓ 28 ⓑ 29 ⓑ 30 ⓒ 31 ⓑ

32 ⓓ

Model 10

1 ⓐ 2 ⓓ 3 ⓑ 4 ⓒ 5 ⓓ

6 ⓒ 7 ⓑ

8

$\because f$ is continuous at $x = 2$

$\therefore f(2^+) = f(2^-) = f(2) = 4$

$\therefore \lim_{x \to 2^+} (x^2 + a x - 2) = 4$

$\therefore 4 + 2a - 2 = 4 \qquad \therefore a = 1$

$, \lim_{x \to 2^-} (5a + bx) = 4 \qquad \therefore 5(1) + b(2) = 4$

$\therefore b = \frac{-1}{2}$

9

$\because 26 + b + c = 70 \qquad \therefore b + c = 44 \qquad \therefore b = 44 - c$

$, \because a^2 = b^2 + c^2 - 2bc \cos A$

$\therefore (26)^2 = (44 - c)^2 + c^2 - 2(44 - c) c \cos 60$

$\therefore 676 = 1936 - 88c + c^2 + c^2 - 44c + c^2$

$\therefore 3c^3 - 132c + 1260 = 0$

$\therefore c^2 - 44c + 420 = 0 \qquad \therefore (c - 30)(c - 14) = 0$

$\therefore c = 30$ cm. , then $b = 14$ cm.

or $c = 14$ cm. , then $b = 30$ cm.

$\therefore$ The area of $\Delta ABC = \frac{1}{2} bc \sin A$

$= \frac{1}{2} \times 420 \times \sin 60°$

$= 105\sqrt{3}$ cm².

10 ⓓ 11 ⓓ 12 ⓒ 13 ⓓ 14 ⓐ

15 ⓐ 16 ⓑ 17 ⓑ 18 ⓑ 19 ⓓ

20 ⓒ

21

$\because |x - 3| = |9 - 2x| \qquad \therefore x - 3 = 9 - 2x$

$\therefore 3x = 12 \qquad \therefore x = 4$

or $x - 3 = -9 + 2x \qquad \therefore x = 6$

$\therefore$ The S.S. $= \{4, 6\}$

22 ⓒ 23 ⓓ 24 ⓓ 25 ⓓ 26 ⓑ

27 ⓒ

28

$\because f(x) = 7^{x+1} \qquad \therefore f(2x - 1) + f(x - 2) = 50$

$\therefore 7^{2x-1+1} + 7^{x-2+1} = 50$

$\therefore 7^{2x} + 7^{x-1} = 50 \qquad \therefore 7(7^x)^2 + 7^x - 350 = 0$

$\therefore (7^x - 7)(7 \times 7^x + 50) = 0 \qquad \therefore 7^x = 7$

$\therefore x = 1$ or $7 \times 7^x = -50$ (refused)

29 ⓒ 30 ⓓ 31 ⓐ 32 ⓑ

Answers of multiple choice examinations

Model 1

1 (a)

2 (a)

Solution :

$\because \log_2 x = \log_4 9$

$\therefore \frac{\log x}{\log 2} = \frac{\log 9}{\log 4} = \frac{2 \log 3}{2 \log 2}$

$\therefore \log x = \log 3 \quad \therefore x = 3$

3 (d)

Solution :

$\lim_{x \to 0} \frac{\sqrt{1 - \cos^2 x}}{x} = \lim_{x \to 0} \frac{\sqrt{\sin^2 x}}{x} = \lim_{x \to 0} \frac{|\sin x|}{x}$

$, \because \lim_{x \to 0^+} \frac{|\sin x|}{x} = 1 \quad , \lim_{x \to 0^-} \frac{|\sin x|}{x} = -1$

$\therefore$ The limit is not exist.

4 (c)

Solution :

$\cos B = \frac{3^2 + 5^2 - 7^2}{2 \times 3 \times 5} = -\frac{1}{2}$

$\therefore m(\angle B) = 120°$

5 (b)

Solution :

$\log_{xy}\left(\frac{1}{x}\right) + \log_{xy}\left(\frac{1}{y}\right) = \log_{xy}\left(\frac{1}{xy}\right)$

$= \log_{xy}(xy)^{-1}$

$= -\log_{xy}(xy) = -1$

6 (b)

Solution :

$\lim_{x \to 16} \frac{\sqrt[4]{x^7} - 128}{x - 16} = \lim_{x \to 16} \frac{x^{\frac{7}{4}} - 16^{\frac{7}{4}}}{x - 16}$

$= \frac{7}{4} \times 16^{\frac{3}{4}} = 14$

7 (c)

Solution :

$f(0^+) = \lim_{x \to 0^+} \frac{2x + \sin x}{\tan 2x + 4x} = \lim_{x \to 0} \frac{2 + \frac{\sin x}{x}}{\frac{\tan 2x}{x} + 4}$

$= \frac{2 + 1}{2 + 4} = \frac{1}{2}$

$, f(0^-) = \lim_{x \to 0^-} \left(5x + \frac{1}{2}\right) = \frac{1}{2}$

$\because f(0^+) = f(0^-) = \frac{1}{2}$

$\therefore \lim_{x \to 0} f(x) = \frac{1}{2}$

8 (a)

Solution :

$y^2 = (20)^2 + (16)^2 - 2(20)(16) \times 0.4$

$\therefore y^2 = 400 \qquad \therefore y = 20$ cm.

9 (b)

10 (b)

Solution :

$y = 3^x$, by replacing the two variables

$\therefore x = 3^y \qquad \therefore \log_3 x = y$

$\therefore f^{-1}(x) = \log_3 x$

11 (d)

Solution :

$x + 3 \geq 0$

$\therefore x \geq -3$

$\therefore$ The domain $= [-3, \infty[$

12 (d)

Solution :

$\lim_{x \to \frac{\pi}{4}} \frac{\tan x}{x} = \frac{\tan\left(\frac{\pi}{4}\right)}{\frac{\pi}{4}} = \frac{1}{\frac{\pi}{4}} = \frac{4}{\pi}$

13 (c)

Solution :

$\lim_{n \to \infty} \frac{\sqrt[n]{3}}{9} = \lim_{n \to \infty} \frac{3^{\frac{1}{n}}}{9} = \frac{1}{9}$

14 (b)

Solution :

$m(\angle B) = 180° - (40° + 80°) = 60°$

$\therefore \frac{a+b+c}{\sin A + \sin B + \sin C} = 2r$

$\therefore \frac{20}{\sin 80° + \sin 60° + \sin 40°} = 2r$

$\therefore 2r \simeq 8.02 \qquad \therefore r \simeq 4$ cm.

---

15 (d)

Solution :

$\because \log_X 3 = -2 \qquad \therefore X^{-2} = 3$

$\therefore (X^{-2})^{\frac{-1}{2}} = 3^{\frac{-1}{2}} \qquad \therefore X = \frac{1}{\sqrt{3}}$

---

16 (a)

Solution :

$f(X) = X^2 + 3 \ , \ g(X) = -2X + 6$

$\therefore (g \circ f)(1) = g(f(1))$

$= g(4) = -2$

---

17 (a)

Solution :

$\lim_{X \to 0} \frac{1 - \cos X}{5X} = \frac{1}{5} \lim_{X \to 0} \frac{1 - \cos X}{X} = \frac{1}{5} \times 0 = 0$

---

18 (a)

---

19 (a)

Solution :

$f(-X) = \frac{-X - \tan(-X)}{|-X|}$

$= \frac{-X + \tan X}{|X|} = -\frac{X - \tan X}{|X|} = -f(X)$

$\therefore f(-X) = -f(X) \qquad \therefore$ The function f is odd.

---

20 (d)

Solution :

$|X| < a$

$\therefore -a < X < a$

$\therefore X \in]-a \ , a[$

21 (c)

---

22 (c)

Solution :

$3^{X+1} + 3^X = 36$

$\therefore 3^X (3^1 + 1) = 36$

$\therefore 3^X (4) = 36$

$\therefore 3^X = 9 = 3^2$

$\therefore X = 2$

---

23 (a)

---

24 (c)

Solution :

$f(-2) = a - (-2)^2 = a - 4$

$\because$ The function is continuous at $X = -2$

$\therefore f(-2^+) = f(-2^-) = f(-2)$

$\therefore -2 = a - 4$

$\therefore a = 2$

---

25 (b)

Solution :

$\because \frac{c}{\sin C} = 2r$

$\therefore \frac{5c}{\sin C} = 10r$

---

26 (b)

---

27 (d)

Solution :

$\because A + B + C = 180°$

$\therefore A + B = 180° - C$

$\therefore \cos(A + B) = \cos(180° - C) = -\cos C$

---

28 (d)

Solution :

By dividing both of numerator and denominator by X^2

We get : $\lim_{X \to \infty} \frac{1 - \frac{5}{X}}{\frac{2}{X} + 3} = \frac{1}{3}$

Model 2

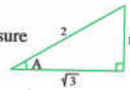
1 (a)

---

2 (a)

$\because$ a is the smallest side in length

$\therefore (\angle A)$ is the smallest angle in measure



$\cos A = \frac{(4\sqrt{3})^2 + (8)^2 - (4)^2}{2 \times 4\sqrt{3} \times 8} = \frac{\sqrt{3}}{2}$

$\therefore \sin A = \frac{1}{2}$

---

3 (d)

Solution :

$\because 5^{X+1} = 7^{X+1} \qquad \therefore X + 1 = 0$

$\therefore 3^{X+1} = 3^0 = 1$

---

4 (a)

---

5 (c)

---

6 (d)

Solution :

$\because \frac{\sin L}{3} = \frac{2 \sin M}{3} = \frac{\sin N}{4} \qquad \therefore \frac{\sin L}{6} = \frac{\sin M}{3} = \frac{\sin N}{8}$

$\therefore \ell : m : n = 6 : 3 : 8$

---

7 (b)

Solution :

$y = \sqrt[5]{X}$, by replacing the two variables.

$\therefore X = \sqrt[5]{y} \qquad \therefore X^5 = y$

$\therefore f^{-1}(X) = X^5$

---

8 (c)

Solution :

$\lim_{X \to 0} \frac{(X+2)^5 - 32}{X} = \lim_{(X+2) \to 2} \frac{(X+2)^5 - 2^5}{(X+2) - 2}$

$= 5 \times 2^4 = 80$

---

9 (c)

Solution :

$D_1 = \text{Domain of } f = \mathbb{R}$

, $(g \circ f) X = g(f(X)) = \sqrt{X - 1}$

$\therefore$ The value of X which makes $f(X)$ in the domain of g is $D_2 = [1 \ , \infty[$

$\therefore$ The domain of $(g \circ f) = D_1 \cap D_2 = [1 \ , \infty[$

---

10 (c)

Solution :

$\frac{7}{\sin 30°} = 2r \qquad$ *i.e.* $2r = 14$ cm.

$\therefore$ The diameter length = 14 cm.

---

11 (b)

Solution :

$\log_X (X + 6) = 2$

$\therefore X^2 = X + 6$

$\therefore X^2 - X - 6 = 0$

$\therefore (X + 2)(X - 3) = 0$

$\therefore X = -2$ (refused) , $X = 3$

---

12 (b)

---

13 (b)

Solution :

$\lim_{X \to 3} \frac{X^2 - 7X + 12}{X - 3} = \lim_{X \to 3} \frac{(X-3)(X-4)}{X-3} = -1$

---

14 (c)

Solution :

$\because$ The curve passes through the point $\left(\frac{1}{8} , \frac{-1}{2}\right)$

$\therefore f\left(\frac{1}{8}\right) = \frac{-1}{2}$

$\therefore \log_4 \left(1 - \frac{1}{8} a\right) = \frac{-1}{2}$

$\therefore (4)^{-\frac{1}{2}} = 1 - \frac{1}{8} a$

$\therefore \frac{1}{2} = 1 - \frac{1}{8} a \qquad \therefore \frac{1}{8} a = \frac{1}{2}$

$\therefore a = 4$

---

15 (a)

Solution :

By dividing both of numerator and denominator by X^3

We get : $\lim_{X \to \infty} \frac{\left(2 + \frac{1}{X}\right)\left(4 - \frac{1}{X}\right)^2}{\left(2 + \frac{3}{X}\right)^3} = \frac{2 \times (4)^2}{(2)^3} = 4$

16 (d)

17 (d)

Solution :

$\because X - 2 \geq 0$ $\qquad \therefore X \geq 2$

$\therefore$ The domain $[2 , \infty[- \{3\}$

18 (a)

19 (a)

Solution :

$\because \log_3 X \times \log_2 3 = 5$

$\therefore \frac{\log X}{\log 3} \times \frac{\log 3}{\log 2} = 5$

$\therefore \log X = 5 \log 2$

$\therefore \log X = \log 2^5$

$\therefore X = 32$

$\therefore$ The solution set $= \{32\}$

20 (a)

Solution :

$\lim_{X \to 0} 5 X \csc 2 X$

$= \lim_{X \to 0} \frac{5 X}{\sin 2 X} = \frac{5}{2}$

21 (b)

22 (b)

Solution :

$X^{\frac{3}{2}} = 8$

$\therefore (X^{\frac{3}{2}})^{\frac{2}{3}} = 8^{\frac{2}{3}}$ $\qquad \therefore X = 4$

23 (c)

Solution :

$f(0) = k^2$

$, \lim_{X \to 0} f(X) = \lim_{X \to 0} \frac{\sin 9 X}{\tan 16 X} = \lim_{X \to 0} \frac{\frac{\sin 9 X}{X}}{\frac{\tan 16 X}{X}} = \frac{9}{16}$

$\because$ The function is continuous at $X = 0$

$\therefore k^2 = \frac{9}{16}$ $\qquad \therefore k = \pm \frac{3}{4}$

24 (c)

25 (a)

Solution :

let $\frac{a+b}{13} = \frac{b+c}{11} = \frac{c+a}{12} = k$

$\therefore a + b = 13 k$ (1)

$, b + c = 11 k$ (2)

$, c + a = 12 k$ (3)

By adding (1) , (2) and (3) :

$\therefore 2 a + 2 b + 2 c = 36 k$

$\therefore a + b + c = 18 k$ (4)

by subtracting (1) from (4) :

$\therefore c = 5 k$, then $a = 7 k$

$\therefore b = 6 k$

$\therefore \cos A = \frac{36 k^2 + 25 k^2 - 49 k^2}{2 (6 k)(5 k)} = \frac{12 k^2}{60 k^2} = \frac{1}{5}$

26 (c)

Solution :

$\lim_{X \to 1} \frac{X^{6\frac{1}{2}} - X^{\frac{1}{2}}}{X^{3\frac{1}{2}} - X^{\frac{1}{2}}} = \lim_{X \to 1} \frac{X^{\frac{1}{2}} (X^6 - 1)}{X^{\frac{1}{2}} (X^3 - 1)}$

$= \lim_{X \to 1} \frac{X^6 - 1^6}{X^3 - 1^3} = \frac{6}{3} (1)^3 = 2$

27 (d)

Solution :

$\therefore \lim_{X \to 3} f(X) = 16$

$\therefore f(3^-) = f(3^+) = 16$

$\therefore 2 (3) + b = 16$, then $b = 10$

$, 3 (3)^2 + 3 (a) - 2 = 16$, then $3 a = -9$

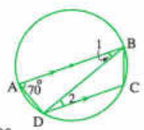
$\therefore a = -3$

$\therefore a + b = 10 - 3 = 7$

28 (a)

Solution :

In ΔABD :



$\because \overline{AB}$ is a diameter

$\therefore m(\angle ADB) = 90°$

$\therefore m(\angle 1) = 180° - (70° + 90°) = 20°$

$\therefore m(\angle 1) = m(\angle 2) = 20°$ alternative angles.

$\because$ ABCD is a cyclic quadrilateral.

$\therefore m(\angle C) = 180° - 70° = 110°$

In Δ BCD : $m(\angle DBC) = 180° - (110° + 20°) = 50°$

$\therefore \frac{10}{\sin 20°} = \frac{CD}{\sin 50°} = \frac{BD}{\sin 110°}$

$\therefore CD \simeq 22.4$ cm. ,

$BD \simeq 27.47$ cm.

$\therefore$ The perimeter of (Δ BDC)

$= 22.4 + 27.47 + 10$

$\simeq 60$ cm.

Model 3

1 (c)

2 (c)

Solution :

$\lim_{X \to \infty} \frac{7 X^{-2} + 5 X^{-1} - 1}{4 X^{-2} + 3 X^{-1} + 2} = \lim_{X \to \infty} \frac{\frac{7}{X^2} + \frac{5}{X} - 1}{\frac{4}{X^2} + \frac{3}{X} + 2} = \frac{-1}{2}$

3 (c)

Solution :

$\because m(\angle D) + m(\angle H) + m(\angle E) = 180°$

$\therefore D + H = 180° - E$

$\therefore \cos (D + H) = \cos (180° - E) = -\cos E$

$= -\sin (90° + E)$

4 (b)

5 (b)

Solution :

$\because$ limit is exists and the denominator $= 0$ at $X = 4$

$\therefore$ The numerator $= 0$ at $X = 4$

$\therefore 16 + 16 k = 0$ $\qquad \therefore k = -1$

6 (a)

Solution :

$\because 5 \times 3^{X-2} = 45$

$\therefore 3^{X-2} = 9 = 3^2$

$\therefore X - 2 = 2$ $\qquad \therefore X = 4$

$\therefore 7^{2-X} = 7^{2-4} = 7^{-2} = \frac{1}{49}$

7 (b)

Solution :

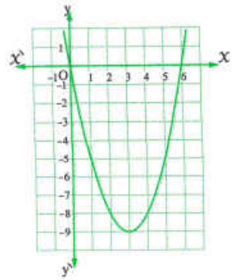
$\lim_{X \to 0} \left(\cos 3 X - \frac{\tan 4 X}{X}\right) = 1 - 4 = -3$

8 (d)

Solution :

From the graph

the range $= [-9 , \infty[$



9 (a)

Solution :

$\log_{Xy} \frac{1}{X} + \log_{Xy} \frac{1}{y} = \log_{Xy} \frac{1}{Xy} = \log_{Xy} (X y)^{-1}$

$= -\log_{Xy} X y = -1$

10 (a)

11 (c)

Solution :

$\lim_{X \to 3} \frac{|X - 6| - 3}{X^2 - 9} = \lim_{X \to 3} \frac{-X + 6 - 3}{(X - 3)(X + 3)}$

$= \lim_{X \to 3} \frac{-(X - 3)}{(X - 3)(X + 3)} = \frac{-1}{6}$

12 (b)

Solution :

$m(\angle Z) = 180° - (75° + 45°) = 60°$

$\therefore z : y = \sin z : \sin y$

$= \sin 60° : \sin 45° = \sqrt{6} : 2$

13 (d)

Solution :

$f(X) = \frac{|X - 2|}{X - 2} = \begin{cases} 1 & , X > 2 \\ -1 & , X < 2 \end{cases}$

$\therefore$ The range $= \{1 , -1\}$

14 (c)

Solution :

$\because \lim_{x \to -2} f(x)$ is exist. $\therefore f(-2^+) = f(-2^-)$

$\therefore 3(-2) - 2 = k(-2) - 6$

$\therefore -8 = -2k - 6$ $\therefore k = 1$

---

15 (c)

Solution :

$|2x - 1| \le 5$ $\therefore 5 \le 2x - 1 \le 5$

$\therefore -4 \le 2x \le 6$ $\therefore 2 \le x \le 3$

$\therefore$ S.S. $= [-2, 3]$

---

16 (b)

Solution :

$\lim_{x \to 0} \frac{2x^2 + \tan 3x}{5x + \sin 7x} = \lim_{x \to 0} \frac{2x + \frac{\tan 3x}{x}}{5 + \frac{\sin 7x}{x}}$

$= \frac{3}{5+7} = \frac{1}{4}$

---

17 (b)

Solution :

$c = 70 - (23 + 15) = 32$ cm.

$\because$ c is the greatest side in length

$\therefore (\angle C)$ is the greatest angle in measure

$\therefore \cos C = \frac{(23)^2 + (15)^2 - (32)^2}{2 \times 23 \times 15}$

$\therefore m(\angle C) \approx 113° \grave{2}$

---

18 (a)

---

19 (c)

Solution :

$\frac{f(x+2)}{f(x-2)} + \frac{f(2x+1)}{f(2x-1)} = \frac{3^{x+2+1}}{3^{x-2+1}} + \frac{3^{2x+1+1}}{3^{2x-1+1}}$

$= \frac{3^{x+3}}{3^{x-1}} + \frac{3^{2x+2}}{3^{2x}}$

$= 3^4 + 3^2 = 90$

---

20 (b)

Solution :

by dividing each of numerator and denominator by x^2

We get : $\lim_{x \to \infty} \frac{2 + \frac{3}{x} - \frac{1}{x^2}}{\frac{4}{x} - 5 + \frac{2}{x^2}} = \frac{-2}{5}$

21 (a)

Solution :

$\because \angle M$ is acute angle

$\therefore h = 20 \sin 60° = 10\sqrt{3}$ cm.

$, \because 10\sqrt{3} < 20 < m$

$\therefore$ The right choice is m = 21 cm.

---

22 (d)

Solution :

$\because f^{-1}(3) = 1$ $\therefore f(1) = 3$

$\therefore \log_c (7(1) + 1) = 3$

$\therefore C^3 = 8$ $\therefore C = 2$

---

23 (a)

Solution :

$\lim_{x \to \frac{\pi}{2}} \frac{6 \cos x}{2x - \pi} = \lim_{x \to \frac{\pi}{2}} \frac{6 \sin\left(\frac{\pi}{2} - x\right)}{2\left(x - \frac{\pi}{2}\right)}$

$= -3 \lim_{x \to \frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\frac{\pi}{2} - x} = -3 \times 1 = -3$

$\therefore f(x)$ to be continuous at $x = \frac{\pi}{2}$

$\therefore f\left(\frac{\pi}{2}\right) = \lim_{x \to \frac{\pi}{2}} f(x)$ *i.e.* $f\left(\frac{\pi}{2}\right) = -3$

---

24 (a)

Solution :

$\lim_{x \to 1} \frac{\sqrt[5]{x} + \sqrt[3]{x} - 2}{x^2 - 1} = \lim_{x \to 1} \frac{x^{\frac{1}{5}} + x^{\frac{1}{3}} - 1 - 1}{x^2 - 1}$

$= \lim_{x \to 1} \frac{x^{\frac{1}{5}} - 1^{\frac{1}{5}}}{x^2 - 1^2} + \lim_{x \to 1} \frac{x^{\frac{1}{3}} - 1^{\frac{1}{3}}}{x^2 - 1^2}$

$= \frac{1}{10} \times (1)^{\frac{1}{5} - 2} + \frac{1}{6} \times (1)^{\frac{1}{3} - 2} = \frac{4}{15}$

---

25 (d)

Solution :

$\frac{y}{\sin Y} = \frac{z}{\sin Z} = 2r = 8$ $\therefore \frac{yz}{\sin Y \sin Z} = 8 \times 8 = 64$

26 (a)

Solution :

$\because 8^y = 9$ $\therefore 2^{3y} = 9 = 3^2$ (1)

$, \because 4^x = 3$ $\therefore 2^{2x} = 3$ (2)

From (1) , (2) : $2^{3y} = (2^{2x})^2$

$\therefore 3y = 4x$ *i.e.* $x = \frac{3}{4} y$

$\therefore \frac{x+y}{x-y} = \frac{\frac{3}{4}y + y}{\frac{3}{4}y - y} = -7$

---

27 (a)

Solution :

$a = BC = \sqrt{(-2+1)^2 + (5+2)^2} = 5\sqrt{2}$ unit length

$, b = AC = \sqrt{(4+1)^2 + (3+2)^2} = 5\sqrt{2}$ unit length

$, c = AB = \sqrt{(4+2)^2 + (3-5)^2} = 2\sqrt{10}$ unit length

$\therefore \cos \angle B = \frac{\left(5\sqrt{2}\right)^2 + \left(2\sqrt{10}\right)^2 - \left(5\sqrt{2}\right)^2}{2 \times 5\sqrt{2} \times 2\sqrt{10}} = \frac{\sqrt{5}}{5}$

---

28 (b)

Solution :

From the graph g (0) = 2

, then $f(2) = 0$

$\therefore (f \circ g)(0) = 0$

Model 4

1 (b)

---

2 (a)

Solution :

$f(-x) = \frac{\sin(-x)}{-x} = \frac{-\sin x}{-x} = \frac{\sin x}{x} = f(x)$

$\therefore f(-x) = f(x)$

$\therefore$ The function is even

---

3 (c)

Solution :

$\log 15 = \log(3 \times 5) = \log 3 + \log 5 = x + y$

---

4 (d)

5 (b)

Solution :

$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

$\therefore b^2 + c^2 - a^2 = 2bc \times \cos A$

$\because \cos A = \sin(90° - A)$

$\therefore b^2 + c^2 - a^2 = 2bc \times \sin(90° - A)$

---

6 (c)

---

7 (b)

Solution :

$\because \frac{a}{\sin A} = 2r$

$\therefore 2r = 6$ $\therefore r = 3$ cm.

---

8 (b)

---

9 (d)

Solution :

$f(1) = 2a$

$, \lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$

$= \lim_{x \to 1} \frac{(x-1)(x+1)}{x-1} = 2$

$\because$ The function is continuous at $x = 1$

$\therefore f(1) = \lim_{x \to 1} f(x)$ $\therefore 2 = 2a$ $\therefore a = 1$

---

10 (c)

---

11 (c)

Solution :

$\because$ The triangle is equilateral.

$\therefore \frac{9}{\sin 60°} = 2r$ $\therefore r = 3\sqrt{3}$

$\therefore$ The area of the circle $= \pi r^2 = \pi\left(3\sqrt{3}\right)^2 = 27\pi$ cm^2.

---

12 (d)

Solution :

$\lim_{x \to 1} \frac{x^2 - k^2}{x + 2} = -1$ $\therefore \frac{1 - k^2}{1 + 2} = -1$

$\therefore 1 - k^2 = -3$ $\therefore k^2 = 4$

$\therefore k = \pm 2$

13 (b)

Solution :

by dividing both of numerator and denominator by x^2

We get : $\lim\limits_{x \to \infty} \dfrac{\frac{2}{x}+\frac{3}{x^2}}{5+\frac{4}{x^2}} = \frac{0}{5} = 0$

14 (b)

Solution :

$\lim\limits_{x \to 2} \dfrac{x^5 - 32}{x^2+3x-10} = \lim\limits_{x \to 2} \dfrac{x^5-2^5}{(x-2)(x+5)}$

$= \lim\limits_{x \to 2} \dfrac{x^5-2^5}{x-2} \times \lim\limits_{x \to 2} \dfrac{1}{x+5}$

$= 5 \times 2^4 \times \frac{1}{7} = \frac{80}{7}$

15 (c)

16 (b)

Solution :

$y = \sqrt[5]{x}$, by replacing the two variables

We get : $x = \sqrt[5]{y}$ $\quad \therefore x^5 = y$

$\therefore f^{-1}(x) = x^5$

17 (b)

18 (b)

19 (d)

Solution :

$\because 2 \sin A = 3 \sin B = 4 \sin C$ (divide by 12)

$\therefore \frac{\sin A}{6} = \frac{\sin B}{4} = \frac{\sin C}{3}$

$\therefore a : b : c = 6 : 4 : 3$

20 (c)

Solution :

$\because$ The function is odd. $\quad \therefore f(-a) = -f(a)$

$\therefore f(a) + f(-a) = f(a) + \left(-f(a)\right) = 0$

21 (b)

Solution :

let the measures of the angles of this triangle are

$8x , 3x , x$

$\therefore 8x + 3x + x = 180°$ $\quad \therefore x = 15°$

$\therefore$ The measure of the angles are 120° , 45° , 15°

$\therefore$ The ratio between the longest two sides of this triangle = sin 120° : sin 45°

$= \frac{\sqrt{3}}{2} : \frac{\sqrt{2}}{2} = \sqrt{6} : 2$

22 (d)

Solution :

$(f \circ g)(2) = f(g(2))$

$= f(3^2) = 4(3^2) - 5 = 31$

23 (c)

24 (c)

Solution :

$\frac{15\sqrt{3}}{\sin 60°} = \frac{d}{\sin 30°}$ $\quad \therefore d = 15$ cm.

25 (b)

26 (c)

27 (c)

Solution :

$\lim\limits_{x \to \frac{\pi}{2}} (2x - \cos x) = 2\left(\frac{\pi}{2}\right) - \cos 90°$

$= \pi - 0 = \pi$

28 (b)

Model 5

1 (d)

2 (c)

Solution :

$\frac{5\sqrt{3}}{\sin 60°} = 2r$ $\quad \therefore 2r = 10$ cm.

i.e. The diameter length = 10 cm.

3 (d)

Solution :

By dividing each of numerator and denominator by x

$\therefore \lim\limits_{x \to \infty} \dfrac{a+\frac{6}{x}}{2+\frac{7}{x}} = 4$

$\therefore \frac{a}{2} = 4$ $\quad \therefore a = 8$

4 (c)

5 (b)

Solution :

let the greatest angle θ

$\therefore \cos\theta = \frac{3^2+5^2-7^2}{2 \times 3 \times 5}$ $\quad \therefore m(\angle \theta) = 120°$

6 (d)

Solution :

$(f \circ g)(2) = f(g(2))$

$= f(9) = 4(9) - 5 = 31$

7 (b)

Solution :

$y = 7x$

by replacing the two variables

We get : $x = 7y$ $\quad \therefore y = \frac{1}{7}x = \frac{x}{7}$

8 (d)

9 (b)

Solution :

$\lim\limits_{x \to 3} \dfrac{x^2 - 7x + 12}{x-3} = \lim\limits_{x \to 3} \dfrac{\cancel{(x-3)}(x-4)}{\cancel{x-3}}$

$= 3 - 4 = -1$

10 (a)

Solution :

$\because \sqrt{x^2 - 4x + 4} > 0$ $\quad \therefore \sqrt{(x-2)^2} > 0$

$|x - 2| > 0$

For any value of $x \neq 2$

We get $|x - 2| > 0$ $\quad \therefore$ The solution set is $\mathbb{R} - \{2\}$

11 (a)

Solution :

$\because \angle A$ is an acute

, h = 3 sin 60 ≃ 2.6 cm.

, $\because a > b$

$\therefore$ There's only one solution.

12 (c)

Solution :

$\lim\limits_{x \to 0} \dfrac{\sqrt{x+1}-1}{x} \times \dfrac{\sqrt{x+1}+1}{\sqrt{x+1}+1}$

$= \lim\limits_{x \to 0} \dfrac{(x+1)-1}{x\left(\sqrt{x+1}+1\right)} = \lim\limits_{x \to 0} \dfrac{1}{\sqrt{x+1}+1} = \frac{1}{2}$

13 (b)

14 (a)

Solution :

$x^3 = 8 = 2^3$ $\quad \therefore x = 2$

15 (c)

Solution :

$\frac{15\sqrt{3}}{\sin 60°} = \frac{d}{\sin 30°}$ $\quad \therefore d = 15$ cm.

16 (c)

Solution :

$\lim\limits_{x \to 0} \dfrac{(x+2)^5 - 32}{x} = \lim\limits_{(x+2) \to 2} \dfrac{(x+2)^5 - 2^5}{(x+2)-2}$

$= 5 \times 2^{5-1} = 80$

17 (d)

18 (a)

Solution :

$f(x) = 4^{x-1}$

$\therefore f(x+1) = 4^{x+1-1} = 4^x$

19 (b)

Solution :

$a^2 = b^2 + c^2 - 2bc \cos A$

$\therefore a^2 = (2)^2 + (2.5)^2 - 2 \times 2 \times 2.5 \times \frac{2}{5}$

$\therefore a = 2.5$ cm.

$\therefore a = c = 2.5$ cm. , b = 2 cm.

$\therefore$ The triangle is isosceles.

20 (b)

Solution :

$\log_x (x + 6) = 2$

$\therefore x^2 = x + 6$ $\quad \therefore x^2 - x - 6 = 0$

$\therefore (x + 2)(x - 3) = 0$

$\therefore x = -2$ (refused) $\quad$ or $x = 3$

21 (b)

22 (c)

Solution :

$\frac{X}{\sin X} = 2r$ $\quad\therefore \frac{2X}{\sin X} = 4r$

23 (a)

Solution :

$f(0^-) = \lim_{X \to 0^-} \frac{\sin^2 2X}{X^2} = \lim_{X \to 0} \left(\frac{\sin 2X}{X}\right)^2 = 2^2 = 4$

, $f(0^+) = \lim_{X \to 0^+} (2a + 3\cos X) = 2a + 3\cos 0$

$= 2a + 3$

$\because \lim_{X \to 0} f(X)$ is exist.

$\therefore 2a + 3 = 4$ $\quad\therefore 2a = 1$ $\quad\therefore a = \frac{1}{2}$

24 (a)

Solution :

$\lim_{X \to 0} 5X \csc 2X = \lim_{X \to 0} \frac{5X}{\sin 2X} = \frac{5}{2}$

25 (d)

Solution :

$\log X + \log 5 = 2$

$\therefore \log 5X = 2$ $\quad\therefore 10^2 = 5X$

$\therefore 100 = 5X$ $\quad\therefore X = 20$

26 (d)

Solution :

$\lim_{X \to 1} \frac{X^2 + kX + m}{2X - 1} = 3$

$\therefore \frac{1 + k + m}{2 - 1} = 3$ $\quad\therefore k + m = 2$

27 (c)

28 (c)

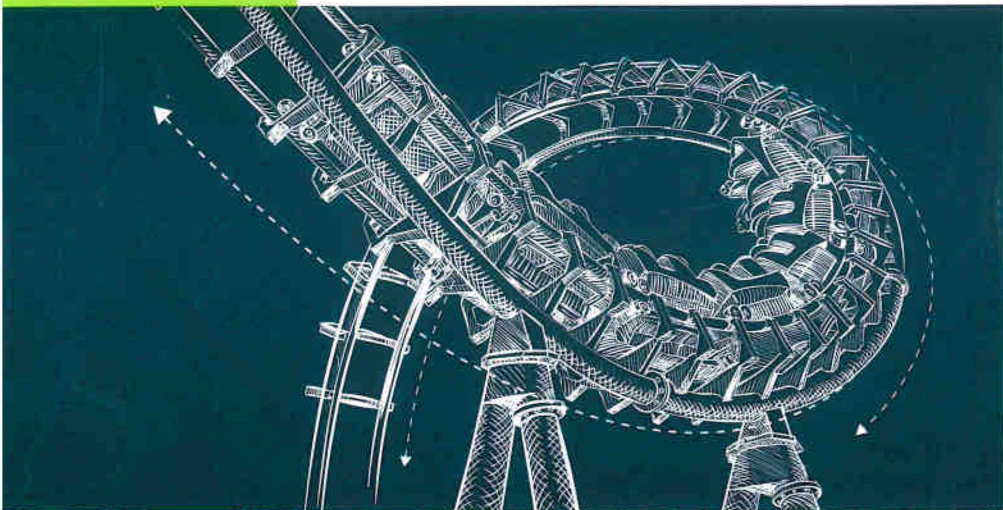
Solution :

sum $= 5000\left(1 + \frac{5}{100}\right)^7 = 7035.5$ L.E.

SCIENTIFIC SECTION

Pure Mathematics

By a group of supervisors



FIRST TERM

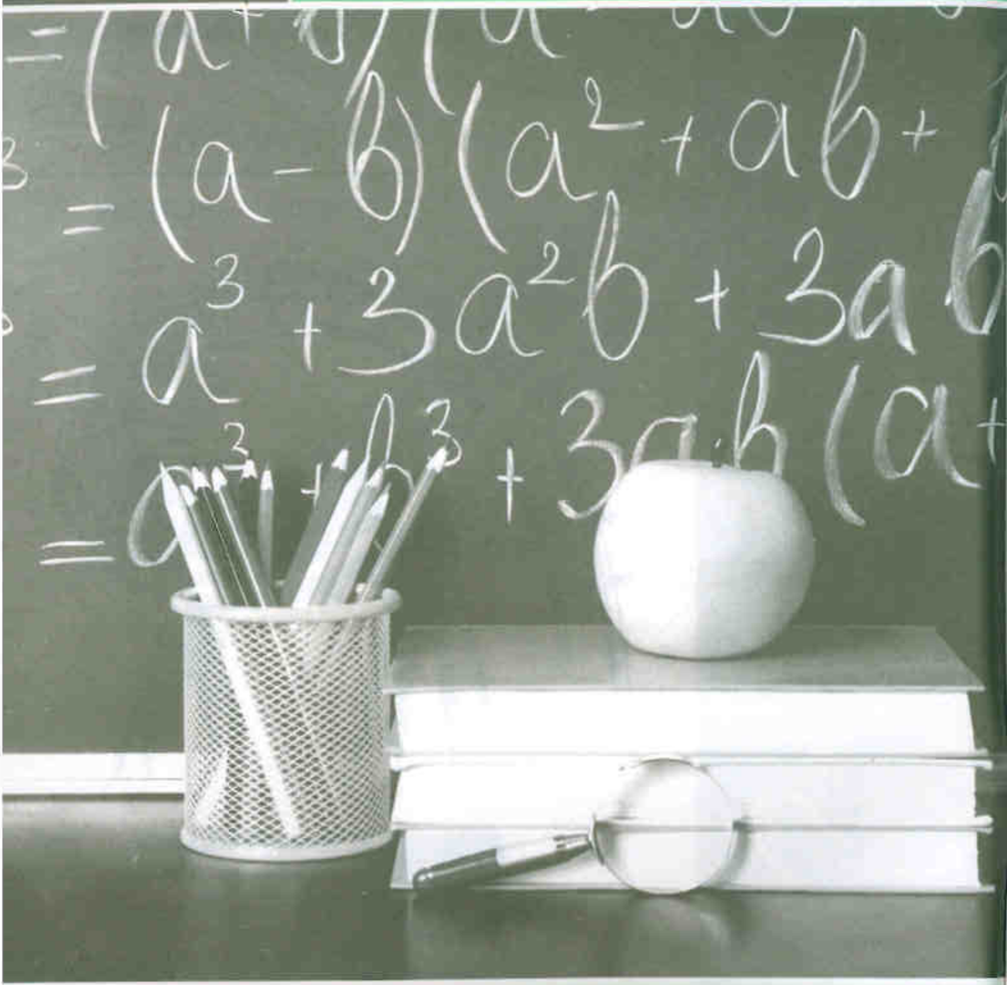
2 SEC.

2022

GUIDE ANSWERS

EL-MOASSER

FIRST Answers of Algebra



Answers of "Unit One"

Exercise on Pre-requirements

First Multiple choice questions

(1) d (2) b (3) d (4) c
(5) c (6) a (7) c

Second Essay questions

1

$\because f(x) = a^2 x^2 + 10\,a\,x + 25$

$, g(x) = 9x^2 + 30x + c - 4$

$, \because f(x) = g(x)$ $\therefore 10\,a = 30$ $\therefore a = 3$

$, c - 4 = 25$ $\therefore c = 29$

2

(1) $\because a + b = 5$ (1) $, a + c = 3$ (2)

$, b = -2$

by substituting the value of b in (1)

$\therefore a - 2 = 5$ $\therefore a = 7$

by substituting the value of a in (2)

$\therefore 7 + c = 3$ $\therefore c = -4$

(2) $\because a + b = 1$ (1)

$, a + c = 0$ (2)

$, b = -2$ $\therefore a - 2 = 1$ $\therefore a = 3$

$\therefore 3 + c = 0$ $\therefore c = -3$

(3) $\because a + 2b = 7$ (1)

$, -c = 5$ $\therefore c = -5$ $, a - b = 4$ (2)

From (1) , (2) : $\therefore b = 1 , a = 5$

Exercise 1

First Multiple choice questions

(1) c (2) d (3) d (4) c (5) a (6) d
(7) d (8) b (9) a (10) d (11) c (12) b
(13) c (14) d (15) c (16) b (17) b (18) a
(19) b (20) d (21) a (22) d (23) a (24) d
(25) b (26) b (27) c (28) c (29) d (30) b
(31) c (32) d (33) c (34) d (35) b (36) c
(37) d (38) d (39) a (40) b (41) c (42) c
(43) c (44) First : b Second : c (45) b
(46) d (47) d (48) c

Second Essay questions

1

(1) Function. (2) Not function.
(3) Function. (4) Not function.
(5) Function. (6) Not function.
(7) Function. (8) Not function.
(9) Function. (10) Function.

2

(1) Function. (2) Not function.
(3) Not function. (4) Function.
(5) Not function. (6) Function.

3

(1) $\mathbb{R} - \{-3 , 3\}$ (2) $\mathbb{R} - \{1 , 2\}$
(3) $\mathbb{R} - \{0\}$ (4) $\mathbb{R} - \{3\}$
(5) $\mathbb{R} - \left\{1 , \frac{-2}{3}\right\}$ (6) $\mathbb{R} - \{-1\}$

4

(1) $\mathbb{R} - \left\{\frac{5}{2}\right\}$ (2) $]3 , \infty[$
(3) $]-\infty , 1[$ (4) $\mathbb{R} -]-4 , 4[$
(5) $[-2 , 2]$ (6) $]-3 , 3[$
(7) $\mathbb{R}$ (8) $\mathbb{R} - \{2\}$
(9) $[0 , \infty[- \{1\}$

5

(1) $\mathbb{R}$ (2) $]-\infty , 4[$ (3) $[0 , 6]$

6

(1) The domain $= \mathbb{R}$, the range $= \{2\}$

, the function is constant on its domain $\mathbb{R}$

(2) The domain $= \mathbb{R}$, the range $= \{-2 , 3\}$

, the function is constant on $]-\infty , 0[,]0 , \infty[$

(3) The domain $= \mathbb{R} - \{0\}$, the range $= \mathbb{R} - [-2 , 2]$

, the function is decreasing on $]-\infty , 0[,]0 , \infty[$

(4) The domain $= \mathbb{R} - \{1\}$, the range $=]-2 , \infty[$
, the function is decreasing on $]-\infty , 1[$
, increasing on $]1 , \infty[$

(5) The domain $= \mathbb{R} - \{2\}$, the range $= \mathbb{R} - \{2\}$
, the function is increasing on $]-\infty , 2[,]2 , \infty[$

(6) The domain $= [0 , \infty[$, the range $= [0 , 4]$
, the function is increasing on $]0 , 2[$
, constant on $]2 , \infty[$

(7) The domain $= \mathbb{R} - \{-1 , 2\}$, the range $= \{3\}$
, the function is constant on its domain.

(8) The domain $= \mathbb{R} -]0 , 1]$, the range
$=]3 , \infty[\cup \{2\}$, constant on $]-\infty , 0[$
and increasing on $]1 , \infty[$

(9) The domain $= \mathbb{R}$, the range $=]-\infty , 4]$
, the function is increasing on $]-\infty , -2[,]0 , 2[$
and decreasing on $]-2 , 0[,]2 , \infty[$

(10) The domain $= \mathbb{R}$, the range $= \mathbb{R} -]1 , 2]$
, the function is decreasing on $]-\infty , 0[,]1 , \infty[$
and increasing on $]0 , 1[$

(11) The domain $= \mathbb{R} - \{0\}$, the range $= \mathbb{R} - [-2 , 2]$
, the function is increasing on $]-\infty , 0[,]0 , \infty[$

(12) The domain $= [-2 , \infty[$, the range $= [0 , \infty[$
, the function is increasing on $]-2 , 0[$ and
decreasing on $]0 , 2[,]2 , \infty[$

Third Higher skills

(1) (d) (2) (d) (3) (d)
(4) (c) (5) (a) (6) (d)

Instructions to solve :

(1) $\because X$ is the number of sides
$\therefore X$ is an integer more than 2
$\therefore$ The domain $= \mathbb{Z}^+ - \{1 , 2\}$

(2) Put $\sqrt[3]{X} - 2 = 0$ $\therefore \sqrt[3]{X} = 2$ $\therefore X = 8$
$\therefore$ The domain $= \mathbb{R} - \{8\}$

(3) $\because 3X \geq 0$ $\therefore X \geq 0$ $\therefore X \in [0 , \infty[$
, Put $\sqrt{3X} - X = 0$ $\therefore \sqrt{3X} = X$
$\therefore 3X = X^2$ $\therefore X^2 - 3X = 0$
$\therefore X(X-3) = 0$ $\therefore X = 0$ or $X = 3$
$\therefore$ The domain $=]0 , \infty[- \{3\}$

(4) $\because X - 1 \geq 0$ $\therefore X \geq 1$ $\therefore X \in [1 , \infty[$
, Put $\sqrt{X-1} = 3$ $\therefore X - 1 = 9$ $\therefore X = 10$
$\therefore$ The domain $= [1 , \infty[- \{10\}$

(5) $\because X - 1 \geq 0$ $\therefore X \geq 1$ $\therefore X \in [1 , \infty[$
Put $\sqrt{X-1} + 3 = 0$ (contradiction)
$\therefore$ The domain $= [1 , \infty[$

(6) $\because f(X) = \sqrt{-(X^2 - 6X + 9)} = \sqrt{-(X-3)^2}$
$\therefore -(X-3)^2 \geq 0$ $\therefore (X-3)^2 \leq 0$
which is not true unless they are equal
i.e. $X = 3$ $\therefore$ The domain $= \{3\}$

Exercise 2

First Multiple choice questions

(1) a	(2) a	(3) b	(4) b	(5) c	(6) c
(7) a	(8) d	(9) c	(10) c	(11) d	(12) d
(13) c	(14) c	(15) d	(16) b	(17) b	(18) d
(19) b	(20) c	(21) a	(22) c	(23) d	(24) c
(25) b	(26) b	(27) c	(28) b	(29) d	(30) d
(31) a	(32) b	(33) a	(34) b	(35) a	(36) c
(37) a	(38) c	(39) b	(40) d	(41) c	(42) b
(43) d	(44) b	(45) b			

Second Essay questions

1

(1) $(f+g)(X) = X^2 - 16$
, the domain $= [-4 , 8] \cap [-7 , 4] = [-4 , 4]$

(2) $(f-g)(X) = X^2 - 2X - 8$
, the domain $= [-4 , 4]$

(3) $(f \cdot g)(X) = (X-4)(X^2 - X - 12)$
, the domain $= [-4 , 4]$

(4) $\left(\frac{f}{g}\right)(X) = \frac{X^2 - X - 12}{X-4} = \frac{(X-4)(X+3)}{X-4} = X + 3$
, the domain $= [-4 , 4[$

(5) $\left(\frac{g}{f}\right)(X) = \frac{X-4}{(X-4)(X+3)} = \frac{1}{X+3}$
, the domain $= [-4 , 4[- \{-3\}$

2

$f(x) = x^2 - 4$, its domain $= \mathbb{R}$

, $g(x) = \sqrt{x-1}$, its domain $= [1, \infty[$

$\therefore$ The common domain $= [1, \infty[$

(1) $(f+g)(x) = x^2 - 4 + \sqrt{x-1}$

, the domain $= [1, \infty[$

$(f \cdot g)(x) = (x^2 - 4)\left(\sqrt{x-1}\right)$

, the domain $= [1, \infty[$

, $\left(\frac{f}{g}\right)(x) = \frac{x^2-4}{\sqrt{x-1}}$, the domain $=]1, \infty[$

, $\left(\frac{g}{f}\right)(x) = \frac{\sqrt{x-1}}{x^2-4}$, the domain $= [1, \infty[- \{2\}$

(2) $(f+g)(5) = 5^2 - 4 + \sqrt{5-1} = 23$

, $(f \cdot g)(2) = (2^2 - 4)\left(\sqrt{2-1}\right) = 0$

, $\left(\frac{f}{g}\right)(3) = \frac{3^2-4}{\sqrt{3-1}} = \frac{5\sqrt{2}}{2}$

, $\left(\frac{g}{f}\right)(-2)$ is not defined because $-2 \notin$ domain.

3

First :

(1) $(f+g)(x) = x^2 - 4x + \sqrt{x+2}$

, the domain $= \mathbb{R} \cap [-2, \infty[= [-2, \infty[$

(2) $(g-h)(x) = \sqrt{x+2} - \sqrt{4-x}$

, the domain $= [-2, \infty[\cap]-\infty, 4] = [-2, 4]$

(3) $(f \cdot h)(x) = (x^2 - 4x)\left(\sqrt{4-x}\right)$

, the domain $= \mathbb{R} \cap]-\infty, 4] =]-\infty, 4]$

(4) $\left(\frac{h}{f}\right)(x) = \frac{\sqrt{4-x}}{x^2-4x} = \frac{\sqrt{4-x}}{x(x-4)}$

, the domain $=]-\infty, 4[- \{0\}$

Second :

(1) $(g-h)(1) = \sqrt{1+2} - \sqrt{4-1} = 0$

(2) $(f \cdot h)(5)$ is not defined because $5 \notin$ domain.

(3) $\left(\frac{h}{f}\right)(3) = \frac{\sqrt{4-3}}{3^2 - 4 \times 3} = -\frac{1}{3}$

4

(1) $\mathbb{R} - \{0, -2\}$ (2) $\mathbb{R} - \{-1, 1\}$

(3) $]-\infty, 1[$ (4) $]\frac{1}{2}, \infty[$

(5) $[2, \infty[$ (6) $[4, \infty[- \{6\}$

(7) $[3, \infty[$ (8) $]5, \infty[$

5

(1) $(f \cdot g)(x) = \begin{cases} x^2 - 2x & , x \ge 2 \\ -x^2 + 2x & , x < 2 \end{cases}$

, the domain $= \mathbb{R}$

(2) $\left(\frac{f}{g}\right)(x) = \begin{cases} 1 - \frac{2}{x} & , x \ge 2 \\ -1 + \frac{2}{x} & , x < 2 , x \ne 0 \end{cases}$

, the domain $= \mathbb{R} - \{0\}$

(3) $\left(\frac{g}{f}\right)(x) = \begin{cases} \frac{x}{x-2} & , x > 2 \\ \frac{x}{-x+2} & , x < 2 \end{cases}$

, the domain $= \mathbb{R} - \{2\}$

6

(1) $\mathbb{R} -]3, 5]$ (2) $[2, \infty[- \{3\}$

(3) $[-2, \infty[- \{7, -1\}$ (4) $\mathbb{R} - \left\{1, 2, \frac{7}{4}\right\}$

7

(1) $(f \circ g)(2) = f\left(g(2)\right) = f(2^2 - 5)$

$= f(-1) = 3 \times (-1) + 1 = -2$

(2) $(g \circ f)(-3) = g\left(f(-3)\right) = g\left(3(-3) + 1\right)$

$= g(-8) = (-8)^2 - 5 = 59$

(3) $(g \circ k)(1) = g\left(k(1)\right) = g(1^3)$

$= g(1) = 1^2 - 5 = -4$

(4) $(k \circ f)(-2) = k\left(f(-2)\right) = k\left(3(-2) + 1\right)$

$= k(-5) = (-5)^3 = -125$

8

(1) $(f \circ g)(x) = f\left(g(x)\right) = f(x+3) = \frac{1}{x+3}$

$\because D_1 =$ domain of $g = \mathbb{R}$

, the value of x which makes $g(x)$ in the domain of f is $D_2 = \mathbb{R} - \{-3\}$

$\therefore$ The domain of $(f \circ g) = D_1 \cap D_2 = \mathbb{R} - \{-3\}$

(2) $(g \circ f)(x) = g\left(f(x)\right) = g\left(\frac{1}{x}\right) = \frac{1}{x} + 3$

, $D_1 =$ domain of $f = \mathbb{R} - \{0\}$

, the value of x which makes $f(x)$ in the domain of g is $D_2 = \mathbb{R} - \{0\}$

$\therefore$ The domain of $(g \circ f)(x) = D_1 \cap D_2 = \mathbb{R} - \{0\}$

9

(1) $(f \circ g)(x) = f(g(x)) = f(x^2) = \frac{1}{x^2 - 1}$

, $\because D_1$ = domain of $g = \mathbb{R}$

, $\because$ the value of x which makes $g(x)$ in the domain of f is $D_2 = \mathbb{R} - \{-1, 1\}$

$\therefore$ The domain of $(f \circ g) = D_1 \cap D_2 = \mathbb{R} - \{-1, 1\}$

(2) $(g \circ f)(x) = g(f(x)) = g\left(\frac{1}{x-1}\right) = \left(\frac{1}{x-1}\right)^2$

$= \frac{1}{x^2 - 2x + 1}$

, $\because D_1$ = the domain of $f = \mathbb{R} - \{1\}$

, the value of x which makes $f(x)$ in the domain of g is $D_2 = \mathbb{R} - \{1\}$

$\therefore$ The domain of $(g \circ f) = D_1 \cap D_2 = \mathbb{R} - \{1\}$

10

(1) $\because (f \circ g)(x) = f(g(x)) = f(3x) = (3x)^2 + 6$

$= 9x^2 + 6$

$\therefore (f \circ g)(3) = 9 \times 3^2 + 6 = 87$

(2) $\because 9x^2 + 6 = 42$ $\therefore 9x^2 = 36$

$\therefore x^2 = 4$ $\therefore x = \pm 2$

11

$\because (f \circ g)(x) = f(g(x)) = f(\sqrt{x-2})$

$= (\sqrt{x-2})^2 - 3 = x - 2 - 3 = x - 5$

, $\because D_1$ = domain of $g = [2, \infty[$

, the value of x which makes $g(x)$ in the domain of f is $D_2 = \mathbb{R}$

$\therefore$ The domain $(f \circ g) = D_1 \cap D_2 = [2, \infty[$

, $(f \circ g)(3) = 3 - 5 = -2$

12

$\because (f \circ g)(x) = f(g(x)) = f\left(\frac{2}{x-3}\right) = \sqrt{\frac{2}{x-3} + 1}$

$= \sqrt{\frac{x-1}{x-3}}$

$\because D_1$ = domain of $g = \mathbb{R} - \{3\}$

, put $\frac{x-1}{x-3} \geq 0$

$\therefore$ The value of x which makes $g(x)$ in the domain of f is $D_2 = \mathbb{R} -]1, 3]$

$\therefore$ The domain of $(f \circ g) = D_1 \cap D_2 = \mathbb{R} -]1, 3]$

13

(1) $\because (f \circ g)(x) = f(g(x)) = f(\sqrt{4-x})$

$= \sqrt{\sqrt{4-x} - 2}$

, $\because D_1$ = domain of $g =]-\infty, 4]$

, put $\sqrt{4-x} - 2 \geq 0$ $\therefore \sqrt{4-x} \geq 2$

$\therefore 4 - x \geq 4$ $\therefore x \leq 0$

$\therefore$ The value of x which makes $g(x)$ in the domain of f is $D_2 =]-\infty, 0]$

$\therefore$ The domain of $(f \circ g) = D_1 \cap D_2 =]-\infty, 0]$

(2) $\because (g \circ f)(x) = g(f(x)) = g(\sqrt{x-2})$

$= \sqrt{4 - \sqrt{x-2}}$

, D_1 = the domain of $f = [2, \infty[$, put $4 - \sqrt{x-2} \geq 0$

$\therefore \sqrt{x-2} \leq 4$ $\therefore x - 2 \leq 16$

$\therefore x \leq 18$

$\therefore$ The value of x which makes $f(x)$ in the domain of g is $D_2 =]-\infty, 18]$

$\therefore$ The domain of $(g \circ f) = D_1 \cap D_2 = [2, 18]$

14

(1) $\because (f \circ g)(x) = f(g(x)) = f(\sqrt{x^2 - 4})$

$= \sqrt{\sqrt{x^2 - 4} - 1}$

, put $x^2 - 4 \geq 0$

$\therefore D_1$ = domain of $g = \mathbb{R} -]-2, 2[$

, put $\sqrt{x^2 - 4} - 1 \geq 0$ $\therefore \sqrt{x^2 - 4} \geq 1$

$\therefore x^2 - 4 \geq 1$ $\therefore x^2 - 5 \geq 0$

, the value of x which makes $g(x)$ in the domain of f is $D_2 = \mathbb{R} -]-\sqrt{5}, \sqrt{5}[$

$\therefore$ The domain of $(f \circ g) = D_1 \cap D_2$

$= \mathbb{R} -]-\sqrt{5}, \sqrt{5}[$

(2) $\because (g \circ f)(x) = g(f(x))$

$= g(\sqrt{x-1}) = \sqrt{(\sqrt{x-1})^2 - 4}$

$= \sqrt{x - 1 - 4} = \sqrt{x-5}$

, $\because D_1$ = domain of $f = [1, \infty[$

, the value of x which makes $f(x)$ in the domain of g is $D_2 = [5, \infty[$

$\therefore$ The domain of $(g \circ f) = D_1 \cap D_2 = [5, \infty[$

15

$f(x) = \sqrt{x-4}$, $g(x) = x^3$

(There are other solutions)

16

Let $f(x) = a x + b$

$\because (f \circ f)(x) = 16x + 15$

$\therefore f\left(f(x)\right) = 16x + 15$

$\therefore f(a x + b) = 16x + 15$

$\therefore a(a x + b) + b = 16x + 15$

$\therefore a^2 x + a b + b = 16x + 15$

$\therefore a^2 = 16$ and so $a = \pm 4$

, $a b + b = 15$ $\therefore b = \frac{15}{a+1}$

at $a = 4$ $\therefore b = 3$

, at $a = -4$ $\therefore b = -5$

$\therefore f(x) = 4x + 3$ or $f(x) = -4x - 5$

Third Higher skills

1

(1) (d) (2) (d) (3) (d) (4) (c) (5) (c)
(6) (d) (7) (a) (8) (d) (9) (c)

Instructions to solve 1 :

(1) $\because$ The domain of $f = [1, \infty[$, the domain of $g =]-\infty, 1]$

$\therefore$ The domain of $(f+g) = [1, \infty[\cap]-\infty, 1] = \{1\}$

(2) When $0 < x < 1$

$(f+g)(x) = f(x) + g(x) = (-x+2) + (x+2)$
$= 4$

(3) $(g \circ f)(-1) = g\left(f(-1)\right)$

[Notice that : $f(-1) = (-1)^2 + 1 = 2$]

$= g(2) = (2)^2 + 2 = 6$

(4) $\because f\left(g(x)\right) = f(x)$

$\therefore f(2x+3) = f(x)$ $\therefore (2x+3)^2 = x^2$

$\therefore 4x^2 + 12x + 9 - x^2 = 0$

$\therefore 3x^2 + 12x + 9 = 0$ $\therefore x^2 + 4x + 3 = 0$

$\therefore (x+1)(x+3) = 0$ $\therefore x = -1$ or $x = -3$

$\therefore$ The solution set $= \{-1, -3\}$

(5) $\because (f \circ g)(x) = 3x + 2$

$\therefore f\left(g(x)\right) = 3x + 2$ $\therefore 2g(x) + 1 = 3x + 2$

$\therefore 2g(x) = 3x + 1$ $\therefore g(x) = \frac{3}{2}x + \frac{1}{2}$

(6) $\because (g \circ f)(x) = x + 2$

$\therefore g\left(f(x)\right) = x + 2$ $\therefore g(2x - 3) = x + 2$

Put $2x - 3 = y$

$\therefore x = \frac{y+3}{2}$ $\therefore g(y) = \frac{y+3}{2} + 2 = \frac{y+7}{2}$

$\therefore g(x) = \frac{x+7}{2}$

(7) $\because (g \circ f)(0) = (f \circ g)(1)$

$\therefore g\left(f(0)\right) = f\left(g(1)\right)$

Notice that : $f(0) = 2$, $g(1) = \frac{1-k}{2}$

$\therefore g(2) = f\left(\frac{1-k}{2}\right)$

$\therefore \frac{2-k}{5} = \frac{1-k}{2} + 2$ $\therefore \frac{2-k}{5} = \frac{5-k}{2}$

$\therefore 4 - 2k = 25 - 5k$

$\therefore 3k = 21$ $\therefore k = 7$

(8) $\because (f \circ f \circ f)(1) = f\left(f\left(f(1)\right)\right)$

[From the graph $f(1) = 5$]

$= f\left(f(5)\right)$ [From the graph $f(5) = 0$]

$= f(\text{zero})$ [From the graph $f(0) = 2$]

(9) $(f \circ h)(3) - (f \circ h)(8) = f\left(h(3)\right) - f\left(h(8)\right)$

$= f(2) - f(4)$

$= 4 - 8 = -4$

2

(1) $(f \circ f)(x) = f\left(f(x)\right) = f\left(\frac{1}{x}\right) = 1 \div \frac{1}{x} = x$

(2) $(f \circ f \circ f)(x) = f\left(f\left(f(x)\right)\right) = f\left(f\left(\frac{1}{x}\right)\right)$

$= f(x) = \frac{1}{x}$

(3) From the previous , notice that

the double composite of f gives x and the triple composite of f gives $\frac{1}{x}$ and so on

$\therefore (f \circ f \circ f \circ \ldots$ to n times$)(x)$

$= \begin{cases} x \text{ if n is even} \\ \frac{1}{x} \text{ if n is odd} \end{cases}$

Exercise 3

First Multiple choice questions

(1) d	(2) a	(3) a	(4) a	(5) b	(6) a
(7) c	(8) b	(9) b	(10) a	(11) c	(12) a
(13) d	(14) a	(15) d	(16) b	(17) b	(18) d
(19) d	(20) b	(21) c	(22) c	(23) a	(24) c
(25) b	(26) c	(27) c	(28) c	(29) b	(30) d
(31) d	(32) c	(33) a	(34) c	(35) a	(36) d
(37) c	(38) c	(39) a	(40) a	(41) b	(42) d
(43) d	(44) d	(45) b	(46) b		

Second Essay questions

1

Figure (1) : symmetric about X-axis , y-axis and the origin point.

Figure (2) : symmetric about X-axis.

Figure (3) : symmetric about the origin point.

Figure (4) : symmetric about the origin point.

Figure (5) : symmetric about the y-axis.

Figure (6) : symmetric about the origin point.

2

(1) Odd	(2) Neither even nor odd
(3) Even	(4) Neither even nor odd
(5) Odd	(6) Odd

3

Figure (1) :

* The range $= \{-2 , 2\}$ * Neither even nor odd

Figure (2) :

* The range $= \mathbb{R} - [-2 , 2]$ * odd

Figure (3) :

* The range $=]-\infty , 3]$ * even

Figure (4) :

* The range $= [0 , 4]$ * even

Figure (5) :

* The range $= \mathbb{R} - \{0\}$ * odd

Figure (6) :

* The range $= \mathbb{R}$ * odd

4

Figure (1) :

$f(X) = X^3 + X$

$\because$ The domain $= \mathbb{R}$

, the curve is symmetric about the origin point.

$\therefore$ The function f is odd.

algebraically verifying :

$\forall X , -X \in \mathbb{R}$

$\because f(-X) = (-X)^3 + (-X) = -X^3 - X$

$= -(X^3 + X) = -f(X)$

$\therefore$ The function f is odd.

Figure (2) :

$f(X) = X^3 - 2$

$\because$ The domain $= \mathbb{R}$

, the curve is neither symmetric about y-axis nor symmetric about the origin point.

$\therefore$ The function f is neither even nor odd.

algebraically verifying :

$\forall X , -X \in \mathbb{R}$

$\because f(-X) = (-X)^3 - 2 = -X^3 - 2 = -(X^3 + 2)$

$\therefore f(-X) \neq f(X) \neq -f(X)$

$\therefore$ The function f is neither even nor odd.

Figure (3) :

$f(X) = 2 - X^2$

$\because$ The domain $= [-2 , 2]$

, the curve is symmetric about y-axis

$\therefore$ The function f is even.

algebraically verifying :

$\forall X , -X \in [-2 , 2]$

$\because f(-X) = 2 - (-X)^2 = 2 - X^2 = f(X)$

$\therefore$ The function is even.

5

First :

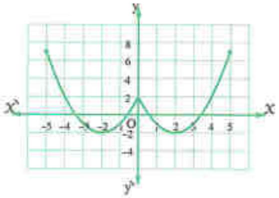


Fig. (1)

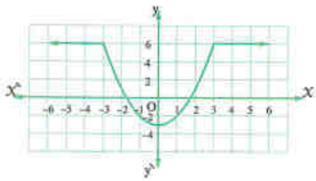


Fig. (3)

Second :

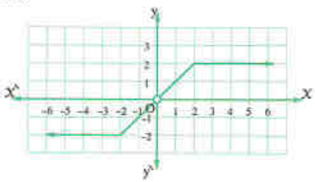


Fig. (2)

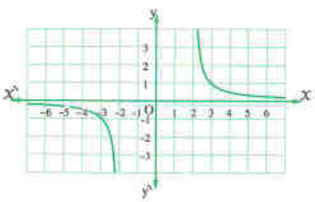


Fig. (4)

Third :

Figure (1) :

The domain $= [-5, 5]$, the range $= [-2, 7]$

, not one-to-one

Figure (2) :

The domain $= \mathbb{R} - \{0\}$, the range $= [-2, 2] - \{0\}$

, not one-to-one

Figure (3) :

The domain $= \mathbb{R}$, the range $= [-3, 6]$

, not one-to-one

Figure (4) :

The domain $= \mathbb{R} - [-2, 2]$, the range $= \mathbb{R} - \{0\}$

, one-to-one

6

(1) $f(-x) = 5 = f(x)$ $\quad \therefore f$ is even.

(2) $f(-x) = (-x)^4 + (-x)^2 - 1$

$= x^4 + x^2 - 1 = f(x)$

$\therefore f$ is even.

(3) $f(-x) = 3(-x) - 4(-x)^3 = -3x + 4x^3$

$= -(3x - 4x^3) = -f(x)$

$\therefore f$ is odd.

(4) $f(-x) = (-x)^2 - 3(-x) + 4$

$= x^2 + 3x + 4 \neq f(x) \neq -f(x)$

$\therefore f$ is neither even nor odd.

(5) $f(-x) = (-x)^3 \left((-x)^2 - 1\right)$

$= -x^3 (x^2 - 1) = -f(x)$

$\therefore f$ is odd.

(6) $f(-x) = (-x - 3)^2 - 7 = (x + 3)^2 - 7$

$\neq f(x) \neq -f(x)$

$\therefore f$ is neither even nor odd.

(7) $f(-x) = \dfrac{(-x)^2 + 2}{-x - 3}$

$= \dfrac{x^2 + 2}{-x - 3} = \dfrac{x^2 + 2}{-(x + 3)} \neq f(x) \neq -f(x)$

$\therefore f$ is neither even nor odd.

Another solution :

$\because$ The domain of the function $f = \mathbb{R} - \{-3\}$

$\therefore 3 \in$ the domain of the function

, $-3 \notin$ the domain

$\therefore$ The function is neither even nor odd.

(8) $f(-x) = \dfrac{2(-x)^3 - (-x)^5}{-x} = \dfrac{-2x^3 + x^5}{-x}$

$= \dfrac{2x^3 - x^5}{x} = f(x)$

$\therefore f$ is even.

(9) $f(-x)=\frac{(-x)^2}{1-x}=\frac{x^2}{1-x}\neq f(x)\neq -f(x)$

$\therefore$ f is neither even nor odd.

(10) The domain of $f=[-3,\infty[$

$\therefore$ for each $x\in[-3,\infty[$,

It is not necessary to find $-x\in[-3,\infty[$

$\therefore$ The function f is neither even nor odd.

(11) $f(-x)=\left((-x)^2+1\right)^3=(x^2+1)^3=f(x)$

$\therefore$ f is even.

(12) $f(-x)=\sqrt{(-x)^2+6}=\sqrt{x^2+6}=f(x)$

$\therefore$ f is even.

(13) $f(-x)=\sqrt[3]{(-x)^3+(-x)}=\sqrt[3]{-(x^3+x)}$
$=-\sqrt[3]{x^3+x}=-f(x)$

$\therefore$ f is odd.

(14) $f(-x)=(-x)^3-\frac{1}{-x}=-x^3+\frac{1}{x}=-\left(x^3-\frac{1}{x}\right)$
$=-f(x)$

$\therefore$ f is odd.

(15) $f(-x)=\left(-x-\frac{2}{-x}\right)^3=-\left(x-\frac{2}{x}\right)^3$
$=-f(x)$

$\therefore$ f is odd.

(16) $f(-x)=\left(\frac{(-x)^2}{3}-\frac{5}{(-x)^4}\right)^5$
$=\left(\frac{x^2}{3}-\frac{5}{x^4}\right)^5=f(x)$

$\therefore$ f is even.

(17) $f(-x)=\left(\frac{2+x}{3-x}\right)^2+\left(\frac{2-x}{3+x}\right)^2=f(x)$

$\therefore$ f is even.

(18) $f(-x)=\left(\frac{1+x}{1-x}\right)^2-\left(\frac{1-x}{1+x}\right)^2=-f(x)$

$\therefore$ f is odd.

(19) $f(-x)=\left(\frac{-x-1}{-x+1}\right)^5+\left(\frac{-x+1}{-x-1}\right)^5$
$=\left(\frac{x+1}{x-1}\right)^5+\left(\frac{x-1}{x+1}\right)^5=f(x)$

$\therefore$ f is even.

(20) $f(-x)=[(-x)^3+1]^4-[(-x)^3-1]^4$
$=[-(x^3-1)]^4-[-(x^3+1)]^4$
$=(x^3-1)^4-(x^3+1)^4=-f(x)$

$\therefore$ f is odd.

(21) $f(-x)=-x\cos(-x)=-x\cos x=-f(x)$

$\therefore$ f is odd.

(22) $f(-x)=\frac{-3x}{\tan(-x)}=\frac{-3x}{-\tan x}=\frac{3x}{\tan x}=f(x)$

$\therefore$ f is even.

(23) $f(-x)=\frac{(-x)^3\times\sin(-3x)}{1+(-x)^4}$
$=\frac{x^3\sin 3x}{1+x^4}=f(x)$

$\therefore$ f is even.

(24) $f(-x)=\frac{3(-x)^2-\cos(-x)}{(-x)^3+6x}=\frac{3x^2-\cos x}{-x^3+6x}$
$=\frac{3x^2-\cos x}{-(x^3-6x)}=-f(x)$

$\therefore$ f is odd.

(25) $f(-x)=(-x)^2\left(\sin(-x)\right)^3$
$=x^2(-\sin x)^3=-x^2\sin^3 x=-f(x)$

$\therefore$ f is odd.

(26) $f(-x)=-x\sin(-x)^3=-x(-\sin x^3)$
$=x\sin x^3=f(x)$

$\therefore$ f is even.

(27) $f(-x)=\frac{(-x)^2+\tan(-x)}{(-x)^4+\sin(-x)}$
$=\frac{x^2-\tan x}{x^4-\sin x}\neq f(x)\neq -f(x)$

$\therefore$ f is neither even nor odd.

(28) $f(-x)=\sin(-x)^2-\left(\sin(-x)\right)^2$
$=\sin x^2-(-\sin x)^2$
$=\sin x^2-\sin^2 x=f(x)$

$\therefore$ f is even.

(29) $f(-x)=(-x)^4+\sin^6(-x)$
$=x^4+(-\sin x)^6$
$=x^4+\sin^6 x=f(x)$

$\therefore$ The function is even.

(30) $f(-x)=(-x)^7+\tan^5(-x)$
$=-x^7+(-\tan x)^5$
$=-x^7-\tan^5 x=-f(x)$

$\therefore$ The function is odd.

(31) $f(-x)=\frac{\sin(-3x)\cos(-2x)}{\sec(-x)}$
$=\frac{-\sin 3x\cos 2x}{\sec x}=-f(x)$

$\therefore$ f is odd.

(32) $\because f(x) = 2^{2x} + \cos x + 2^{-2x}$

$\therefore f(-x) = 2^{-2x} + \cos(-x) + 2^{2x}$

$= 2^{-2x} + \cos x + 2^{2x} = f(x)$

$\therefore f$ is even.

(33) $f(x) = \left((\cos x + \sin x) - 1\right)\left((\cos x + \sin x) + 1\right)$

$= (\cos x + \sin x)^2 - 1$

$= \cos^2 x + \sin^2 x + 2\sin x \cos x - 1$

$= 1 + 2\sin x \cos x - 1 = 2\sin x \cos x$

$\therefore f(-x) = 2\sin(-x)\cos(-x)$

$= -2\sin x \cos x = -f(x)$

$\therefore f$ is odd.

(34) $f(-x) = \begin{cases} -2x & , -x \geq 0 \\ 2x & , -x < 0 \end{cases}$

$= \begin{cases} -2x & , x \leq 0 \\ 2x & , x > 0 \end{cases}$

$= \begin{cases} 2x & , x \geq 0 \\ -2x & , x < 0 \end{cases}$

$= f(x)$

$\therefore f$ is even.

(35) $f(-x) = \begin{cases} -2x + (-x)^2 & , -x \leq 0 \\ -2x - (-x)^2 & , -x > 0 \end{cases}$

$= \begin{cases} -(2x - x^2) & , x \geq 0 \\ -(2x + x^2) & , x < 0 \end{cases}$

$= \begin{cases} -(2x + x^2) & , x \leq 0 \\ -(2x - x^2) & , x > 0 \end{cases}$

$= -\begin{cases} 2x + x^2 & , x \leq 0 \\ 2x - x^2 & , x > 0 \end{cases}$

$= -f(x)$

$\therefore f$ is odd.

7

Figure (1) :

Notice that each horizontal straight line intersects the curve at one point only so , the function f is one-to-one.

algebraically verifying :

Let a , b $\in$ the domain of the function f

$\therefore f(a) = 2 - 2a$, $f(b) = 2 - 2b$

put $f(a) = f(b)$ $\therefore 2 - 2a = 2 - 2b$

$\therefore -2a = -2b$ $\therefore a = b$

$\therefore$ The function f is one-to-one.

Figure (2) :

Notice that each horizontal straight line intersects the curve at one point only so , the function f is one to one.

algebraically verifying :

Let a , b $\in$ the domain of the function f

$\therefore f(a) = 4 - a^3$, $f(b) = 4 - b^3$

, put $f(a) = f(b)$ $\therefore 4 - a^3 = 4 - b^3$

$\therefore -a^3 = -b^3$ $\therefore a = b$

$\therefore$ The function f is one-to-one.

Figure (3) :

Notice that each horizontal straight line intersects the curve at one point only so , the function f is one-to-one.

algebraically verifying :

Let a , b $\in$ the domain of the function f

$\therefore f(a) = \frac{a-1}{a-2}$, $f(b) = \frac{b-1}{b-2}$,

put $f(a) = f(b)$ $\therefore \frac{a-1}{a-2} = \frac{b-1}{b-2}$

$\therefore (a-1)(b-2) = (a-2)(b-1)$

$\therefore ab - 2a - b + 2 = ab - a - 2b + 2$

$\therefore -2a - b = -a - 2b$

$\therefore 2b - b = 2a - a$ $\therefore b = a$

$\therefore$ The function f is one-to-one.

8

(1) Let a , b $\in$ the domain of the function f

, $\because f(a) = 2a - 3$, $f(b) = 2b - 3$

, put $f(a) = f(b)$ $\therefore 2a - 3 = 2b - 3$

$\therefore 2a = 2b$ $\therefore a = b$

$\therefore f$ is one-to-one.

(2) Let a , b $\in$ the domain of the function f

, $\because f(a) = 4 - a^3$, $f(b) = 4 - b^3$

, put $f(a) = f(b)$ $\therefore 4 - a^3 = 4 - b^3$

$\therefore -a^3 = -b^3$ $\therefore a = b$

$\therefore f$ is one-to-one.

(3) Let a , b $\in$ the domain of the function f

, $\because f(a) = \frac{3}{2a+5}$, $f(b) = \frac{3}{2b+5}$

, put $f(a) = f(b)$ $\therefore \frac{3}{2a+5} = \frac{3}{2b+5}$

$\therefore 2a + 5 = 2b + 5$ $\therefore a = b$

$\therefore f$ is one-to-one.

(4) Let a , b ∈ the domain of the function f

$\therefore f(a) = \frac{3a-5}{4a+3}$, $f(b) = \frac{3b-5}{4b+3}$

, put $f(a) = f(b)$ $\therefore \frac{3a-5}{4a+3} = \frac{3b-5}{4b+3}$

$\therefore (3a-5)(4b+3) = (4a+3)(3b-5)$

$\therefore 12ab + 9a - 20b - 15$

$= 12ab - 20a + 9b - 15$

$\therefore 9a - 20b = -20a + 9b$ $\therefore 29a = 29b$

$\therefore a = b$ $\therefore f$ is one-to-one.

9

(1) Let a , b ∈ the domain of the function f

, $\because f(a) = 3$, $f(b) = 3$ $\therefore f(a) = f(b)$

$\therefore f$ is not one-to-one.

(2) Let a , b ∈ the domain of the function f

, $\because f(a) = (a+3)^2$, $f(b) = (b+3)^2$

, put $f(a) = f(b)$ $\therefore (a+3)^2 = (b+3)^2$

$\therefore a + 3 = \pm(b+3)$ $\therefore a + 3 = b + 3$

$\therefore a = b$ or $a + 3 = -b - 3$

$\therefore a = -b - 6$

$\therefore$ a has 2 values : b , $-b-6$

$\therefore f$ is not one-to-one.

(3) Let a , b ∈ the domain of the function f

, $\because f(a) = a^2 - 5a + 6$, $f(b) = b^2 - 5b + 6$

, put $f(a) = f(b)$

$\therefore a^2 - 5a + 6 = b^2 - 5b + 6$

$\therefore a^2 - 5a = b^2 - 5b$

$\therefore a^2 - b^2 - 5a + 5b = 0$

$\therefore (a-b)(a+b) - 5(a-b) = 0$

$\therefore (a-b)(a+b-5) = 0$

$\therefore a - b = 0$ $\therefore a = b$

or $a + b - 5 = 0$ $\therefore a = -b + 5$

$\therefore$ a has 2 values : b , $-b+5$

$\therefore f$ is not one-to-one.

(4) Let a , b ∈ the domain of the function f

, $\because f(a) = \frac{1}{a^2-4}$, $f(b) = \frac{1}{b^2-4}$

, put $f(a) = f(b)$ $\therefore \frac{1}{a^2-4} = \frac{1}{b^2-4}$

$\therefore a^2 - 4 = b^2 - 4$ $\therefore a^2 = b^2$

$\therefore a = \pm b$ $\therefore$ a has 2 values : b , $-b$

$\therefore f$ is not one-to-one.

10

(1) Let a , b ∈ the domain of the function f

, $\because f(a) = a^3 + 1$, $f(b) = b^3 + 1$

put $f(a) = f(b)$ $\therefore a^3 + 1 = b^3 + 1$

$\therefore a^3 = b^3$ $\therefore a = b$

$\therefore f$ is one-to-one.

(2) Let a , b ∈ the domain of the function f

, $\because f(a) = a^2 - 1$, $f(b) = b^2 - 1$

, put $f(a) = f(b)$ $\therefore a^2 - 1 = b^2 - 1$

$\therefore a^2 = b^2$ $\therefore a = \pm b$

$\therefore$ a has 2 values : b , $-b$

$\therefore f$ is not one-to-one.

(3) Let a , b ∈ the domain of the function f

, $\because f(a) = 2a^2 - a - 3$, $f(b) = 2b^2 - b - 3$

, put $f(a) = f(b)$

$\therefore 2a^2 - a - 3 = 2b^2 - b - 3$

$\therefore 2a^2 - 2b^2 - a + b = 0$

$\therefore 2(a^2 - b^2) - (a - b) = 0$

$\therefore 2(a-b)(a+b) - (a-b) = 0$

$\therefore (a-b)\big(2(a+b) - 1\big) = 0$

$\therefore a - b = 0$ $\therefore a = b$

or $2(a+b) - 1 = 0$ $\therefore a + b = \frac{1}{2}$

$\therefore a = \frac{1}{2} - b$ $\therefore$ a has 2 values : b , $\frac{1}{2} - b$

$\therefore f$ is not one-to-one.

(4) Let a , b ∈ the domain of the function f

, $\because f(a) = a^4 + 2a^2 + 1$,

$f(b) = b^4 + 2b^2 + 1$

, put $f(a) = f(b)$

$\therefore a^4 + 2a^2 + 1 = b^4 + 2b^2 + 1$

$\therefore (a^2+1)^2 = (b^2+1)^2$

$\therefore a^2 + 1 = b^2 + 1$ $\therefore a^2 = b^2$

$\therefore a = \pm b$ $\therefore$ a has 2 values : b , $-b$

$\therefore f$ is not one-to-one.

(5) Let a , b ∈ the domain of the function f

, ∵ $f(a) = \frac{2a+1}{a-2}$, $f(b) = \frac{2b+1}{b-2}$

, put $f(a) = f(b)$ ∴ $\frac{2a+1}{a-2} = \frac{2b+1}{b-2}$

∴ $(2a+1)(b-2) = (2b+1)(a-2)$

∴ $2ab - 4a + b - 2 = 2ab + a - 4b - 2$

∴ $-4a + b = a - 4b$

∴ $5b = 5a$ ∴ $a = b$

∴ f is one-to-one.

(6) Let a , b ∈ the domain of the function f

, ∵ $f(a) = \frac{3-a}{a+2}$, $f(b) = \frac{3-b}{b+2}$

, put $f(a) = f(b)$ ∴ $\frac{3-a}{a+2} = \frac{3-b}{b+2}$

∴ $(3-a)(b+2) = (a+2)(3-b)$

∴ $3b + 6 - ab - 2a = 3a - ab + 6 - 2b$

∴ $3b - 2a = 3a - 2b$ ∴ $5b = 5a$

∴ $a = b$ ∴ f is one-to-one.

11

From (1) to (4) neither even nor odd.

12

(1) ∵ The function is odd. ∴ $f(-5) = -f(5)$

∴ The expression $= \frac{-7f(5) + 3f(5)}{-2f(5)}$

$= \frac{-4f(5)}{-2f(5)} = 2$

(2) ∵ The function is even. ∴ $f(-5) = f(5)$

∴ The expression $= \frac{7f(5) + 3f(5)}{2f(5)}$

$= \frac{10f(5)}{2f(5)} = 5$

13

(1) $(f+g)(X) = (1-X)^2 + (1+X)^2$

∴ $(f+g)(-X) = (1+X)^2 + (1-X)^2 = (f+g)(X)$

∴ $(f+g)$ is even.

(2) $(f-g)(X) = (1-X)^2 - (1+X)^2$

∴ $(f-g)(-X) = (1+X)^2 - (1-X)^2$

$= -(f-g)(X)$

∴ $(f-g)$ is odd.

(3) $(f \cdot g)(X) = (1-X)^2(1+X)^2$

∴ $(f \cdot g)(-X) = (1+X)^2(1-X)^2 = (f \cdot g)(X)$

∴ $(f \cdot g)$ is even.

(4) $\left(\frac{f}{g}\right)(X) = \frac{(1-X)^2}{(1+X)^2}$

∴ $\left(\frac{f}{g}\right)(-X) = \frac{(1+X)^2}{(1-X)^2} \neq \left(\frac{f}{g}\right)(X) \neq -\left(\frac{f}{g}\right)(X)$

∴ $\left(\frac{f}{g}\right)$ is neither even nor odd.

14

∵ f_1 , f_2 are even and g_1 , g_2 are odd

(1) $f_1 + g_2$ is neither even nor odd.

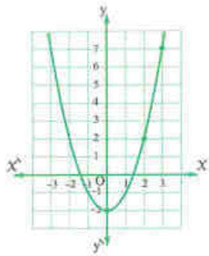
(2) $f_1 - f_2$ is even. (3) $g_1 + g_2$ is odd.

(4) $f_1 \cdot g_2$ is odd. (5) $g_1 \cdot g_2$ is even.

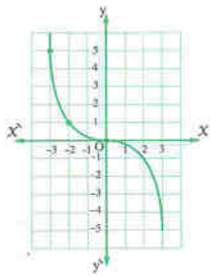
(6) $\frac{f_2}{f_1}$ is even.

15

(1)



(2)



Third Higher skills

(1) (c)	(2) (c)	(3) (c)	(4) (b)
(5) (a)	(6) (b)	(7) (a)	(8) (a)

Instructions solving :

(1) $\because$ The point (2 , 3) belong to the function f

$\therefore$ The point (2 , – 1) $\notin$ the function

, $\because$ f is an odd function and (2 , 3) belongs to it

$\therefore$ The point (5 , 3) $\notin$ the function

$\therefore$ The point which could belong to the function is (3 , 2)

(2) $\because$ The function is one-to-one passes through the two points (2 , a) , (3 , b)

$\therefore$ It must be a $\neq$ b always.

(3) $\because$ f is even $\quad \therefore f(X) = f(-X)$

$\therefore f(X) + X^2 f(X) = 3$

$\therefore f(X)[1 + X^2] = 3$

$\therefore f(X) = \frac{3}{1+X^2} \quad \therefore f(1) = 1\frac{1}{2}$

(4) $\because$ f is odd , $f(1) = k \quad \therefore f(-1) = -k$

, $\because f(X+2) = f(X) + f(2)$

Put $X = -1$

$\therefore f(1) = f(-1) + f(2) \quad \therefore k = -k + f(2)$

$\therefore f(2) = 2k$

Put $X = 1$

$\therefore f(3) = f(1) + f(2) = k + 2k = 3k$

(5) $\because (f+g)(-X) = \frac{1+(-X)}{1-(-X)} + \frac{1-(-X)}{1+(-X)}$

$= \frac{1-X}{1+X} + \frac{1+X}{1-X} = (f+g)(X)$

$\therefore (f+g)$ is an even function

, $\because (f \cdot g)(X) = \frac{1+X}{1-X} \times \frac{1-X}{1+X} = 1$

$\therefore (f \cdot g)$ is an even function too.

(6) $\because g(-X) = f(-X) + f(X) = g(X)$

$\therefore$ g is always even.

(7) $\because 3f(X) + 2f(-X) = X^3 - \sin X \quad (1)$

Put $-X$ instead of X

$\therefore 3f(-X) + 2f(X) = (-X)^3 - \sin(-X)$

$\therefore 3f(-X) + 2f(X) = -X^3 + \sin X \quad (2)$

By adding (1) , (2) :

$\therefore 5f(X) + 5f(-X) = \text{zero}$

$\therefore f(X) = -f(-X)$

$\therefore$ The function is odd.

(8) $f(X) = X^3$, $g(X) = X^2 + 1$

(1) $(f \times g)(X) = X^3(X^2+1)$

$\therefore (f \times g)(-X) = (-X)^3\left((-X)^2+1\right)$

$= -X^3(X^2+1)$

$\therefore (f \times g)(-X) = -(f \times g)(X)$

$\therefore (f \times g)$ is odd.

(2) $(f \circ g)(X) = f\left(g(X)\right) = f(X^2+1)$

$= (X^2+1)^3$

$\therefore (f \circ g)(-X) = \left((-X)^2+1\right)^3 = (X^2+1)^3$

$= (f \circ g)(X)$

$\therefore (f \circ g)$ is an even function.

(3) $(g \circ f)(X) = g\left(f(X)\right) = g(X^3) = X^6 + 1$

$\therefore (g \circ f)(-X) = \left((-X)^6+1\right) = X^6 + 1$

$= (g \circ f)(X)$

$\therefore (g \circ f)$ is an even function.

$\therefore$ The answer is (a)

Exercise 4

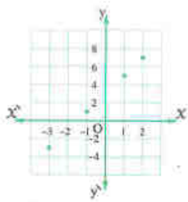
First Multiple choice questions

(1) a	(2) c	(3) d	(4) b	(5) d	(6) d
(7) b	(8) c	(9) c	(10) c	(11) d	(12) b
(13) b	(14) b	(15) c	(16) b	(17) b	

Second Essay questions

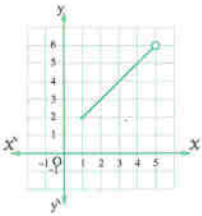
1

(1)



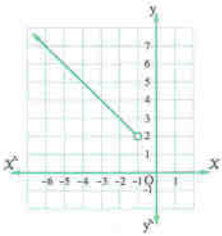
The range = $\{-3 , 1 , 5 , 7\}$

(2)



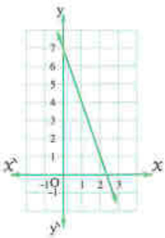
The range = [2 , 6[

(3)



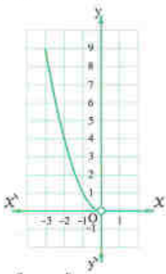
The range =]2 , ∞[

(4)



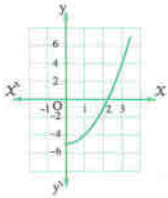
The range = ℝ

(5)



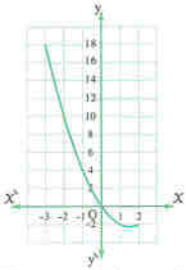
The range =]0 , ∞[

(6)



* The range = [– 5 , ∞[

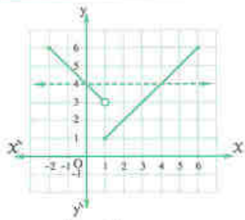
(7)



* Notice : The vertex of the curve is $\left(1.5 , -2\frac{1}{4}\right)$
* The range = $\left[-2\frac{1}{4} , 18\right]$

2

(1)



* The range = [1 , 6]
* The function is decreasing on]– 2 , 1[and increasing on]1 , 6[

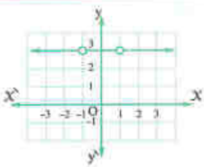
(2) The function is not one-to-one because there is a horizontal line cuts the function at more than one point.

3

(1) $\because f(X) = \frac{3(X^2-1)}{(X^2-1)}$

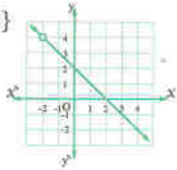
$\therefore f(X) = 3 , X \neq \pm 1$

* The domain $= \mathbb{R} - \{1 , -1\}$



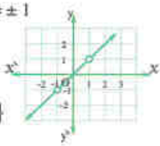
* The range = $\{3\}$
* The function is constant on its domain.
* The function is even.
* The axis of symmetry is $X = 0$

(2) $g(X) = \frac{(2-X)(2+X)}{X+2} = 2 - X$, $X \neq -2$



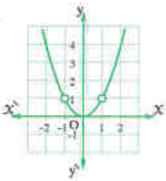
* The domain = $\mathbb{R} - \{-2\}$
* The range = $\mathbb{R} - \{4\}$
* The function is decreasing on its domain.
* The function is neither even nor odd.

(3) $g(X) = \frac{X(X^2-1)}{(X^2-1)} = X$, $X \neq \pm 1$



* The domain $= \mathbb{R} - \{-1 , 1\}$
* The range = $\mathbb{R} - \{-1 , 1\}$
* The function is increasing on $\mathbb{R} - \{-1 , 1\}$
* The function is odd.
* The function is symmetric about origin.

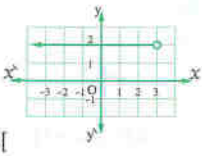
(4) * $f(X) = \frac{X^2(X^2-1)}{(X^2-1)} = X^2$ where $X \neq \pm 1$



* The domain $= \mathbb{R} - \{1 , -1\}$
* The range $= [0 , \infty[- \{1\}$
* The function is decreasing on $]-\infty , 0[- \{-1\}$ and increasing on $]0 , \infty[- \{1\}$
* The function is even.
* The axis of symmetry is $X = 0$

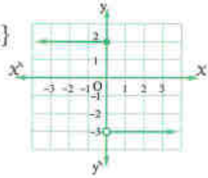
4

(1) * The domain $=]-\infty , 3[$



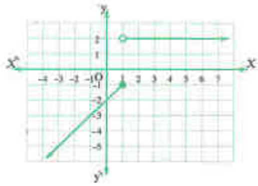
* The range = $\{2\}$
* The function is constant on $]-\infty , 3[$
* The function is neither even nor odd.
* The function has neither point of symmetry nor axis of symmetry.

(2) * The domain = $\mathbb{R}$



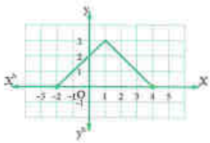
* The range = $\{-3 , 2\}$
* The function is constant on $]-\infty , 0[$, $]0 , \infty[$
* The function is neither even nor odd.
* The function has neither point of symmetry nor axis of symmetry.

(3)



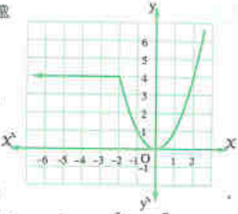
* The range = $]-\infty , -1] \cup \{2\}$
* The function is constant on $]1 , \infty[$ and increasing on $]-\infty , 1[$
* The function is neither even nor odd.
* The function has neither point of symmetry nor axis of symmetry.

(4)



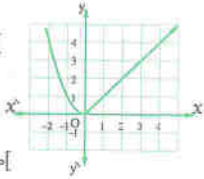
* The domain = $[-2 , 4]$
* The range = $[0 , 3]$
* The function is increasing on $]-2 , 1[$ and decreasing on $]1 , 4[$
* The function is neither odd nor even.
* The axis of symmetry is the straight line $X = 1$

(5) * The domain = $\mathbb{R}$



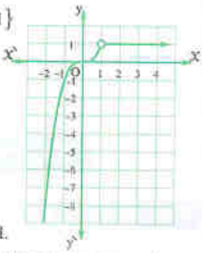
* The range $= [0, \infty[$
* The function is constant on $]-\infty, -2[$ and decreasing on $]-2, 0[$ and increasing on $]0, \infty[$
* The function is neither even nor odd.
* The function has neither point of symmetry nor axis of symmetry.

(6) * The domain = $\mathbb{R}$



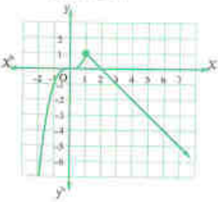
* The range $= [0, \infty[$
* The function is decreasing on $]-\infty, 0[$ and increasing on $]0, \infty[$
* The function is neither even nor odd.
* The function has neither point of symmetry nor axis of symmetry.

(7) * The domain $= \mathbb{R} - \{1\}$



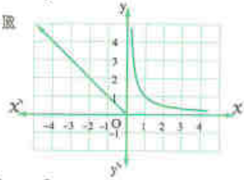
* The range $=]-\infty, 1]$
* The function is increasing on $]-\infty, 1[$ and constant on $]1, \infty[$
* The function is neither even nor odd.
* The function has neither point of symmetry nor axis of symmetry.

(8)



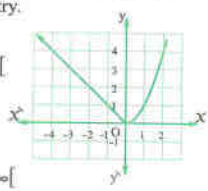
* The domain = $\mathbb{R}$ * The range $=]-\infty, 1]$
* The function is increasing on $]-\infty, 1[$ and decreasing on $]1, \infty[$
* The function is neither even nor odd.
* The function has neither point of symmetry nor axis of symmetry.

(9) * The domain = $\mathbb{R}$



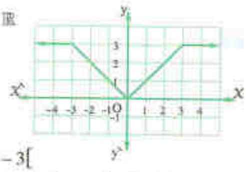
* The range $= [0, \infty[$
* The function is decreasing on each of $]-\infty, 0[$, $]0, \infty[$
* The function is neither even nor odd.
* The function has neither point of symmetry nor axis of symmetry.

(10) * The domain = $\mathbb{R}$



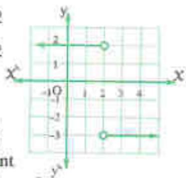
* The range $= [0, \infty[$
* The function is decreasing on $]-\infty, 0[$ and increasing on $]0, \infty[$
* The function is neither even nor odd.
* The function has neither point of symmetry nor axis of symmetry.

(11) * The domain = $\mathbb{R}$



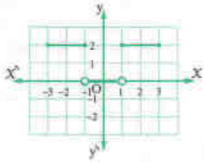
* The range $= [0, 3]$
* The function is constant on each of $]-\infty, -3[$, $]3, \infty[$ and decreasing on $]-3, 0[$ and increasing on $]0, 3[$
* The function is even.
* The axis of symmetry is $X = 0$

(12) * $f(X) = \begin{cases} -3, & X > 2 \\ 2, & X < 2 \end{cases}$



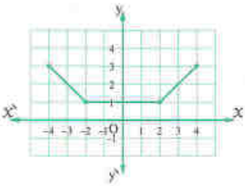
* The domain $= \mathbb{R} - \{2\}$
* The range $= \{2, -3\}$
* The function is constant on each of $]-\infty, 2[$, $]2, \infty[$
* The function is neither even nor odd.
* The function has neither point of symmetry nor axis of symmetry.

(13) * The domain $= [-3, 3]$



* The range $= \{0, 2\}$
* The function is constant on each $]-3, -1[$, $]-1, 1[$, $]1, 3[$
* The function is even.
* The axis of symmetry is the straight line $x = 0$

(14) * The domain $= [-4, 4]$



* The range $= [1, 3]$
* The function is decreasing on $]-4, -2[$, constant on $]-2, 2[$ and increasing on $]2, 4[$
* The function is even.
* The axis of symmetry is the straight line $x = 0$

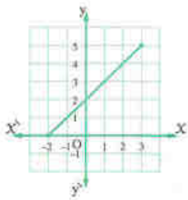
5

$f_1(x) = 3x - 1$, its domain $= \mathbb{R}$

, $f_2(x) = 3 - 2x$, its domain $= [-2, 3]$

, $(f_1 + f_2)(x) = x + 2$, its domain $= [-2, 3]$

x	-2	-1	0	1	2	3
$(f_1 + f_2)(x)$	0	1	2	3	4	5



* The domain $= [-2, 3]$
* The function is increasing on $]-2, 3[$

6

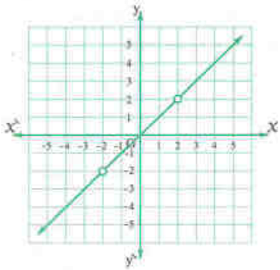
$\because f(x) = x^3 - 4x$, its domain $= \mathbb{R}$

, $g(x) = x^2 - 4$, its domain $= \mathbb{R}$

$\therefore \frac{f}{g}(x) = \frac{x^3 - 4x}{x^2 - 4} = \frac{x(x^2 - 4)}{x^2 - 4} = x$

where $x \neq \pm 2$

, the domain of $\frac{f}{g} = \mathbb{R} - \{2, -2\}$



* The range $= \mathbb{R} - \{2, -2\}$
* The function is odd.
* The function is increasing on $\mathbb{R} - \{2, -2\}$
* The function is one-to-one.

Exercise 5

First Multiple choice questions

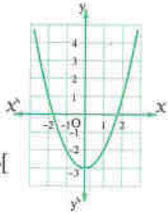
(1) a	(2) b	(3) c	(4) b	(5) d	(6) c
(7) c	(8) a	(9) d	(10) b	(11) c	(12) c
(13) a	(14) c	(15) b	(16) c	(17) c	(18) b
(19) a	(20) b	(21) b	(22) b	(23) b	(24) d
(25) b	(26) c	(27) c	(28) b	(29) c	(30) c
(31) a	(32) c	(33) c	(34) d	(35) b	(36) c
(37) b	(38) c	(39) d	(40) c	(41) d	(42) a
(43) c	(44) a	(45) b	(46) b	(47) c	(48) a
(49) d	(50) b	(51) d	(52) c		

Second Essay questions

1

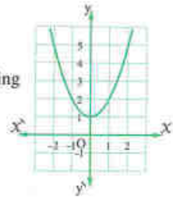
(1) * The domain = $\mathbb{R}$

* The range = $[-3 , \infty[$
* The function is decreasing on $]-\infty , 0[$ and increasing on $]0 , \infty[$
* The function is even.
* The equation of the axis of symmetry is $X = 0$



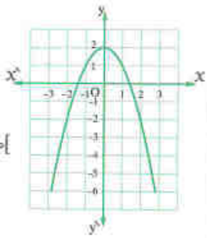
(2) * The domain = $\mathbb{R}$

* The range = $[1 , \infty[$
* The function is decreasing on $]-\infty , 0[$ and increasing on $]0 , \infty[$
* The function is even.
* The equation of the axis of symmetry is $X = 0$



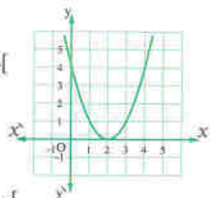
(3) * The domain = $\mathbb{R}$

* The range $=]-\infty , 2]$
* The function is decreasing on $]0 , \infty[$ and increasing on $]-\infty , 0[$
* The function is even.
* The equation of the axis of symmetry is $X = 0$



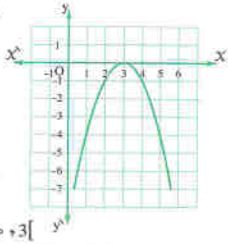
(4) * The domain = $\mathbb{R}$

* The range = $[0 , \infty[$
* The function is decreasing on $]-\infty , 2[$ and increasing on $]2 , \infty[$
* The function is neither even nor odd.
* The equation of the axis of symmetry is $X = 2$



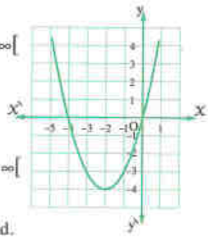
(5) * The domain = $\mathbb{R}$

* The range $=]-\infty , 0]$
* The function is decreasing on $]3 , \infty[$ and increasing on $]-\infty , 3[$
* The function is neither even nor odd.
* The equation of the axis of symmetry is $X = 3$



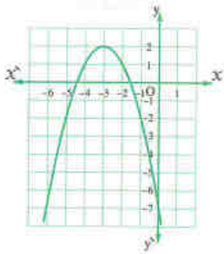
(6) * The domain = $\mathbb{R}$

* The range = $[-4 , \infty[$
* The function is decreasing on $]-\infty , -2[$ and increasing on $]-2 , \infty[$
* The function is neither even nor odd.
* The equation of the axis of symmetry is $X = -2$



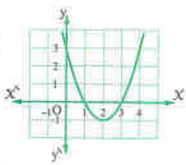
(7) * The domain = $\mathbb{R}$

* The range $=]-\infty , 2]$
* The function is decreasing on $]-3 , \infty[$ and increasing on $]-\infty , -3[$
* The function is neither even nor odd.
* The equation of the axis of symmetry is $X = -3$

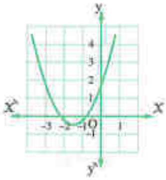


(8) * The domain = $\mathbb{R}$

* The range = $]-1 , \infty[$
* The function is decreasing on $]-\infty , 2[$ and increasing on $]2 , \infty[$
* The function is neither even nor odd.
* The equation of the axis of symmetry is $X = 2$

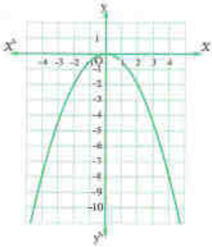


(9) * The domain = $\mathbb{R}$



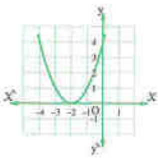
* The range = $\left[\frac{-1}{2}, \infty\right[$
* The function is decreasing on $\left]-\infty, \frac{-3}{2}\right[$ and increasing on $\left]\frac{-3}{2}, \infty\right[$
* The function is neither even nor odd.
* The equation of the axis of symmetry is $X = \frac{-3}{2}$

(10) * The domain = $\mathbb{R}$



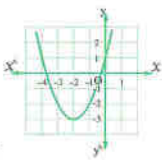
* The range $=]-\infty, 0]$
* The function is decreasing on $]0, \infty[$ and increasing on $]-\infty, 0[$
* The function is even.
* The equation of the axis of symmetry is $X = 0$

(11) * $g(X) = (X + 2)^2$



* The domain = $\mathbb{R}$
* The range = $[0, \infty[$
* The function is decreasing on $]-\infty, -2[$ and increasing on $]-2, \infty[$
* The function is neither even nor odd.
* The equation of the axis of symmetry is $X = -2$

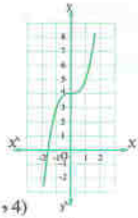
(12) * $g(X) = (X + 2)^2 - 3$



* The domain = $\mathbb{R}$
* The range = $[-3, \infty[$
* The function is decreasing on $]-\infty, -2[$ and increasing on $]-2, \infty[$
* The function is neither even nor odd.
* The equation of the axis of symmetry is $X = -2$

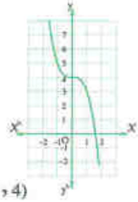
2

(1) * The domain = $\mathbb{R}$



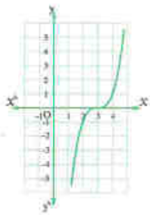
* The range = $\mathbb{R}$
* The function is increasing on $\mathbb{R}$
* The function is neither even nor odd.
* The point of symmetry is (0 , 4)

(2) * The domain = $\mathbb{R}$



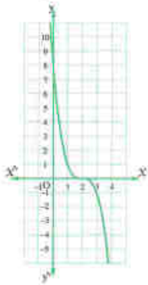
* The range = $\mathbb{R}$
* The function is decreasing on $\mathbb{R}$
* The function is neither even nor odd.
* The point of symmetry is (0 , 4)

(3) * The domain = $\mathbb{R}$



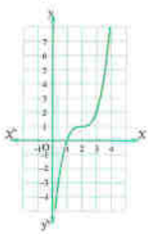
* The range = $\mathbb{R}$
* The function is increasing on $\mathbb{R}$
* The function is neither even nor odd.
* The point of symmetry is (3 , 0)

(4) $g(X) = -(X - 2)^3$



* The domain = $\mathbb{R}$
* The range = $\mathbb{R}$
* The function is decreasing on $\mathbb{R}$
* The function is neither even nor odd.
* The point of symmetry is (2 , 0)

(5) * The domain = $\mathbb{R}$



* The range = $\mathbb{R}$
* The function is increasing on $\mathbb{R}$
* The function is neither even nor odd.
* The point of symmetry is (2 , 1)

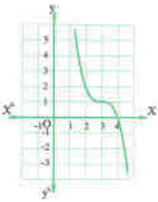
(6) * The domain = $\mathbb{R}$

* The range = $\mathbb{R}$
* The function is decreasing on $\mathbb{R}$
* The function is neither even nor odd.
* The point of symmetry is (1 , 2)



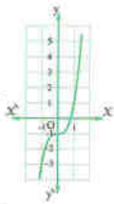
(7) * $g(X) = -(X-3)^2 + 1$

* The domain = $\mathbb{R}$
* The range = $\mathbb{R}$
* The function is decreasing on $\mathbb{R}$
* The function is neither even nor odd
* The point of symmetry is (3 , 1)



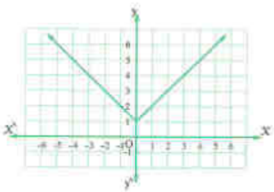
(8) * The domain = $\mathbb{R}$

* The range = $\mathbb{R}$
* The function is increasing on $\mathbb{R}$
* The function is neither even nor odd.
* The point of symmetry is (0 , – 1)



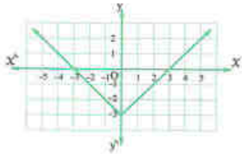
3

(1)



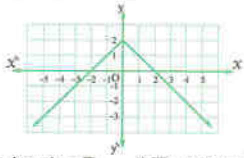
* The domain = $\mathbb{R}$
* The range = $[1 , \infty[$
* The function is decreasing on $]-\infty , 0[$ and increasing on $]0 , \infty[$
* The function is even.
* The equation of the axis of symmetry is $X = 0$

(2)



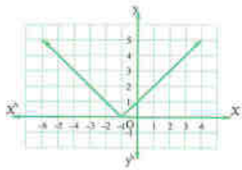
* The domain = $\mathbb{R}$ * The range = $[-3 , \infty[$
* The function is decreasing on $]-\infty , 0[$ and increasing on $]0 , \infty[$
* The function is even.
* The equation of the axis of symmetry is $X = 0$

(3)



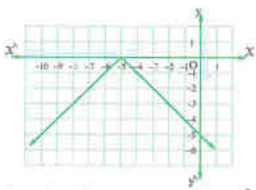
* The domain = $\mathbb{R}$ * The range = $]-\infty , 2]$
* The function is decreasing on $]0 , \infty[$ and increasing on $]-\infty , 0[$
* The function is even.
* The equation of the axis of symmetry is $X = 0$

(4)



* The domain = $\mathbb{R}$ * The range = $[0 , \infty[$
* The function is decreasing on $]-\infty , -1[$ and increasing on $]-1 , \infty[$
* The function is neither even nor odd.
* The equation of the axis of symmetry $X = -1$

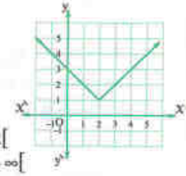
(5)



* The domain = $\mathbb{R}$ * The range = $]-\infty , 0]$

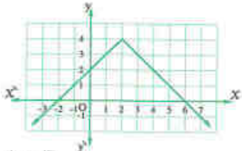
* The function is decreasing on $]-5, \infty[$ and increasing on $]-\infty, -5[$
* The function is neither even nor odd.
* The equation of the axis of symmetry $X = -5$

(6) * $g(X) = |X - 2| + 1$



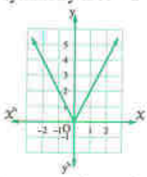
* The domain = $\mathbb{R}$
* The range = $[1, \infty[$
* The function is decreasing on $]-\infty, 2[$ and increasing on $]2, \infty[$
* The function is neither even nor odd.
* The equation of the axis of symmetry is $X = 2$

(7)



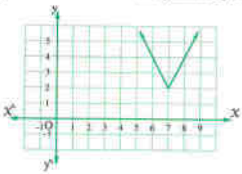
* The domain = $\mathbb{R}$
* The range = $[-\infty, 4]$
* The function is decreasing on $]2, \infty[$ and increasing on $]-\infty, 2[$
* The function is neither even nor odd.
* The equation of the axis of symmetry is $X = 2$

(8) * The domain = $\mathbb{R}$



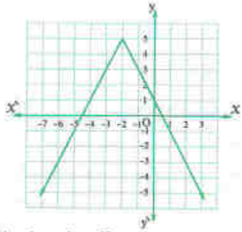
* The range = $[0, \infty[$
* The function is decreasing on $]-\infty, 0[$ and increasing on $]0, \infty[$
* The function is even.
* The equation of the axis of symmetry is $X = 0$

(9)



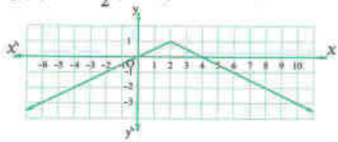
* The domain = $\mathbb{R}$ * The range = $[2, \infty[$
* The function is decreasing on $]-\infty, 7[$ and increasing on $]7, \infty[$
* The function is neither even nor odd.
* The equation of the axis of symmetry $X = 7$

(10)



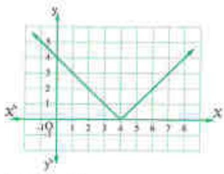
* The domain = $\mathbb{R}$
* The range = $]-\infty, 5]$
* The function is decreasing on $]-2, \infty[$ and increasing on $]-\infty, -2[$
* The function is neither even nor odd.
* The equation of the axis of symmetry is $X = -2$

(11) * $g(X) = 1 - \frac{1}{2}|X - 2|$



* The domain = $\mathbb{R}$
* The range = $]-\infty, 1]$
* The function is decreasing on $]2, \infty[$ and increasing on $]-\infty, 2[$
* The function is neither even nor odd.
* The equation of the axis of symmetry is $X = 2$

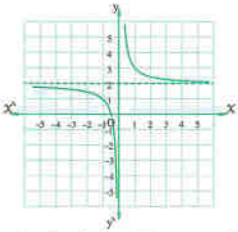
(12) $\because g(X) = \sqrt{X^2 - 8X + 16} = \sqrt{(X-4)^2} = |X - 4|$



* The domain = $\mathbb{R}$
* The range = $[0, \infty[$
* The function is decreasing on $]-\infty, 4[$ and increasing on $]4, \infty[$
* The function is neither even nor odd.
* The equation of the axis of symmetry is $X = 4$

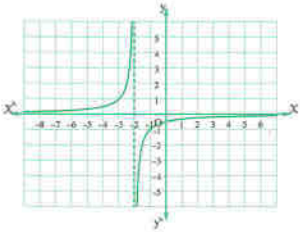
4

(1)



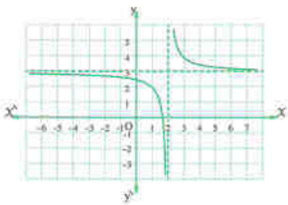
* The domain $= \mathbb{R} - \{0\}$ * The range $= \mathbb{R} - \{2\}$
* The function is decreasing on $]0 , \infty[$, $]-\infty , 0[$
* The function is neither even nor odd.
* The point of symmetry is $(0 , 2)$

(2)



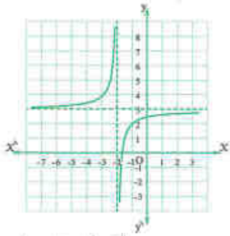
* The domain $= \mathbb{R} - \{-2\}$
* The range $= \mathbb{R} - \{0\}$
* The function is increasing on $]-\infty , -2]$
 , $]-2 , \infty[$
* The function is neither even nor odd.
* The point of symmetry is $(-2 , 0)$

(3)



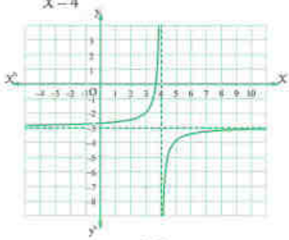
* The domain $= \mathbb{R} - \{2\}$
* The range $= \mathbb{R} - \{3\}$
* The function is decreasing on $]-\infty , 2[$, $]2 , \infty[$
* The function is neither even nor odd.
* The point of symmetry is $(2 , 2)$

(4)



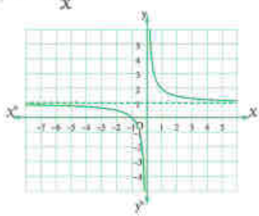
* The domain $= \mathbb{R} - \{-2\}$
* The range $= \mathbb{R} - \{3\}$
* The function is increasing on $]-\infty , -2[$, $]-2 , \infty[$
* The function is neither even nor odd.
* The point of symmetry is $(-2 , 3)$

(5) $g(X) = \frac{-1}{X-4} - 3$



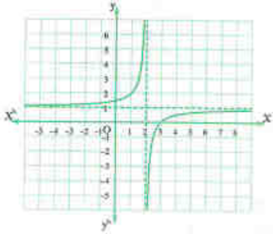
* The domain $= \mathbb{R} - \{4\}$
* The range $= \mathbb{R} - \{-3\}$
* The function is increasing on $]-\infty , 4[$, $]4 , \infty[$
* The function is neither even nor odd.
* The point of symmetry is $(4 , -3)$

(6) $g(X) = 1 + \frac{1}{X}$



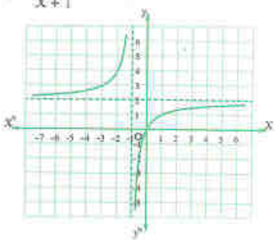
* The domain $= \mathbb{R} - \{0\}$ * The range $= \mathbb{R} - \{1\}$
* The function is decreasing on $]-\infty , 0[$, $]0 , \infty[$
* The function is neither even nor odd.
* The point of symmetry is $(0 , 1)$

(7) $g(x) = \frac{(x-2)-1}{x-2} = \frac{-1}{x-2} + 1$



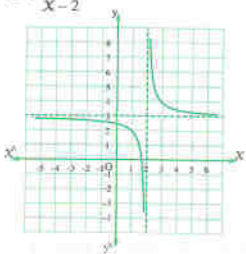
* The domain = $\mathbb{R} - \{2\}$ * The range = $\mathbb{R} - \{1\}$
* The function is increasing on $]-\infty , 2[$, $]2 , \infty[$
* The function is neither even nor odd.
* The point of symmetry is (2 , 1)

(8) $g(x) = \frac{-2}{x+1} + 2$



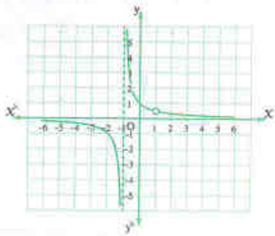
* The domain = $\mathbb{R} - \{-1\}$ * The range = $\mathbb{R} - \{2\}$
* The function is increasing on $]-\infty , -1[$, $]-1 , \infty[$
* The function is neither even nor odd.
* The point of symmetry is (− 1 , 2)

(9) $g(x) = 3 + \frac{1}{x-2}$



* The domain = $\mathbb{R} - \{2\}$
* The range = $\mathbb{R} - \{3\}$
* The function is decreasing on $]-\infty , 2[$, $]2 , \infty[$
* The function is neither even nor odd.
* The point of symmetry is (2 , 3)

(10) $g(x) = \frac{x-1}{(x-1)(x+1)} = \frac{1}{x+1}$, $x \neq 1$



* The domain = $\mathbb{R} - \{1 , -1\}$
* The range = $\mathbb{R} - \{0 , \frac{1}{2}\}$
* The function is decreasing on $]-\infty , -1[$, $]-1 , 1[$, $]1 , \infty[$
* The function is neither even nor odd.
* There is no point of symmetry.

5

* $g(x) = |x-3|$ * $h(x) = |x| - 2$
* $z(x) = |x+3| + 2$

6

(1) $f(x) = -1$ (2) $f(x) = x - 2$

(3) $f(x) = (x+2)^2 - 3$ (4) $f(x) = -(x-1)^2 + 2$

(5) $f(x) = x^3$ (6) $f(x) = (x-2)^3$

(7) $f(x) = (x+1)^3 - 3$ (8) $f(x) = |x-2| - 3$

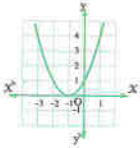
(9) $f(x) = -|x-1| + 3$ (10) $f(x) = \frac{-1}{x}$

(11) $f(x) = \frac{1}{x} + 2$ (12) $f(x) = \frac{-1}{x+2} - 2$

7

(1) $f_1(x) = f(x+1) = (x+1)^2$

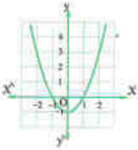
* The domain = $\mathbb{R}$
* The range = $[0, \infty[$



(2) $f_2(x) = f(x) - 1$

$= x^2 - 1$

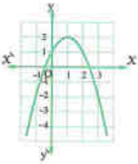
* The domain = $\mathbb{R}$
* The range = $[-1, \infty[$



(3) $f_3(x) = 2 - f(x-1)$

$= 2 - (x-1)^2$

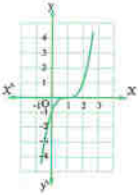
* The domain = $\mathbb{R}$
* The range = $]-\infty, 2]$



(4) $g_1(x) = g(x-1)$

$= (x-1)^3$

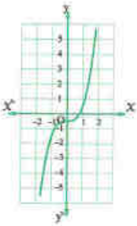
* The domain = $\mathbb{R}$
* The range = $\mathbb{R}$



(5) $g_2(x) = g(x) - \frac{1}{2}$

$= x^3 - \frac{1}{2}$

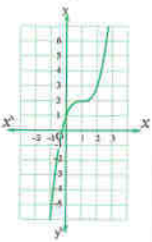
* The domain = $\mathbb{R}$
* The range = $\mathbb{R}$



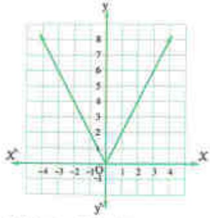
(6) $g_3(x) = g(x-1) + 2$

$= (x-1)^3 + 2$

* The domain = $\mathbb{R}$
* The range = $\mathbb{R}$



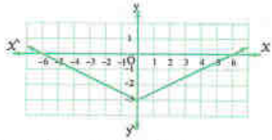
(7)



$k_1(x) = 2\,k(x) = 2|x|$

* The domain = $\mathbb{R}$ * The range = $[0, \infty[$

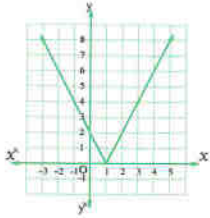
(8)



$k_2(x) = \frac{1}{2}\,k(x) - 3 = \frac{1}{2}|x| - 3$

* The domain = $\mathbb{R}$ * The range = $]-3, \infty[$

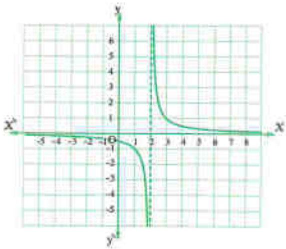
(9)



$k_3(x) = 2\,k(x-1) = 2|x-1|$

* The domain = $\mathbb{R}$ * The range = $[0, \infty[$

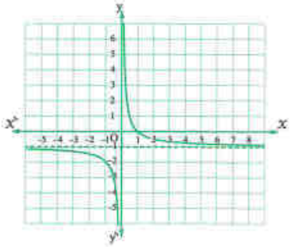
(10)



$n_1(x) = n(x-2) = \frac{1}{x-2}$

* The domain = $\mathbb{R} - \{2\}$ * The range = $\mathbb{R} - \{0\}$

(11)

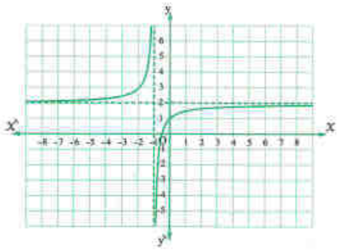


$n_2(X) = n(X) - 1 = \frac{1}{X} - 1$

* The domain $= \mathbb{R} - \{0\}$

* The range $= \mathbb{R} - \{-1\}$

(12)



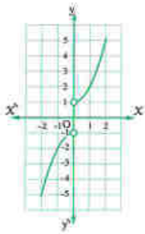
$n_3(X) = 2 - n(X + 1) = 2 - \frac{1}{X + 1}$

* The domain $= \mathbb{R} - \{-1\}$

* The range $= \mathbb{R} - \{2\}$

8

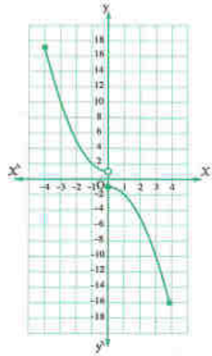
(1)



* The range $= \mathbb{R} - [-1 , 1]$

* The function is increasing on $\mathbb{R} - \{0\}$

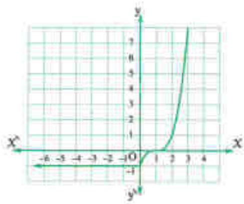
(2)



* The range $= [-17 , 17] -]-1 , 1]$

* The function is decreasing on $]-4 , 0[$, $]0 , 4[$

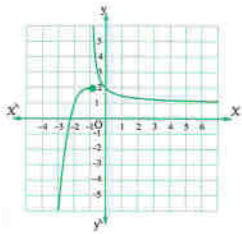
(3)



* The range $= [-1 , \infty[$

* The function is constant on $]-\infty , 0[$ and increasing on $]0 , \infty[$

(4)



* The range $= \mathbb{R}$

* The function is decreasing on $]-1 , \infty[$ and increasing on $]-\infty , -1[$

9

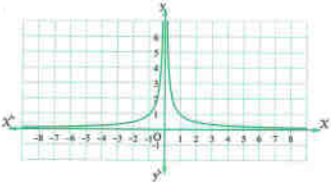
Fig. (1) : $f(x) = |(x-3)^2 - 1|$

Fig. (2) : $g(x) = |x^3|$

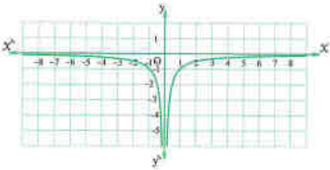
Fig. (3) : $h(x) = \dfrac{1}{|x-2|}$

10

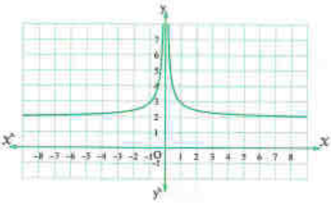
(1)



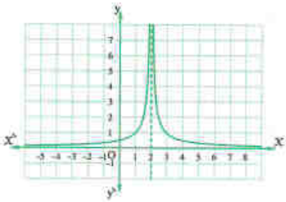
(2)



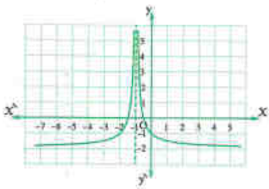
(3)



(4)

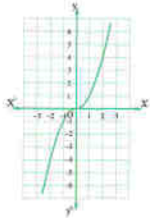


(5)

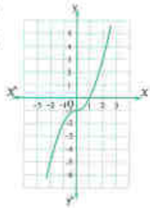


11

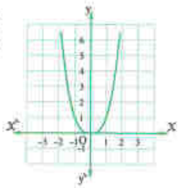
(1) $f(x) = \begin{cases} x^2 & , \ x \geq 0 \\ -x^2 & , \ x < 0 \end{cases}$



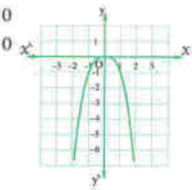
(2) $f(x) = \begin{cases} x^2 - 1 & , \ x \geq 0 \\ -x^2 - 1 & , \ x < 0 \end{cases}$



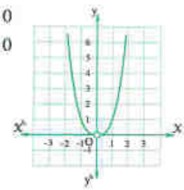
(3) $f(x) = \begin{cases} x^3 & , \ x \geq 0 \\ -x^3 & , \ x < 0 \end{cases}$



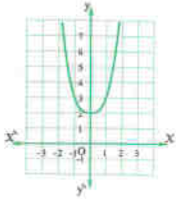
(4) $f(x) = \begin{cases} -x^3 & , \ x \geq 0 \\ x^3 & , \ x < 0 \end{cases}$



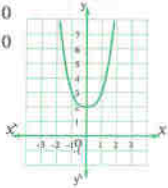
(5) $f(x) = \begin{cases} x^3 & , \ x > 0 \\ -x^3 & , \ x < 0 \end{cases}$



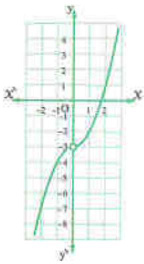
(6) $f(X) = \begin{cases} X^3 + 2 & , \quad X \geq 0 \\ -X^3 + 2 & , \quad X < 0 \end{cases}$



(7) $f(X) = \begin{cases} X^3 + 2 & , \quad X \geq 0 \\ -X^3 + 2 & , \quad X < 0 \end{cases}$

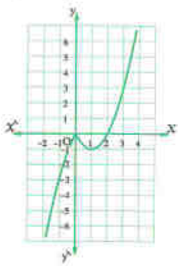


(8) $f(X) = \begin{cases} X^2 - 3 & , \quad X > 0 \\ -X^2 - 3 & , \quad X < 0 \end{cases}$



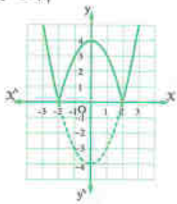
(9) $f(X) = \begin{cases} X^2 - 2X & , \quad X \geq 0 \\ -X^2 + 2X & , \quad X < 0 \end{cases}$

$= \begin{cases} (X-1)^2 - 1 & , \quad X \geq 0 \\ -(X-1)^2 + 1 & , \quad X < 0 \end{cases}$



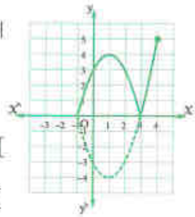
12

(1) $f(X) = |X^2 - 4|$



* The range = $[0 , \infty[$
* The function is decreasing on $]-\infty , -2[$, $]0 , 2[$ and increasing on $]-2 , 0[$, $]2 , \infty[$

(2) $f(X) = |X^2 - 2X - 3|$, $X \in [-1 , 4]$



* The range = $[0 , 5]$
* The function is increasing on $]-1 , 1[$, $]3 , 4[$ and decreasing on $]1 , 3[$

Third Higher skills

1

(1) (c) (2) (b) (3) (a) (4) (c) (5) (b)
(6) (b) (7) (b) (8) (b) (9) (c)

Instructions to solve 1 :

(1) $\because$ The curve g (X) is the same as the curve $f(X)$ by translation 3 units to the right

$\therefore$ Each point of the intersection points of the curve with the X-axis move 3 units to the right too

$\therefore X \in \{-3 + 3 , 1 + 3 , 0 + 3\}$

i.e. $X \in \{0 , 4 , 3\}$

(2) $\because$ The range of the quadratic functions = $[1 , \infty[$

$\therefore b - 2 = 1 \qquad \therefore b = 3$

, $\because$ the curve passes through the point (3 , 2)

$\therefore f(3) = 2 \qquad \therefore (3 - a + 1)^2 + 1 = 2$

$\therefore (4 - a)^2 = 1 \qquad \therefore 4 - a = \pm 1$

$\therefore a = 4 \pm 1 \qquad \therefore a = 3 \text{ or } a = 5$

(3) $\because$ The curve of $f(X)$ is the same as the curve of g (X) by translation one unit to the left

$\therefore$ The function is increasing on $]-1, \infty[$

(4) The curve $y = 3(X-5)^2 + 7$ by translation 3 units to the right and one unit downwards

$\therefore y = 3(X-5-3)^2 + 7 - 1$

$\therefore y = 3(X-8)^2 + 6$

(5) $\because (f \circ f)(X) = f\left(f(X)\right) = f(|X| + 2)$
$= ||X| + 2| + 2 = |X| + 4$

$\therefore$ The range of the function $(f \circ f)$ is $[4, \infty[$

(6) $\because (f \circ g)(X) = f\left(g(X)\right) = |\frac{1}{2}X^2 - 8|$

$\therefore$ The range of the function $(f \circ g) = [0, \infty[$

(7) $\because |f(X)|$ is the same as the curve of $f(X)$ where f is odd after reflecting the part below X-axis upward

i.e. The function is $|f(X)|$ will be even.

(8) $\because$ The function is symmetric about the origin

$\therefore g(X) = X^3 - 2$

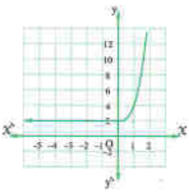
$\therefore$ g is an increasing function.

(9) $\because$ The function is symmetric about y-axis

$\therefore g(X) = -X^3 + 2$

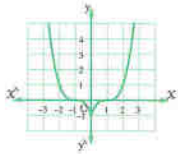
2

(1) $f(X) = \begin{cases} X^3 + X^3 + 2 & , X \geq 0 \\ X^3 - X^3 + 2 & , X < 0 \end{cases}$
$= \begin{cases} 2X^3 + 2 & , X \geq 0 \\ 2 & , X < 0 \end{cases}$



* The domain $= \mathbb{R}$, the range $= [2, \infty[$
* The function is constant on $]-\infty, 0[$, is increasing on $]0, \infty[$
* The function is neither odd nor even.

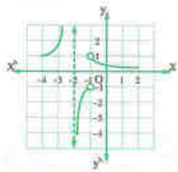
(2) $f(X) = \begin{cases} (X-1)^3 & , X \geq 0 \\ (-X-1)^3 & , X < 0 \end{cases}$
$= \begin{cases} (X-1)^3 & , X \geq 0 \\ -(X+1)^3 & , X < 0 \end{cases}$



* The domain $= \mathbb{R}$
* The range $= [-1, \infty[$
* The function is decreasing on $]-\infty, 0[$ and increasing on $]0, \infty[$
* The function is even.

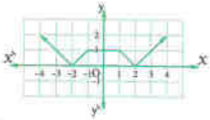
(3) $f(X) = \begin{cases} \dfrac{X+1}{(X+2)(X+1)} & , X > -1 \\ \dfrac{-(X+1)}{(X+2)(X+1)} & , X < -1, X \neq -2 \end{cases}$

$= \begin{cases} \dfrac{1}{X+2} & , X > -1 \\ -\dfrac{1}{X+2} & , X < -1, X \neq -2 \end{cases}$



* The domain $= \mathbb{R} - \{-1, -2\}$, the range $= \mathbb{R} - [-1, 0]$
* The function is increasing on $]-\infty, -2[$, $]-2, -1[$ and decreasing on $]-1, \infty[$
* The function is neither even nor odd.

(4) $f(X) = \begin{cases} -X-2 & , X < -2 \\ X+2 & , -2 \leq X \leq -1 \\ 1 & , -1 < X < 1 \\ -X+2 & , 1 \leq X < 2 \\ X-2 & , X \geq 2 \end{cases}$

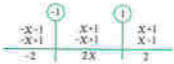


* The domain = $\mathbb{R}$, the range = $[0 , \infty[$

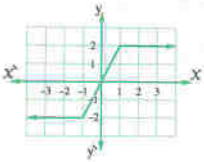
* The function is decreasing on $]-\infty , -2[$, $]1 , 2[$, increasing on $]-2 , -1[$, $]2 , \infty[$ and constant on $]-1 , 1[$

* The function is even.

(5) $f(X) = |X+1| - |X-1|$



$$= \begin{cases} 2 & , X > 1 \\ 2X & , -1 \le X \le 1 \\ -2 & , X < -1 \end{cases}$$



* The domain = $\mathbb{R}$, the range = $[-2 , 2]$

* The function is constant on each of $]-\infty , -1[$, $]1 , \infty[$ and increasing on $]-1 , 1[$

* The function is odd.

Exercise 6

First Multiple choice questions

(1) b	(2) c	(3) c	(4) a	(5) c	(6) c
(7) c	(8) b	(9) b	(10) d	(11) a	(12) b
(13) c	(14) b	(15) c	(16) c	(17) d	(18) a
(19) b	(20) b	(21) c	(22) c	(23) b	(24) d
(25) d	(26) d	(27) b	(28) c	(29) c	(30) b
(31) d	(32) d	(33) d	(34) b	(35) b	(36) c
(37) c	(38) a	(39) b	(40) b	(41) a	(42) b
(43) c	(44) c	(45) d	(46) d		

Second Essay questions

Exercises on solving absolute value equations

1

(1) $4|X| = 20$ $\quad \therefore |X| = 5$

$\therefore X = \pm 5$ $\quad \therefore$ The S.S. $= \{5 , -5\}$

(2) $|2X-3| = 7$ $\quad \therefore 2X - 3 = \pm 7$

$\therefore 2X - 3 = 7$ $\quad \therefore 2X = 10$

$\therefore X = 5$ or $2X - 3 = -7$

$\therefore 2X = -4$ $\quad \therefore X = -2$

$\therefore$ The S.S. $= \{5 , -2\}$

(3) $|X+2| = 1$ $\quad \therefore X + 2 = \pm 1$

$\therefore X = -1$ or $X = -3$

$\therefore$ The S.S. $= \{-1 , -3\}$

(4) $3|X| = 3$ $\quad \therefore |X| = 1$

$\therefore X = \pm 1$ $\quad \therefore$ The S.S. $= \{1 , -1\}$

(5) When $X \ge 2$:

$\therefore X - 2 = 3X - 4$ $\quad \therefore 2X = 2$

$\therefore X = 1 \notin [2 , \infty[$

When $X < 2$:

$\therefore -X + 2 = 3X - 4$ $\quad \therefore 4X = 6$

$\therefore X = \frac{3}{2} \in]-\infty , 2[$ $\quad \therefore$ The S.S. $= \left\{\frac{3}{2}\right\}$

(6) When $X \ge -2$:

$\therefore X + 2 = X - 1$ (refused)

When $X < -2$:

$\therefore -X - 2 = X - 1$ $\quad \therefore 2X = -1$

$\therefore X = -\frac{1}{2} \notin]-\infty , -2[$ $\quad \therefore$ The S.S. $= \varnothing$

(7) When $X \ge -2$:

$\therefore X + 2 = -X + 2$ $\quad \therefore 2X = 0$

$\therefore X = 0 \in [-2 , \infty[$

When $X < -2$:

$\therefore -X - 2 = -X + 2$ (refused)

$\therefore$ The S.S. $= \{0\}$

(8) $\because 2|X| = 5X - 21$

When $X \ge 0$:

$\therefore 2X = 5X - 21$ $\quad \therefore 3X = 21$

$\therefore X = 7 \in [0 , \infty[$

When $x < 0$:

$\therefore -2x = 5x - 21$ $\quad \therefore 7x = 21$

$\therefore x = 3 \notin]-\infty , 0[$ $\quad \therefore$ The S.S. $= \{7\}$

(9) $|x + 5| = |x - 3|$

$\therefore x + 5 = \pm (x - 3)$

$\therefore x + 5 = x - 3$ (refused)

or $x + 5 = -x + 3$ $\quad \therefore 2x = -2$

$\therefore x = -1$ (satisfy) $\quad \therefore$ The S.S. $= \{-1\}$

(10) $\because |2(x - 3)| = |x - 3|$

$\therefore 2|x - 3| = |x - 3|$ $\quad \therefore |x - 3| = 0$

$\therefore x - 3 = 0$ $\quad \therefore x = 3$

$\therefore$ The S.S. $= \{3\}$

(11) $|x - 1| = 2|x - 2|$

$\therefore x - 1 = \pm 2(x - 2)$ $\quad \therefore x - 1 = 2x - 4$

$\therefore x = 3$ (satisfy)

or $x - 1 = -2x + 4$ $\quad \therefore 3x = 5$

$\therefore x = \frac{5}{3}$ (satisfy) $\quad \therefore$ The S.S. $= \left\{3 , \frac{5}{3}\right\}$

(12) When $x \geq 0$:

$\therefore x^2 - 5x + 6 = 0$

$\therefore (x - 2)(x - 3) = 0$

$\therefore x = 2 \in [0 , \infty[$ or $x = 3 \in [0 , \infty[$

When $x < 0$: $\quad \therefore x^2 + 5x + 6 = 0$

$\therefore (x + 2)(x + 3) = 0$ $\quad \therefore x = -2 \in]-\infty , 0[$

or $x = -3 \in]-\infty , 0[$

$\therefore$ The S.S. $= \{2 , 3 , -2 , -3\}$

(13) $\sqrt{x^2} - 2x = 6$ $\quad \therefore |x| - 2x = 6$

When $x \geq 0$: $\quad \therefore x - 2x = 6$

$\therefore x = -6 \notin [0 , \infty[$

When $x < 0$:

$\therefore -x - 2x = 6$ $\quad \therefore 3x = -6$

$\therefore x = -2 \in]-\infty , 0[$ $\quad \therefore$ The S.S. $= \{-2\}$

(14) $\because \sqrt{x^2 - 4x + 4} = 4$

$\therefore \sqrt{(x - 2)^2} = 4$ $\quad \therefore |x - 2| = 4$

$\therefore x - 2 = \pm 4$

$\therefore x = 6$ or $x = -2$

$\therefore$ The S.S. $= \{6 , -2\}$

(15) $|x| + x = 0$

When $x \geq 0$:

$\therefore x + x = 0$ $\quad \therefore 2x = 0$

$\therefore x = 0 \in [0 , \infty[$

When $x < 0$: $\quad \therefore -x + x = 0$

$\therefore 0 = 0$ and this is satisfying for any value of $x < 0$: $\quad \therefore$ The S.S. $=]-\infty , 0]$

(16) When $x \geq -3$: $\quad \therefore x + 3 + 2x = 0$

$\therefore 3x = -3$ $\quad \therefore x = -1 \in [-3 , \infty[$

When $x < -3$: $\quad \therefore -x - 3 + 2x = 0$

$\therefore x = 3 \notin]-\infty , -3[$ $\quad \therefore$ The S.S. $= \{-1\}$

(17) $\because \sqrt{x^2 - 6x + 9} + 2x = 9$

$\therefore \sqrt{(x - 3)^2} + 2x = 9$ $\quad \therefore |x - 3| + 2x = 9$

When $x \geq 3$:

$\therefore x - 3 + 2x = 9$ $\quad \therefore 3x = 12$

$\therefore x = 4 \in [3 , \infty[$

When $x < 3$: $\quad \therefore -x + 3 + 2x = 9$

$\therefore x = 6 \notin]-\infty , 3[$ $\quad \therefore$ The S.S. $= \{4\}$

(18) $|x - 3|\left(|x - 3| - 1\right) = 0$

$\therefore |x - 3| = 0$ and hence $x = 3$

or $|x - 3| - 1 = 0$ $\quad \therefore x - 3 = \pm 1$

and hence $x = 2$ or $x = 4$

$\therefore$ The S.S. $= \{3 , 2 , 4\}$

(19) $\because 5|x - 3| - 2\sqrt{(x - 3)^2} = 12$

$\therefore 5|x - 3| - 2|x - 3| = 12$

$\therefore 3|x - 3| = 12$ $\quad \therefore |x - 3| = 4$

$\therefore x - 3 = \pm 4$ $\quad \therefore x = 7$ or $x = -1$

$\therefore$ The S.S. $= \{7 , -1\}$

(20) $|x - 1||x + 1| = |x - 1|$

$\therefore |x - 1|\left(|x + 1| - 1\right) = 0$

$\therefore |x - 1| = 0$ $\quad \therefore x = 1$

or $|x + 1| - 1 = 0$ $\quad \therefore x + 1 = \pm 1$

$\therefore x = 0$ or $x = -2$

$\therefore$ The S.S. $= \{1 , 0 , -2\}$

(21) $|x^2 - 1| = 26$ $\quad \therefore x^2 - 1 = \pm 26$

$\therefore x^2 = -25$ (refused)

or $x^2 = 27$ and hence $x = \pm 3\sqrt{3}$

$\therefore$ The S.S. $= \{3\sqrt{3} , -3\sqrt{3}\}$

(22) $\left(|x+1|+2\right)\left(|x+1|-5\right)=0$

$\therefore |x+1|+2=0$ (refused)

or $|x+1|-5=0$

$\therefore x+1=\pm 5$ and hence $x=4$ or $x=-6$

$\therefore$ The S.S. $=\{4 , -6\}$

(23) $|x-5|^2=2|x-5|$

$\therefore |x-5|\left(|x-5|-2\right)=0$

$\therefore |x-5|=0$ and hence $x=5$

or $|x-5|-2=0$ and hence $x-5=\pm 2$

$\therefore x=7$ or $x=3$

$\therefore$ The S.S. $=\{5 , 7 , 3\}$

(24) $x(|x|-1)=0$

$\therefore x=0$ or $|x|-1=0$

$\therefore |x|=1$ $\quad\therefore x=\pm 1$

$\therefore$ The S.S. $=\{0 , 1 , -1\}$

(25) $x|x-5|-6=0$

When $x\geq 5$ $\quad\therefore x^2-5x-6=0$

$\therefore (x-6)(x+1)=0$

$\therefore x=6\in[5 , \infty[$ or $x=-1\notin[5 , \infty[$

When $x<5$ $\quad\therefore x^2-5x+6=0$

$\therefore (x-2)(x-3)=0$ $\quad\therefore x=2\in]-\infty , 5[$

or $x=3\in]-\infty , 5[$

$\therefore$ The S.S. $=\{2 , 3 , 6\}$

(26) $x^2+x-10=\pm 10$ $\quad\therefore x^2+x-10=-10$

$\therefore x^2+x=0$ $\quad\therefore x(x+1)=0$

$\therefore x=0$ or $x=-1$

or $x^2+x-20=0$ $\quad\therefore (x+5)(x-4)=0$

$\therefore x=-5$ or $x=4$

$\therefore$ The S.S. $=\{0 , -1 , -5 , 4\}$

(27) $x|x-2|=4x-8$

When $x>2$: $\therefore x(x-2)=4x-8$

$\therefore x^2-2x-4x+8=0$

$\therefore x^2-6x+8=0$ $\quad\therefore (x-2)(x-4)=0$

$\therefore x=2\notin]2 , \infty[$ or $x=4\in]2 , \infty[$

When $x<2$: $\therefore x(-x+2)=4x-8$

$\therefore -x^2+2x=4x-8$

$\therefore x^2+2x-8=0$ $\quad\therefore (x+4)(x-2)=0$

$\therefore x=-4\in]-\infty , 2[$ or $x=2\notin]-\infty , 2[$

$\therefore$ The S.S. $=\{4 , -4\}$

(28) When $x\geq 0$: $\quad\therefore x^3\times x=8x$

$\therefore x^4-8x=0$ $\quad\therefore x(x^3-8)=0$

$\therefore x=0\in[0 , \infty[$ or $x^3-8=0$

$\therefore x^3=8$ $\quad\therefore x=2\in[0 , \infty[$

When $x<0$: $\quad\therefore x^3\times -x=8x$

$\therefore x^4+8x=0$ $\quad\therefore x(x^3+8)=0$

$\therefore x=0\notin]-\infty , 0[$ or $x^3+8=0$

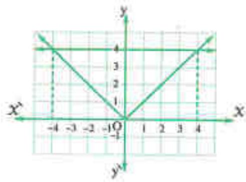
$\therefore x^3=-8$ $\quad\therefore x=-2\in]-\infty , 0[$

$\therefore$ The S.S. $=\{-2 , 0 , 2\}$

2

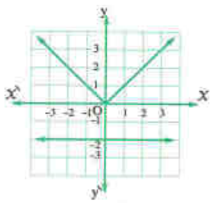
* We shall give the solution graphically and you can verify it algebraically.
* Draw the curves of the two functions $f(x)$, $g(x)$, the x - coordinate of the intersection point of the two curves represents the S.S.

(1) $\because |x|=4$ $\quad\therefore f(x)=|x|$, $g(x)=4$



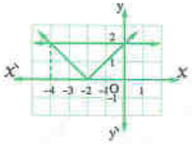
From the graph : The S.S. $=\{4 , -4\}$

(2) $\because |x|=-2$ $\quad\therefore f(x)=|x|$, $g(x)=-2$



From the graph : The S.S. $=\varnothing$

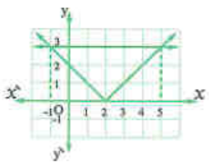
(3) $\because |x+2| = 2$ $\quad \therefore f(x) = |x+2|$, $g(x) = 2$



From the graph : The S.S. = $\{0 , -4\}$

(4) $\because \sqrt{(x-2)^2} = 3$ $\quad \therefore |x-2| = 3$

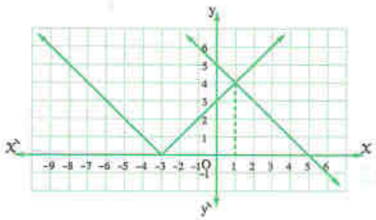
$\therefore f(x) = |x-2|$, $g(x) = 3$



From the graph : The S.S. = $\{5 , -1\}$

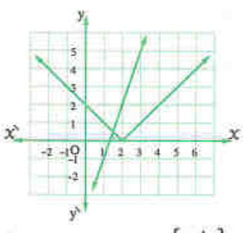
(5) $\because |x+3| = 5 - x$

$\therefore f(x) = |x+3|$, $g(x) = 5 - x$



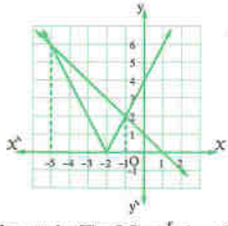
From the graph : The S.S. = $\{1\}$

(6) $f(x) = |x-2|$, $g(x) = 3x - 4$



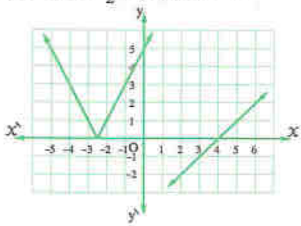
From the graph : The S.S. = $\left\{1\frac{1}{2}\right\}$

(7) $f(x) = 2|x+2|$, $g(x) = 1 - x$



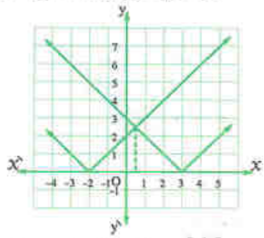
From the graph : The S.S. = $\{-1 , -5\}$

(8) $f(x) = 2\left|x + \frac{5}{2}\right|$, $g(x) = x - 4$



From the graph : The S.S. = $\varnothing$

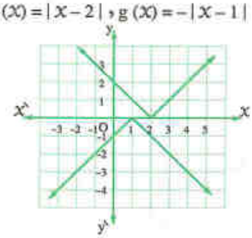
(9) $f(x) = |x+2|$, $g(x) = |x-3|$



From the graph : The S.S. = $\left\{\frac{1}{2}\right\}$

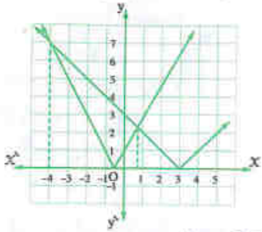
(10) $\because |x-2| = -|x-1|$

$\therefore f(x) = |x-2|$, $g(x) = -|x-1|$



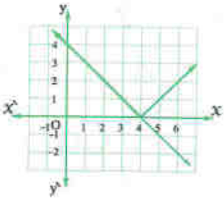
From the graph : The S.S. = $\varnothing$

(11) $f(x) = |x - 3|$, $g(x) = |2x + 1|$



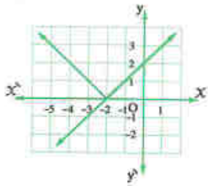
From the graph : The S.S. $= \left\{-4 , \frac{2}{3}\right\}$

(12) $f(x) = |x - 4|$, $g(x) = 4 - x$



From the graph : The S.S. $=]-\infty , 4]$

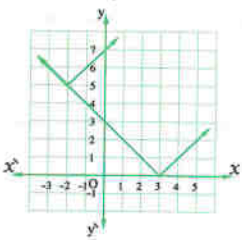
(13) $f(x) = |x + 2|$, $g(x) = x + 2$



From the graph : The S.S. $= [-2 , \infty[$

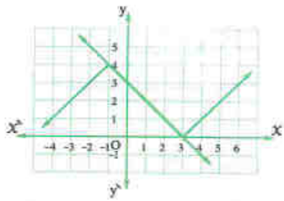
(14) $\because |x - 3| = |x + 2| + 5$

$\therefore f(x) = |x - 3|$, $g(x) = |x + 2| + 5$



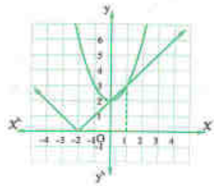
From the graph : The S.S. $=]-\infty , -2]$

(15) $f(x) = |x - 3|$, $g(x) = 4 - |x + 1|$



From the graph : The S.S. $= [-1 , 3]$

(16) $f(x) = |x + 2|$, $g(x) = x^2 + 2$



From the graph : The S.S. $= \{0 , 1\}$

3

$f(-x) = \frac{12}{|-x| + 2} = \frac{12}{|x| + 2} = f(x)$

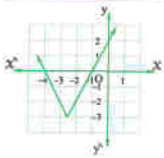
$\therefore$ The function is even.

$\because f(x) = 2$ $\qquad$ $\therefore \frac{12}{|x| + 2} = 2$

$\therefore |x| + 2 = 6$ $\qquad$ $\therefore |x| = 4$

$\therefore x = \pm 4$ $\qquad$ $\therefore$ The S.S. $= \{-4 , 4\}$

4



From the graph :

* The range $= [-3 , \infty[$
* The function is decreasing on $]-\infty , \frac{-5}{2}[$

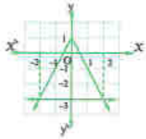
  and increasing on $]\frac{-5}{2} , \infty[$
* The S.S. $= \{-1 , -4\}$
* The algebraic solution :

$\therefore |2x + 5| = 3$ $\qquad$ $\therefore 2x + 5 = \pm 3$

$\therefore 2x + 5 = 3$, then $x = -1$

or $2x + 5 = -3$, then $x = -4$

5



From the graph :

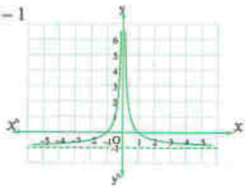
* The range $=]-\infty , 1]$
* The function is increasing on $]-\infty , 0[$ and decreasing on $]0 , \infty[$
* The function is even because it is symmetric about the y-axis
* The S.S. $= \{-2 , 2\}$, you can verify algebraically.

6

$f(-x) = \frac{1-|-x|}{|-x|} = \frac{1-|x|}{|x|} = f(x)$

$\therefore$ The function is even.

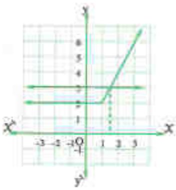
$f(x) = \frac{1-|x|}{|x|} = \frac{1}{|x|} - 1$



From the graph : The S.S. $= \{-1 , 1\}$

7

$f(x) = \begin{cases} 2x & , x \geq 1 \\ 2 & , x < 1 \end{cases}$

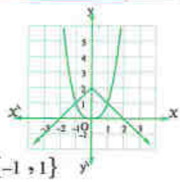


From the graph :

* The range $= [2 , \infty[$
* The function is constant on $=]-\infty , 1[$ and is increasing on $]1 , \infty[$
* The S.S. $= \left\{\frac{3}{2}\right\}$

8

$f(x) = \begin{cases} x^3 & , x \geq 0 \\ -x^3 & , x < 0 \end{cases}$



From the graph : The S.S. $= \{-1 , 1\}$

Exercises on solving absolute value inequalities

9

(1) $-5 \leq x-3 \leq 5$ $\quad \therefore -2 \leq x \leq 8$

$\therefore$ The S.S. $= [-2 , 8]$

(2) $x-3 \geq 5$, then $x \geq 8$

or $x-3 \leq -5$, then $x \leq -2$

$\therefore$ The S.S. $= \mathbb{R} -]-2 , 8[$

(3) $2x+5 > 3$, then $x > -1$

or $2x+5 < -3$, then $x < -4$

$\therefore$ The S.S. $= \mathbb{R} - [-4 , -1]$

(4) $\because |x-3| < 7$ $\quad \therefore -7 < x-3 < 7$

$\therefore -4 < x < 10$ $\quad \therefore$ The S.S. $=]-4 , 10[$

(5) $-1 \leq \frac{x-3}{4} \leq 1$ $\quad \therefore -4 \leq x-3 \leq 4$

$\therefore -1 \leq x \leq 7$ $\quad \therefore$ The S.S. $= [-1 , 7]$

(6) $|3x+2| < -1$ $\quad \therefore$ The S.S. $= \varnothing$

(7) $\because \frac{1}{|2x-5|} \geq 3$ $\quad \therefore |2x-5| \leq \frac{1}{3}$

$\therefore -\frac{1}{3} \leq 2x-5 \leq \frac{1}{3}$

$\therefore 4\frac{2}{3} \leq 2x \leq 5\frac{1}{3}$ $\quad \therefore \frac{7}{3} \leq x \leq \frac{8}{3}$

, $\because |2x-5| = 0$ when $x = \frac{5}{2}$

$\therefore$ The S.S. $= \left[\frac{7}{3} , \frac{8}{3}\right] - \left\{\frac{5}{2}\right\}$

(8) $\because \frac{1}{|3x|} \geq 5$ $\quad \therefore |3x| \leq \frac{1}{5}$

$\therefore -\frac{1}{5} \leq 3x \leq \frac{1}{5}$ $\quad \therefore -\frac{1}{15} \leq x \leq \frac{1}{15}$

, $\because |3x| = 0$ when $x = 0$

$\therefore$ The S.S. $= \left[-\frac{1}{15} , \frac{1}{15}\right] - \{0\}$

(9) $\because \frac{1}{|2x-3|} > 2$ $\quad \therefore |2x-3| < \frac{1}{2}$

$\therefore -\frac{1}{2} < 2x-3 < \frac{1}{2}$ $\quad \therefore \frac{5}{2} < 2x < \frac{7}{2}$

$\therefore \frac{5}{4} < x < \frac{7}{4}$

, $\because |2x-3| = 0$ when $x = \frac{3}{2}$

$\therefore$ The S.S. $= \left]\frac{5}{4} , \frac{7}{4}\right[- \left\{\frac{3}{2}\right\}$

(10) $|2x-3|>4$ $\quad \therefore 2x-3>4$

, then $x>\frac{7}{2}$ or $2x-3<-4$, then $x<-\frac{1}{2}$

$\therefore$ The S.S. $=\mathbb{R}-\left[-\frac{1}{2} , \frac{7}{2}\right]$

(11) $\because \sqrt{(x-1)^2}\geq 4$ $\quad \therefore |x-1|\geq 4$

$\therefore x-1\geq 4$, then $x\geq 5$

or $x-1\leq -4$, then $x\leq -3$

$\therefore$ The S.S. $=\mathbb{R}-]-3 , 5[$

(12) $\because \sqrt{(2x-3)^2}\leq 9$ $\quad \therefore |2x-3|\leq 9$

$\therefore -9\leq 2x-3\leq 9$ $\quad \therefore -6\leq 2x\leq 12$

$\therefore -3\leq x\leq 6$ $\quad \therefore$ The S.S. $=[-3 , 6]$

(13) $2|x-2|<6$ $\quad \therefore |x-2|<3$

$\therefore -3<x-2<3$ $\quad \therefore -1<x<5$

$\therefore$ The S.S. $=]-1 , 5[$

(14) $|3x-2|+2\times 2|3x-2|\geq 20$

$\therefore 5|3x-2|\geq 20$ $\quad \therefore |3x-2|\geq 4$

$\therefore 3x-2\geq 4$

, then $x\geq 2$ or $3x-2\leq -4$, then $x\leq -\frac{2}{3}$

$\therefore$ The S.S. $=\mathbb{R}-\left]\frac{-2}{3} , 2\right[$

(15) $|x+2|+2|x+2|\geq 6$

$\therefore 3|x+2|\geq 6$ $\quad \therefore |x+2|\geq 2$

$\therefore x+2\geq 2$, then $x\geq 5$

or $x+2\leq -2$, then $x\leq -4$

$\therefore$ The S.S. $=\mathbb{R}-]-4 , 0[$

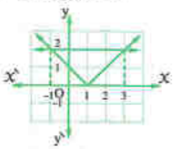
10

(1) $]-4 , 0[$ $\quad$ (2) $\mathbb{R}-[-1 , 5]$ $\quad$ (3) $[-5 , -1]$

11

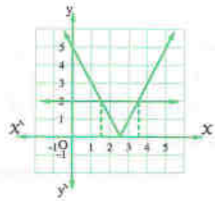
The following is the graphically solution, verify algebraically by yourself :

(1) $f(x)=|x-1|$, $g(x)=2$



From the graph : The S.S. $=]-1 , 3[$

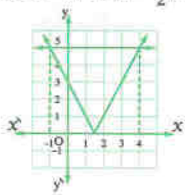
(2) $f(x)=2|x-\frac{5}{2}|$, $g(x)=2$



From the graph : The S.S. $=\mathbb{R}-\left]\frac{3}{2} , \frac{7}{2}\right[$

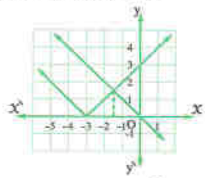
(3) $\because \sqrt{4x^2-12x+9}=\sqrt{(2x-3)^2}=|2x-3|$

$\therefore f(x)=|2x-3|=2|x-\frac{3}{2}|$, $g(x)=5$



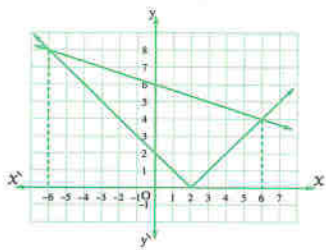
From the graph : The S.S. $=\mathbb{R}-[-1 , 4]$

(4) $f(x)=|x+3|$, $g(x)=-x$



From the graph : The S.S. $=\left]-\infty , -\frac{3}{2}\right[$

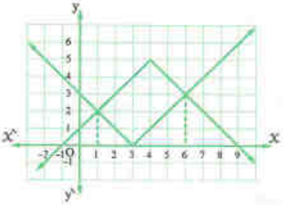
(5) $f(x)=|x-2|$, $g(x)=\frac{-1}{3}x+6$



From the graph : The S.S. $=]-6 , 6[$

(6) $\because |x-3| > 5 - |x-4|$

$\therefore f(x) = |x-3|$, $g(x) = 5 - |x-4|$



From the graph : The S.S. $= \mathbb{R} - [1, 6]$

12

(1) $\because -4 \le x \le 4$ $\therefore |x| \le 4$

(2) $\because 0 < x < 6$, adding (–3) to the terms of the inequality $\therefore -3 < x - 3 < 3$

$\therefore |x-3| < 3$

(3) $\because x \le -2$ or $x \ge 2$ $\therefore |x| \ge 2$

(4) $\because x \in \mathbb{R} - [-2, 6]$

$\therefore x > 6$, $x < -2$, adding (–2) to the terms of the inequality

$\therefore x - 2 > 4$, $x - 2 < -4$ $\therefore |x-2| > 4$

13

(1) Let the mark of the student be x

$\therefore 60 < x < 100$, adding (–80) to the terms of the inequality

$\therefore -20 < x - 80 < 20$ $\therefore |x - 80| < 20$

(2) Let the temperature = x degree

$\therefore 35 < x < 42$, adding (–38.5) to the terms of the inequality

$\therefore -3.5 < x - 38.5 < 3.5$ $\therefore |x - 38.5| < 3.5$

(3) Let the depth that the green algae live in be x metres

$\therefore 0 \le x \le 30$, adding (–15) to the terms of the inequality

$\therefore -15 \le x - 15 \le 15$ $\therefore |x - 15| \le 15$

Miscellaneous exercises

14

(1) Let $|x| - 1 = 0$ $\therefore |x| = 1$

$\therefore x = \pm 1$

$\therefore$ The domain $= \mathbb{R} - \{1, -1\}$

(2) Let $|x| + 1 = 0$ $\therefore |x| = -1$ (contradiction)

$\therefore$ The domain $= \mathbb{R}$

(3) Let $|x-2| - 5 = 0$ $\therefore |x-2| = 5$

$\therefore x - 2 = 5$, then $x = 7$

or $x - 2 = -5$, then $x = -3$

$\therefore$ The domain $= \mathbb{R} - \{7, -3\}$

(4) Let $5 - |x| \ge 0$ $\therefore |x| \le 5$

$\therefore -5 \le x \le 5$ $\therefore$ The domain $= [-5, 5]$

(5) Let $|x| - 5 \ge 0$ $\therefore |x| \ge 5$

$\therefore x \ge 5$ or $x \le -5$

$\therefore$ The domain $= \mathbb{R} -]-5, 5[$

(6) Let $5 - |x| > 0$ $\therefore |x| < 5$

$\therefore -5 < x < 5$ $\therefore$ The domain $=]-5, 5[$

15

(1) $f(-x) = -x|-x| = -x|x| = -f(x)$

$\therefore$ The function is odd.

(2) $f(-x) = (-x)^2|-x| - 1 = x^2|x| - 1 = f(x)$

$\therefore$ The function is even.

(3) $f(-x) = \frac{|1-x| + |1+x|}{|1-x| - |1+x|} = -f(x)$

$\therefore$ The function is odd.

(4) $f(-x) = \frac{(-x)^2 \cos(-2x)}{5 + |-2x|} = \frac{x^2 \cos 2x}{5 + |2x|} = f(x)$

$\therefore$ The function is even.

(5) $f(-x) = 2|-x| \tan(-x) + 2(-x)|\tan(-x)|$

$= 2|x|(-\tan x) - 2x|-\tan x|$

$= -2|x| \tan x - 2x|\tan x| = -f(x)$

$\therefore$ The function is odd.

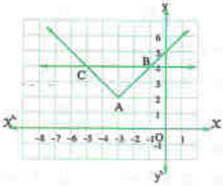
(6) $f(-x) = \sqrt{|-x| + (-x)} = \sqrt{|x| - x}$

$\ne f(x) \ne -f(x)$

$\therefore$ The function is neither even nor odd.

16

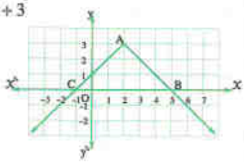
(1) $f(x) = |x+3| + 2$, $g(x) = 4$



$\therefore$ Area of the included region between the two curves f , g = area of Δ ABC

$= \frac{1}{2} \times 4 \times 2 = 4$ square units.

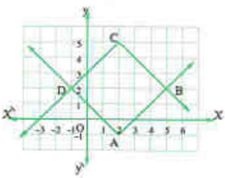
(2) $f(x) = -|x-2| + 3$, $g(x) =$ zero



$\therefore$ Area of the included region between the two curves f , g = area of Δ ABC

$= \frac{1}{2} \times 6 \times 3 = 9$ square units.

(3) $f(x) = |x-2| - 1$, $g(x) = 5 - |x-2|$



$\therefore$ Area of the included region between the two curves f , g = area of the square ABCD

$= \frac{1}{2}(BD)^2 = \frac{1}{2} \times 6^2$

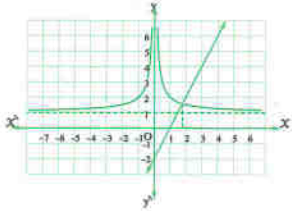
$= 18$ square units.

17

$f(-x) = \frac{|-x|+1}{|-x|} = \frac{|x|+1}{|x|} = f(x)$

$\therefore$ The function is even.

, $f(x) = \frac{|x|+1}{|x|} = 1 + \frac{1}{|x|}$



Graphically :

Put $g(x) = 2x - 2$

From the graph : $x \simeq 1.8$

$\therefore$ The S.S. $= \{1.8\}$

Algebraically : $\because f(x) = \begin{cases} \frac{1}{x} + 1 & , x > 0 \\ \frac{-1}{x} + 1 & , x < 0 \end{cases}$

When $x > 0$:

$\therefore \frac{1}{x} + 1 = 2x - 2 \quad \therefore \frac{1}{x} = 2x - 3$ (multiply by x)

$\therefore 1 = 2x^2 - 3x \quad \therefore 2x^2 - 3x - 1 = 0$

$\therefore x = \frac{3 \pm \sqrt{9 - 4 \times 2 \times -1}}{2 \times 2} = \frac{3 \pm \sqrt{17}}{4}$

$\therefore x \simeq 1.8 \in]0 , \infty[$ or $x \simeq -0.3 \notin]0 , \infty[$

When $x < 0$: $\therefore \frac{-1}{x} + 1 = 2x - 2$

$\therefore \frac{-1}{x} = 2x - 3$ (multiply by x)

$\therefore -1 = 2x^2 - 3x \quad \therefore 2x^2 - 3x + 1 = 0$

$\therefore (2x - 1)(x - 1) = 0$

$\therefore x = \frac{1}{2} \notin]-\infty , 0[$ or $x = 1 \notin]-\infty , 0[$

$\therefore$ The S.S. $= \{1.8\}$

Third Higher skills

1

(1) (d) (2) (d) (3) (c) (4) (c) (5) (d)

(6) (a) (7) (a) (8) (c) (9) (b)

Instructions to solve 1 :

(1) $\because |x+1|^2 + |2x+3| = 0$

$\therefore x + 1 = 0$ and so $x = -1$

In the same time $2x + 3 = 0$ and so $x = \frac{-3}{2}$ this is contradiction

$\therefore$ S.S. $= \varnothing$

(2) $\because |(x-1)(x-3)| = |x-3|$

$\therefore |x-1||x-3| - |x-3| = 0$

$\therefore |x-3|(|x-1|-1) = 0$

$\therefore |x-3| = 0$ and so $x = 3$ or $|x-1| = 1$

$\therefore x-1 = \pm 1$ and so $x = 0$ or 2

$\therefore$ S.S. $= \{0, 2, 3\}$

(3) $\because |x| + |y| \geq |x+y|$

$\therefore \frac{|x|+|y|}{|x+y|} \geq 1$

$\therefore$ The smallest value of the expression $\frac{|x|+|y|}{|x+y|}$ is 1

(4) $\because 32 < 61 < 64$ $\qquad \therefore 2^5 < 2^x < 2^6$

$\therefore 5 < x < 6$

$\therefore |x-6| = -x+6$, $|x-5| = x-5$

$\therefore |x-6| + |x-5| = -x+6+x-5 = 1$

(5) $\because a^2 b > 0$ $\qquad \therefore$ b is positive

, $\because \frac{a}{b} < 0$ $\qquad \therefore$ a is negative

$\therefore \sqrt{a^2} = |a| = -a$, $\sqrt{b^2} = |b| = b$

$\therefore \sqrt{a^2} + \sqrt{b^2} - (b-a) = -a+b-b+a = \text{zero}$

(6) $\because -4 \leq x \leq 6$ $\qquad \therefore -16 \leq 4x \leq 24$

$\therefore -24 \leq 4x - 8 \leq 16$ $\qquad \therefore 0 \leq |4x-8| \leq 24$

$\therefore a = \text{zero}$, $b = 24$ $\qquad \therefore a + b = 24$

(7) $\because x^2 - 3|x| - 10 = 0$ $\qquad \therefore |x|^2 - 3|x| - 10 = 0$

$\therefore (|x|-5)(|x|+2) = 0$

$\therefore |x| = 5$ and so $x = \pm 5$

or $|x| = -2$ (Refused)

$\therefore$ The roots of the equation are ± 5

$\therefore$ The product of the roots equals -25

(8) $\because 3 < |x| < 8$ $\qquad \therefore 3 < x < 8$

$\therefore x = 4, 5, 6, 7$ or $-8 < x < -3$

$\therefore x = -4, -5, -6, -7$

$\therefore$ The number of integer values of x is 8

(9) $\because (g \circ f)(x) = 3$ $\qquad \therefore g(f(x)) = 3$

$\therefore ||2x-5|+1| = 3$ $\qquad \therefore |2x-5| + 1 = \pm 3$

$\therefore |2x-5| = \pm 3 - 1$

$\therefore |2x-5| = -4$ (Refused)

or $|2x-5| = 2$

$\therefore 2x - 5 = \pm 2$ $\qquad \therefore 2x = \pm 2 + 5$

$\therefore 2x = 7$ and so $x = \frac{7}{2}$

or $2x = 3$ and so $x = \frac{3}{2}$

$\therefore$ The solution set $= \left\{\frac{7}{2}, \frac{3}{2}\right\}$

2

(1) $(x+1)(|x|-1) + \frac{1}{2} = \text{zero}$

When $x \geq 0$: $\therefore (x+1)(x-1) + \frac{1}{2} = 0$

$\therefore x^2 - 1 + \frac{1}{2} = 0$ $\qquad \therefore x^2 - \frac{1}{2} = 0$

$\therefore x = \frac{1}{\sqrt{2}} \in [0, \infty[$ or $x = \frac{-1}{\sqrt{2}} \notin [0, \infty[$

When $x < 0$: $\therefore (x+1)(-x-1) + \frac{1}{2} = 0$

$\therefore -(x+1)^2 + \frac{1}{2} = 0$ $\qquad \therefore x + 1 = \pm \frac{1}{\sqrt{2}}$

$\therefore x = \left(-1 + \frac{1}{\sqrt{2}}\right) \in]-\infty, 0[$

or $x = \left(-1 - \frac{1}{\sqrt{2}}\right) \in]-\infty, 0[$

$\therefore$ The S.S. $= \left\{\frac{1}{\sqrt{2}}, -1 + \frac{1}{\sqrt{2}}, -1 - \frac{1}{\sqrt{2}}\right\}$

(2) $\because ||2x+3| - 8| = 5$

$\therefore |2x+3| - 8 = \pm 5$

$\therefore |2x+3| = 8 \pm 5$

$\therefore |2x+3| = 13$ or $|2x+3| = 3$

$\therefore 2x + 3 = \pm 13$	$2x + 3 = \pm 3$
$\therefore 2x = -3 \pm 13$	$\therefore 2x = -3 \pm 3$
$\therefore x = -8$ or 5	$\therefore x = \text{zero}$ or -3

$\therefore$ The solution set $= \{-8, 5, \text{zero}, -3\}$

(3) $\because \frac{2}{|x|} + |x| = 3$ "multiply by $|x|$"

$\therefore |x|^2 - 3|x| + 2 = 0$

$\therefore (|x|-2)(|x|-1) = 0$

$\therefore |x| = 2$, then $x = \pm 2$ or $|x| = 1$, then $x = \pm 1$

$\therefore$ The S.S. $= \{2, -2, 1, -1\}$

(4) $\sqrt{x^2-2+\frac{1}{x^2}+4}=2$ $\therefore\sqrt{x^2+2+\frac{1}{x^2}}=2$

$\therefore\sqrt{\left(x+\frac{1}{x}\right)^2}=2$ $\therefore\left|x+\frac{1}{x}\right|=2$

$\therefore x+\frac{1}{x}=\pm 2$

$\therefore x+\frac{1}{x}=2$, then $x^2-2x+1=0$

$\therefore (x-1)^2=0$ $\therefore x=1$

or $x+\frac{1}{x}=-2$, then $x^2+2x+1=0$

$\therefore (x+1)^2=0$ $\therefore x=-1$

$\therefore$ The S.S. $=\{1 , -1\}$

(5) $\because |x^2-3|+\frac{2}{|x^2-3|}=3$

$\therefore |x^2-3|^2-3|x^2-3|+2=0$

$\therefore (|x^2-3|-1)(|x^2-3|-2)=0$

$\therefore |x^2-3|-1=0$ or $|x^2-3|-2=0$

$\therefore |x^2-3|=1$	$\therefore |x^2-3|=2$
$\therefore x^2-3=\pm 1$	$\therefore x^2-3=\pm 2$
$\therefore x^2=3\pm 1$	$\therefore x^2=3\pm 2$
$\therefore x^2=3\pm 1$	$\therefore x^2=5$ and so $x=\pm\sqrt{5}$

$\therefore x^2=4$ and so $x=\pm 2$

or $x^2=2$ and so $x\pm\sqrt{2}$ or $x^2=1$ and so $x=\pm 1$

$\therefore$ S.S. $=\{2 , -2 , \sqrt{2} , -\sqrt{2} , 1 , -1 , \sqrt{5} , -\sqrt{5}\}$

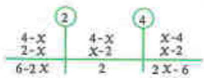
(6) $|x-4|+|x-2|=10$

$\therefore$ at $x>4$

$\therefore 2x-6=10$

$\therefore x=8$ (verified)

, at $x<2$



$\therefore 6-2x=10$

$\therefore x=-2$ (verified)

at $2<x<4$

$\therefore 2=10$ (contradiction)

$\therefore$ The solution set $=\{8 , -2\}$

(7) $\because (|2x-3|-5)(|2x-3|+5)>11$

$\therefore |2x-3|^2-25>11$ $\therefore |2x-3|^2>36$

$\therefore |2x-3|>6$ $\therefore 2x-3>6$ and so $x>\frac{9}{2}$

or $2x-3<-6$ and so $x<\frac{-3}{2}$

$\therefore$ S.S. $=\mathbb{R}-\left[-\frac{3}{2} , \frac{9}{2}\right]$

Answers of Life Applications on Unit One

1

(1) v (10) = 8 × 10 = 80 cm./sec.

(2) v (150) = 80 cm./sec.

(3) v (210) = − 4 × 210 + 880 = 40 cm./sec.

2

$P(\ell)=4\ell$

(1) P (3) = 4 × 3 = 12 length unit.

(2) $P\left(\frac{15}{4}\right)=4\times\frac{15}{4}=15$ length unit.

3

$A(r)=\pi r^2$

(1) $A\left(\frac{1}{2}\right)=\frac{1}{4}\pi$ square unit.

(2) A (5) = 25 π square unit.

4

(1) $f(x)=0.3x+8\times 7$

(2) Yes , the function is one-to-one.

because : let a , b ∈ the domain of the function f

, $\because f(a)=0.3a+8\times 7$

, $f(b)=0.3b+8\times 7$

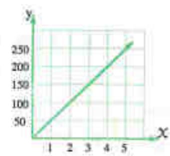
, put $f(a)=f(b)$

$\therefore 0.3a+8\times 7=0.3b+8\times 7$

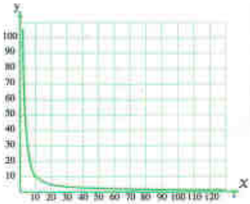
$\therefore 0.3a=0.3b$ $\therefore a=b$

5

$f(x)=50x$



6



$\because d = v \times t$

$, \because t = 3$ min.

$, v = 30$ m./min.

$\therefore d = 3 \times 30 = 90$

$\therefore v = \frac{90}{t}$

$\therefore v \propto \frac{1}{t}$

$\therefore$ v varies inversely with respect to t

$, v = \frac{90}{t}$ when v = 45 m./min.

$\therefore 45 = \frac{90}{t}$ $\qquad \therefore t = \frac{90}{45} = 2$ min.

7

(1) $\because$ The point (0 , 3) belongs to the curve of the function

$\therefore$ (0 , 3) satisfies the equation of the function

$f(X) = a\,(X - 2)^2 + 4$

$\therefore 3 = a\,(0 - 2)^2 + 4$ $\qquad \therefore 3 = 4a + 4$

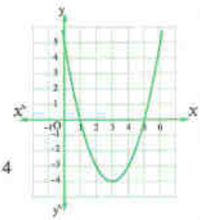
$\therefore 4a = -1$ $\qquad \therefore a = -\frac{1}{4}$

(2) $\because$ The point of the vertex of the curve is (2 , 4)

$\therefore$ The maximum height of the gate = 4 m.

(3) The width of the gate = 2 + 2 = 4 m.

8



(1) From the drawing

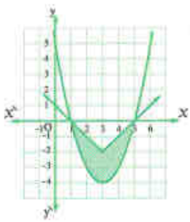
$\ell = 4$ length unit

, z = 4 length unit

$\therefore$ The required area

$= \frac{2}{3}\,\ell z = \frac{2}{3} \times 4 \times 4$

$= \frac{32}{3}$ square unit.



(2) The required area

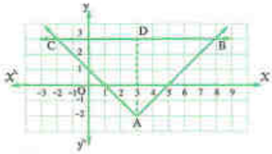
$= \frac{32}{3} - \left(\frac{1}{2} \times 4 \times 2\right)$

$= \frac{20}{3}$ square unit.

9

$f(X) = |X - 3| - 2$

$, g(X) = 3$



$\because$ Length of the unit = 8 metres.

$, \because$ BC = 10 units.

$\therefore$ length of the base = 10 × 8 = 80 metres.

$, \because$ AD = 5 units. $\quad \therefore$ The height = 5 × 8 = 40 metres.

$\therefore$ Area of the land = area of Δ ABC

$= \frac{1}{2} \times 80 \times 40 = 1600$ square meters.

10

The two ways intersect when :

$f(X) = g(X)$ $\qquad \therefore |X - 5| = 5 - \frac{2}{3}X$

, when $X \geq 5 : X - 5 = 5 - \frac{2}{3}X$

$\therefore 1\frac{2}{3}X = 10$ $\qquad \therefore X = 6 \in [5 , \infty[$

, when $X < 5$ $\qquad \therefore -X + 5 = 5 - \frac{2}{3}X$

$\therefore \frac{1}{3}X = 0$ $\qquad \therefore X = 0 \in]-\infty , 5[$

$, \because f(6) = |6 - 5| = 1$

$\therefore$ A (6 , 1) is the point of intersection of the two curves.

$, f(0) = |0 - 5| = 5$

$\therefore$ B (0 , 5) is the point of intersection of the two curves.

$\therefore AB = \sqrt{(6-0)^2 + (1-5)^2} = 2\sqrt{13}$

$, \because$ the unit of length represent 5 km.

$\therefore AB = 5 \times 2\sqrt{13} = 10\sqrt{13} \simeq 36$ km.

11

Let the length of the applicant = X cm.

$\therefore 178 \leq X \leq 192$, adding (– 185) to the terms of the inequality

$\therefore -7 \leq X - 185 \leq 7$

$\therefore |X - 185| \leq 7$

12

(1) After two seconds : $d = 8|5-2| = 24$ cm. ,

[in the motion direction from A to B]

, after 8 seconds : $d = 8|5-8| = 24$ cm.

[in the motion direction from B to C]

(2) $\because 8|5-t| = 16 \qquad \therefore |t-5| = 2$

$\therefore t-5 = -2$, then $t = 3$

[in the motion direction from A to B]

or $t-5 = 2$, then $t = 7$

[in the motion direction from B to C]

(3) $\because 8|t-5| < 8 \qquad \therefore |t-5| < 1$

$\therefore -1 < t-5 < 1 \qquad \therefore 4 < t < 6$

$\therefore t \in]4 , 6[$

13

$\because B = (8 , 4)$

$\because f(8) = \frac{4}{3}|8-5| = 4$

$\therefore$ B lies on the curve of the function f

$\therefore$ The black ball will be fall in the pocket B

Answers of "Unit Two"

Exercise 1

First Multiple choice questions

(1) b	(2) d	(3) b	(4) c	(5) b	(6) c
(7) b	(8) c	(9) b	(10) a	(11) c	(12) d
(13) b	(14) d	(15) c	(16) c	(17) c	(18) b
(19) b	(20) b	(21) d	(22) c	(23) a	(24) a
(25) c	(26) a	(27) b	(28) c	(29) a	(30) d
(31) b	(32) a	(33) b	(34) b	(35) b	(36) d
(37) c	(38) d	(39) a	(40) a	(41) a	(42) d
(43) c	(44) d	(45) a	(46) c		

Second Essay questions

1

(1) $\frac{(27)^{-3}\times(12)^2}{16\times(81)^{-2}} = \frac{(3^3)^{-3}\times(3)^2\times(2^2)^2}{2^4\times(3^4)^{-2}}$

$= \frac{3^{-9}\times3^2\times2^4}{2^4\times3^{-8}} = 3^{-9+2+8} = 3$

(2) $\frac{4^{2n+1}\times2^{1-n}}{8^{n+2}} = \frac{(2^2)^{2n+1}\times2^{1-n}}{(2^3)^{n+2}} = \frac{2^{4n+2}\times2^{1-n}}{2^{3n+6}}$

$= 2^{4n+2+1-n-3n-6}$

$= 2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

(3) $\frac{9^{4n+1}\times4^{2-2n}}{3^{9n+1}\times48^{1-n}} = \frac{(3^2)^{4n+1}\times(2^2)^{2-2n}}{3^{9n+1}\times3^{1-n}\times(2^4)^{1-n}}$

$= \frac{3^{8n+2}\times2^{4-4n}}{3^{9n+1}\times3^{1-n}\times2^{4-4n}}$

$= 3^{8n+2-9n-1-1+n}\times2^{4-4n-4+4n}$

$= 3^0\times2^0 = 1\times1 = 1$

(4) $\frac{125\times(15)^{n-3}\times(25)^{m+n}}{(75)^n\times(5)^{n+2m}}$

$= \frac{5^3\times5^{n-3}\times3^{n-3}\times(5^2)^{m+n}}{3^n\times(5^2)^n\times5^{n+2m}}$

$= \frac{5^3\times5^{n-3}\times3^{n-3}\times5^{2m+2n}}{3^n\times5^{2n}\times5^{n+2m}}$

$= 5^{3+n-3+2m+2n-2n-n-2m}\times3^{n-3-n}$

$= 5^0\times3^{-3} = \frac{1}{27}$

(5) $(18)^{\frac12}\times(12)^{\frac32}\times\frac{1}{(24)^{\frac12}}$

$= 2^{\frac12}\times(3^2)^{\frac12}\times3^{\frac32}\times(2^2)^{\frac32}\times\frac{1}{(2^3)^{\frac12}\times3^{\frac12}}$

$= 2^{\frac12}\times3\times3^{\frac32}\times2^3\times2^{-\frac32}\times3^{-\frac12}$

$= 2^{\frac12+3-\frac32}\times3^{1+\frac32-\frac12} = 2^2\times3^2 = 36$

(6) The expression $= \frac{3^{\frac12}\times3^{\frac16}\times(7^2)^{\frac16}}{(3^2)^{\frac13}\times7^{\frac13}} = \frac{3^{\frac12}\times3^{\frac16}\times7^{\frac13}}{3^{\frac23}\times7^{\frac13}}$

$= 3^{\frac12+\frac16-\frac23}\times7^{\frac13-\frac13}$

$= 3^0\times7^0 = 1\times1 = 1$

2

(1) L.H.S. $= \frac{(3^3)^{y-\frac13}\times7^{\frac{y(y+3)}{y}}}{(3^4)^{y-1}\times3^{5-y}\times7^{5-y}\times(7^2)^{y-1}}$

$= \frac{3^{3y-1}\times7^{y+3}}{3^{4y-4}\times3^{5-y}\times7^{5-y}\times7^{2y-2}}$

$= 3^{3y-1-4y+4-5+y}\times7^{y+3-5+y-2y+2}$

$= 3^{-2}\times7^0 = 3^{-2} = \frac19$

(2) L.H.S. $= \frac{(7^3)^{2x-\frac13}\times(2^2)^{3x+1}}{(7^2\times2^2)^{3x}\times2^2}$

$= \frac{7^{6x-1}\times2^{6x+2}}{7^{6x}\times2^{6x}\times2^2}$

$= 7^{6x-1-6x}\times2^{6x+2-6x-2} = 7^{-1}\times2^0$

$= \frac17\times1 = \frac17$

(3) L.H.S. $= \frac{5^3\times(2^2)^{\frac38}\times2^{-\frac14}\times5^{-\frac14}}{(2^2)^{\frac58}\times2^{-\frac34}\times3^{-\frac34}\times3^{\frac34}\times5^{\frac34}}$

$= \frac{5^3\times2^{\frac34}\times2^{-\frac14}\times5^{-\frac14}}{2^{\frac54}\times2^{-\frac34}\times3^{-\frac34}\times3^{\frac34}\times5^{\frac34}}$

$= 5^{3-\frac14-\frac34}\times2^{\frac34-\frac14-\frac54+\frac34}\times3^{\frac34-\frac34}$

$= 5^2\times2^0\times3^0 = 25\times1\times1 = 25$

(4) L.H.S. $= \frac{3\times2^{x+2}+2^{x+2}}{2^{x+3}-2\times3\times2^x} = \frac{2^{x+2}(3+1)}{2^{x+1}(2^2-3)}$

$= \frac{2^{x+2}\times2^2}{2^{x+1}\times1} = 2^3 = 8$

3

(1) $\because x^{\frac{7}{2}} = 2^7$ $\quad \therefore x = (2^7)^{\frac{2}{7}} = 4$

$\therefore$ S.S. $= \{4\}$

(2) $x^{\frac{5}{2}} = \frac{1}{2^5} = 2^{-5}$ $\quad \therefore x = (2^{-5})^{\frac{2}{5}}$

$\therefore x = 2^{-2}$ $\quad \therefore$ S.S. $= \left\{\frac{1}{4}\right\}$

(3) $\because x^{-\frac{5}{3}} = 2^{\frac{5}{3}}$ $\quad \therefore x = (2^{\frac{5}{3}})^{-\frac{3}{5}}$

$\therefore x = 2^{-1} = \frac{1}{2}$ $\quad \therefore$ S.S. $= \left\{\frac{1}{2}\right\}$

(4) $x^{\frac{4}{3}} = 3^4$

$\therefore x = \pm (3^4)^{\frac{3}{4}} = \pm 3^3 = \pm 27$

$\therefore$ S.S. $= \{27 , -27\}$

(5) $(x-1)^{\frac{5}{3}} = 2^5$ $\quad \therefore x - 1 = (2^5)^{\frac{3}{5}} = 2^3 = 8$

$\therefore x = 9$ $\quad \therefore$ S.S. $= \{9\}$

(6) $2x + 3 = \pm (3^4)^{\frac{3}{4}}$ $\quad \therefore 2x + 3 = \pm 27$

$\therefore 2x = 24$ $\quad$ or $2x = -30$

$\therefore x = 12$ $\quad$ or $x = -15$

$\therefore$ S.S. $= \{12 , -15\}$

(7) $x + 1 = (32^{-\frac{1}{2}})^{-\frac{2}{5}} = 32^{\frac{1}{5}} = (2^5)^{\frac{1}{5}}$

$\therefore x + 1 = 2$ $\quad \therefore x = 1$

$\therefore$ S.S. $= \{1\}$

(8) $x^2 - 5x + 9 = (3^5)^{\frac{2}{5}}$ $\quad \therefore x^2 - 5x + 9 = 9$

$\therefore x^2 - 5x = 0$ $\quad \therefore x(x-5) = 0$

$\therefore x = 0$ or $x = 5$ $\quad \therefore$ S.S. $= \{0 , 5\}$

(9) $\sqrt{x} + 2 = 9$ $\quad \therefore \sqrt{x} = 7$

$\therefore x = 49$ $\quad \therefore$ S.S. $= \{49\}$

(10) $(x^{\frac{2}{5}} - 1)(x^{\frac{2}{5}} - 4) = 0$

$\therefore x^{\frac{2}{5}} = 1$, then $x = \pm 1$

or $x^{\frac{2}{5}} = 4$, then $x = \pm (2^2)^{\frac{5}{2}} = \pm 32$

$\therefore$ S.S. $= \{1 , -1 , 32 , -32\}$

(11) $(x^{\frac{2}{3}} - 1)(x^{\frac{2}{3}} - 9) = 0$

$\therefore x^{\frac{2}{3}} = 1$ $\quad \therefore x = \pm 1$

or $x^{\frac{2}{3}} = 9$ $\quad \therefore x = \pm (3^2)^{\frac{3}{2}} = \pm 27$

$\therefore$ S.S. $= \{1 , -1 , 27 , -27\}$

(12) $(x^{\frac{1}{2}} - 1)(x^{\frac{1}{2}} - 2) = 0$

$\therefore x^{\frac{1}{2}} = 1$ $\quad \therefore x = 1$

or $x^{\frac{1}{2}} = 2$ $\quad \therefore x = (2)^2 = 4$

$\therefore$ S.S. $= \{1 , 4\}$

(13) $x - 8\sqrt{x} + 15 = 0$

$\therefore (\sqrt{x} - 3)(\sqrt{x} - 5) = 0$

$\therefore \sqrt{x} = 3$, then $x = 9$

or $\sqrt{x} = 5$, then $x = 25$

$\therefore$ S.S. $= \{9 , 25\}$

(14) $\because x^{\frac{2}{3}} - 25x^{\frac{1}{3}} - 54 = 0$

$\therefore (x^{\frac{1}{3}} + 2)(x^{\frac{1}{3}} - 27) = 0$ $\quad \therefore x^{\frac{1}{3}} = 27$

$\therefore x = (3^3)^3 = 3^9 = 19683$

or $x^{\frac{1}{3}} = -2$ $\quad \therefore x = (-2)^3 = -8$

$\therefore$ S.S. $= \{19683 , -8\}$

(15) $x^{\frac{5}{3}} - x^{\frac{5}{6}} - 6 = 0$ $\quad \therefore (x^{\frac{5}{6}} - 3)(x^{\frac{5}{6}} + 2) = 0$

$\therefore x^{\frac{5}{6}} = 3$ $\quad \therefore x = 3^{\frac{6}{5}} \approx 3.7$

or $x^{\frac{5}{6}} = -2$ (refused)

$\therefore$ S.S. $= \{3.7\}$

(16) $x^{\frac{4}{5}} - 3x^{\frac{2}{5}} - 4 = 0$

$\therefore (x^{\frac{2}{5}} + 1)(x^{\frac{2}{5}} - 4) = 0$

$\therefore x^{\frac{2}{5}} = -1$ (refused) or $x^{\frac{2}{5}} = 4$

$\therefore x = \pm (2^2)^{\frac{5}{2}} = \pm 2^5 = \pm 32$

$\therefore$ S.S. $= \{32 , -32\}$

(17) Multiply by $\sqrt[5]{x}$ $\quad \therefore (\sqrt[5]{x})^2 - 3\sqrt[5]{x} + 2 = 0$

$\therefore (x^{\frac{1}{5}} - 1)(x^{\frac{1}{5}} - 2) = 0$

$\therefore x^{\frac{1}{5}} = 1$, then $x = 1$

or $x^{\frac{1}{5}} = 2$, then $x = 32$

$\therefore$ S.S. $= \{1 , 32\}$

4

(1) $\because 5^{2x-1} = \frac{1}{125}$ $\quad \therefore 5^{2x-1} = \frac{1}{5^3}$

$\therefore 5^{2x-1} = 5^{-3}$ $\quad \therefore 2x - 1 = -3$

$\therefore 2x = -2$ $\quad \therefore x = -1$

$\therefore$ The S.S. $= \{-1\}$

(2) $5^{x+2} = x^{x+2}$

$\therefore$ Either $x = 5$ or $x + 2 = 0$

$\therefore x = -2$ $\quad \therefore$ The S.S. $= \{-2 , 5\}$

(3) $2^{x^2-9}=1$ $\therefore x^2-9=0$

$\therefore x^2=9$ $\therefore x=\pm 3$

$\therefore$ The S.S. $=\{3 , -3\}$

(4) $x^{x^2-25}=3^{x^2-25}$

$\therefore$ Either $x=\pm 3$ or $x^2-25=0$

$\therefore x^2=25$ $\therefore x=\pm 5$

$\therefore$ The S.S. $=\{3 , -3 , 5 , -5\}$

(5) $\left(3\sqrt{3}\right)^{|x|}=27$ $\therefore \left(3\sqrt{3}\right)^{|x|}=\left(3\sqrt{3}\right)^2$

$\therefore |x|=2$ $\therefore x=\pm 2$

$\therefore$ The S.S. $=\{2 , -2\}$

(6) $\because 3^{|3x-4|}=3^{4x-4}$ $\therefore |3x-4|=4x-4$

, when $x \geq \frac{4}{3}$ $\therefore 3x-4=4x-4$

$\therefore x=0$ (doesn't satisfy)

, when $x<\frac{4}{3}$ $\therefore -3x+4=4x-4$

$\therefore 7x=8$ $\therefore x=\frac{8}{7}$ (satisfies)

$\therefore$ The S.S. $=\left\{\frac{8}{7}\right\}$

(7) $2\times 4^{x+3}=\frac{1}{32}$ $\therefore 4^{x+3}=\frac{1}{64}$

$\therefore 4^{x+3}=4^{-3}$ $\therefore x+3=-3$

$\therefore x=-6$ $\therefore$ The S.S. $=\{-6\}$

(8) $\left(\frac{3}{5}\right)^{2x-1}=\frac{27}{125}$ $\therefore \left(\frac{3}{5}\right)^{2x-1}=\left(\frac{3}{5}\right)^3$

$\therefore 2x-1=3$ $\therefore 2x=4$

$\therefore x=2$ $\therefore$ The S.S. $=\{2\}$

(9) $5^{x-1}\times 7^{1-x}=\frac{25}{49}$ $\therefore 5^{x-1}\times 7^{-(x-1)}=\left(\frac{5}{7}\right)^2$

$\therefore \left(\frac{5}{7}\right)^{x-1}=\left(\frac{5}{7}\right)^2$ $\therefore x-1=2$

$\therefore x=3$ $\therefore$ The S.S. $=\{3\}$

(10) $\because \frac{9^{x+1}\times 4^{x-1}}{36^x}=\frac{3^{2x+2}\times 2^{2x-2}}{2^{2x}\times 3^{2x}}$

$=3^{2x+2-2x}\times 2^{2x-2-2x}$

$=3^2\times 2^{-2}=\left(\frac{3}{2}\right)^2$

$\therefore \left(\frac{3}{2}\right)^2=\left(\frac{3}{2}\right)^x$ $\therefore x=2$

$\therefore$ The S.S. $=\{2\}$

(11) $\frac{12^{3x-2}\times 9^{x+1}}{18^{2x}\times 4^{2x-2}}=9$

$\therefore \frac{(3\times 2^2)^{3x-2}\times (3^2)^{x+1}}{(2\times 3^2)^{2x}\times (2^2)^{2x-2}}=9$

$\therefore \frac{3^{3x-2}\times 2^{6x-4}\times 3^{2x+2}}{2^{2x}\times 3^{4x}\times 2^{4x-4}}=9$

$\therefore 3^{3x-2+2x+2-4x}\times 2^{6x-4-2x-4x+4}=9$

$\therefore 3^x\times 2^0=3^2$ $\therefore 3^x=3^2$

$\therefore x=2$ $\therefore$ The S.S. $=\{2\}$

(12) $\frac{1}{27}\left(\sqrt{3}\right)^{x+2}=1$

$\therefore \frac{1}{\left(\sqrt{3}\right)^6}\times\left(\sqrt{3}\right)^{x+2}=1$

$\therefore \left(\sqrt{3}\right)^{x+2}=\left(\sqrt{3}\right)^6$ $\therefore x+2=6$

$\therefore x=4$ $\therefore$ The S.S. $=\{4\}$

(13) $\left(\sqrt{3}\right)^{x^2-5x}=1$ $\therefore x^2-5x=0$

$\therefore x(x-5)=0$ $\therefore x=0$ or $x=5$

$\therefore$ The S.S. $=\{0 , 5\}$

(14) $\because 5^{x^2-5x}=\frac{16}{10000}=\frac{1}{625}$

$\therefore 5^{x^2-5x}=5^{-4}$ $\therefore x^2-5x=-4$

$\therefore x^2-5x+4=0$ $\therefore (x-1)(x-4)=0$

$\therefore x=1$ or $x=4$ $\therefore$ The S.S. $=\{1 , 4\}$

(15) $5^{x^2}=25^{x+4}$ $\therefore 5^{x^2}=5^{2x+8}$

$\therefore x^2=2x+8$ $\therefore x^2-2x-8=0$

$\therefore (x+2)(x-4)=0$

$\therefore x=-2$ or $x=4$ $\therefore$ The S.S. $=\{-2 , 4\}$

(16) $3^{x^2-42}=\left(\frac{1}{3}\right)^x$ $\therefore 3^{x^2-42}=3^{-x}$

$\therefore x^2-42=-x$ $\therefore x^2+x-42=0$

$\therefore (x-6)(x+7)=0$

$\therefore x-6=0$, then $x=6$

or $x+7=0$, then $x=-7$

$\therefore$ The S.S. $=\{6 , -7\}$

(17) $\left(\sqrt{7}\right)^{|x+2|}=49$ $\therefore \left(\sqrt{7}\right)^{|x+2|}=\left(\sqrt{7}\right)^4$

$\therefore |x+2|=4$ $\therefore x+2=\pm 4$

$\therefore$ Either $x+2=4$, then $x=2$

or $x+2=-4$, then $x=-6$

$\therefore$ The S.S. $=\{2 , -6\}$

(18) $3^{2x-3} \times 7^{6-4x} = 1$ $\quad \therefore 3^{2x-3} \times 7^{-2(2x-3)} = 1$

$\therefore 3^{2x-3} \times \frac{1}{7^{2(2x-3)}} = 1$

$\therefore 3^{2x-3} = (49)^{2x-3}$ $\quad \therefore 2x - 3 = 0$

$\therefore x = \frac{3}{2}$ $\quad \therefore$ The S.S. $= \left\{\frac{3}{2}\right\}$

(19) $\sqrt{9^x - 2 \times 3^{x+1} + 9} = 24$

$\therefore \sqrt{3^{2x} - 2 \times 3 \times 3^x + 9} = 24$

$\therefore \sqrt{(3^x - 3)^2} = 24$ $\quad \therefore |3^x - 3| = 24$

$\therefore 3^x - 3 = -24$ $\quad \therefore 3^x = -21$ (refused)

or $3^x - 3 = 24$ $\quad \therefore 3^x = 27$

$\therefore 3^x = 3^3$ $\quad \therefore x = 3$

$\therefore$ The S.S. $= \{3\}$

(20) $\sqrt{5^{2x-2} - \frac{2}{5} \times 5^x + 1} = 124$

$\therefore \sqrt{5^{2(x-1)} - 2 \times 5^{x-1} + 1} = 124$

$\therefore \sqrt{(5^{x-1} - 1)^2} = 124$ $\quad \therefore |5^{x-1} - 1| = 124$

$\therefore 5^{x-1} - 1 = -124$ $\quad \therefore 5^{x-1} = -123$ (refused)

or $5^{x-1} - 1 = 124$ $\quad \therefore 5^{x-1} = 125 = 5^3$

$\therefore x - 1 = 3$ $\quad \therefore x = 4$

$\therefore$ The S.S. $= \{4\}$

5

(1) $5^{x+1} + 5^{x-1} = 26$ $\quad \therefore 5^x(5 + 5^{-1}) = 26$

$\therefore 5^x \times \frac{26}{5} = 26$ $\quad \therefore 5^x = 5$

$\therefore x = 1$ $\quad \therefore$ The S.S. $= \{1\}$

(2) $3^{x+3} - 3^{x+2} = 162$ $\quad \therefore 3^x(3^3 - 3^2) = 162$

$\therefore 3^x = 9$ $\quad \therefore 3^x = 3^2$

$\therefore x = 2$ $\quad \therefore$ The S.S. $= \{2\}$

(3) $7^{2-x} + 7^{-x} = 50$ $\quad \therefore 7^{-x}(7^2 + 1) = 50$

$\therefore 7^{-x} \times 50 = 50$ $\quad \therefore 7^{-x} = 1$

$\therefore x = 0$ $\quad \therefore$ The S.S. $= \{0\}$

(4) $5^{2x} + 25 = 26 \times 5^x$ $\quad \therefore 5^{2x} - 26 \times 5^x + 25 = 0$

$\therefore (5^x - 25)(5^x - 1) = 0$

$\therefore 5^x = 25 = 5^2$, then $x = 2$

or $5^x = 1$, then $x = 0$

$\therefore$ The S.S. $= \{2 , 0\}$

(5) $2^x + 2^{5-x} = 12$

$\therefore 2^x + \frac{2^5}{2^x} = 12$ (multiplying by 2^x)

$\therefore 2^{2x} + 32 = 12 \times 2^x$

$\therefore 2^{2x} - 12 \times 2^x + 32 = 0$

$\therefore (2^x - 4)(2^x - 8) = 0$

$\therefore 2^x = 4$, then $x = 2$ or $2^x = 8 = 2^3$

, then $x = 3$

$\therefore$ The S.S. $= \{2 , 3\}$

(6) $\left(\frac{1}{2}\right)^{x+1} + \left(\frac{1}{2}\right)^{x+3} + \left(\frac{1}{2}\right)^{x+5} = 84$

$\therefore \left(\frac{1}{2}\right)^x \left(\frac{1}{2} + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^5\right) = 84$

$\therefore \left(\frac{1}{2}\right)^x \times \frac{21}{32} = 84$ $\quad \therefore \left(\frac{1}{2}\right)^x = 84 \times \frac{32}{21}$

$\therefore \left(\frac{1}{2}\right)^x = 128$ $\quad \therefore \left(\frac{1}{2}\right)^x = \left(\frac{1}{2}\right)^{-7}$

$\therefore x = -7$ $\quad \therefore$ The S.S. $= \{-7\}$

(7) $4^x + 2^{x+1} = 8$ $\quad \therefore 2^{2x} + 2(2)^x - 8 = 0$

$\therefore (2^x + 4)(2^x - 2) = 0$

$\therefore 2^x = -4$ (refused) or $2^x = 2$

$\therefore x = 1$ $\quad \therefore$ The S.S. $= \{1\}$

(8) $2^{2x+1} - 33(2)^x + 16 = 0$

$\therefore 2(2)^{2x} - 33(2)^x + 16 = 0$

$\therefore (2(2)^x - 1)(2^x - 16) = 0$

$\therefore$ Either $2(2)^x = 1$ $\quad \therefore 2^{x+1} = 2^0$

$\therefore x + 1 = 0$ $\quad \therefore x = -1$

or $2^x = 16$ $\quad \therefore 2^x = 2^4$

$\therefore x = 4$ $\quad \therefore$ The S.S. $= \{-1 , 4\}$

(9) $5^{2x-2} - 6(5)^{x-1} + 5 = 0$

$\therefore 5^{2(x-1)} - 6(5)^{x-1} + 5 = 0$

$\therefore (5^{x-1} - 1)(5^{x-1} - 5) = 0$

$\therefore$ Either $5^{x-1} = 1$

$\therefore x - 1 = 0$ $\quad \therefore x = 1$ $\quad$ or $5^{x-1} = 5$

$\therefore x - 1 = 1$ $\quad \therefore x = 2$

$\therefore$ The S.S. $= \{1 , 2\}$

(10) $9^{x^2-1} - 36\,(3)^{x^2-3} + 3 = 0$

$\therefore (3)^{2(x^2-1)} - \frac{36}{9}\,(3)^{x^2-1} + 3 = 0$

$\therefore (3)^{2(x^2-1)} - 4\,(3)^{x^2-1} + 3 = 0$

$\therefore (3^{x^2-1} - 1)(3^{x^2-1} - 3) = 0$

$\therefore$ Either $3^{x^2-1} = 1$

$\therefore x^2 - 1 = 0 \quad \therefore x^2 = 1 \quad \therefore x = \pm 1$

or $3^{x^2-1} = 3 \quad \therefore x^2 - 1 = 1 \quad \therefore x^2 = 2$

$\therefore x = \pm\sqrt{2}$

$\therefore$ The S.S. $= \{1, -1, \sqrt{2}, -\sqrt{2}\}$

(11) $10^x - 5^{x-1} \times 2^{x-2} = 950$

$\therefore 10^x - \frac{5^x}{5} \times \frac{2^x}{4} = 950$

$\therefore 10^x - \frac{1}{20}\,(10)^x = 950$ (multiplying by 20)

$\therefore 20\,(10)^x - (10)^x = 20 \times 950$

$\therefore 19\,(10)^x = 20 \times 950 \quad \therefore 10^x = 1000$

$\therefore 10^x = 10^3 \quad \therefore x = 3$

$\therefore$ The S.S. $= \{3\}$

6

(1) $\because y^{\frac{3}{4}} = 2^6 \quad \therefore y = (2^6)^{\frac{4}{3}} = 2^8 = 256$

, $\because 2x^{\frac{5}{3}} = 2 \times 2^5 \quad \therefore x = (2^5)^{\frac{3}{5}} = 2^3 = 8$

$\therefore 5x + 2y = 5 \times 8 + 2 \times 256 = 40 + 512 = 552$

(2) $x^{\frac{3}{2}} = 27 \quad \therefore x = (3^3)^{\frac{2}{3}} = 3^2 = 9$

, $3y^{\frac{2}{3}} = 27 \quad \therefore y^{\frac{2}{3}} = 9$

$\therefore y = \pm (3^2)^{\frac{3}{2}} = \pm 27$

$\therefore x + y = 9 + 27 = 36$

or $x + y = 9 - 27 = -18$

(3) $\because x^{\frac{4}{3}} = 3^4 \quad \therefore x = \pm (3^4)^{\frac{3}{4}} = \pm 27$

$\therefore 9y^{-\frac{2}{3}} = 81$

$\therefore y^{-\frac{2}{3}} = 9 = 3^2 \quad \therefore y^{\frac{2}{3}} = 3^{-2}$

$\therefore y = \pm (3^{-2})^{\frac{3}{2}} = \pm (3)^{-3} = \pm\frac{1}{27}$

$\therefore |2xy| = \left|2 \times (\pm 27) \times \left(\pm\frac{1}{27}\right)\right| = 2$

7

(1) Error because $(-9)^{\frac{2}{2}} \neq \sqrt{(-9)^2}$

There's a common factor between the numerator and denominator in the power.

(2) Error because $x = \pm 3$

such that the power is even.

8

(1) $\because 2^{2x+2y} = 2^7 \quad \therefore 2x + 2y = 7 \quad (1)$

, $\because 5^{x-2y-3} = 1 \quad \therefore x - 2y = 3 \quad (2)$

By adding (1) and (2) :

$\therefore 3x = 10 \quad \therefore x = \frac{10}{3}$

By substituting in (1) : $\therefore y = \frac{1}{6}$

$\therefore$ The S.S. $= \left\{\left(\frac{10}{3}, \frac{1}{6}\right)\right\}$

(2) $3^x \times 5^y = 75 \quad (1)$

, $3^y \times 5^x = 45 \quad (2)$

Dividing (1) by (2) : $\therefore \left(\frac{3}{5}\right)^x \times \left(\frac{5}{3}\right)^y = \frac{75}{45}$

$\therefore \left(\frac{5}{3}\right)^{-x} \times \left(\frac{5}{3}\right)^y = \frac{5}{3}$

$\therefore \left(\frac{5}{3}\right)^{y-x} = \frac{5}{3} \quad \therefore y - x = 1 \quad (3)$

Multiplying (1) by (2) : $\therefore (15)^x \times (15)^y = 75 \times 45$

$\therefore (15)^{x+y} = 15^3 \quad \therefore x + y = 3 \quad (4)$

By adding (3) and (4) : $\therefore 2y = 4 \quad \therefore y = 2$

By substituting in (3) : $\therefore x = 1$

$\therefore$ The S.S. $= \{(1, 2)\}$

(3) $3^x \times 3^y = 3^3 \quad \therefore 3^{x+y} = 3^3$

$\therefore x + y = 3 \quad \therefore y = 3 - x$

, $\because 3^x + 3^y = 12 \quad \therefore 3^x + 3^{3-x} = 12$

Multiplying by 3^x : $\therefore 3^{2x} + 27 = 12\,(3)^x$

$\therefore 3^{2x} - 12\,(3)^x + 27 = 0$

$\therefore (3^x - 3)(3^x - 9) = 0$

$\therefore$ Either $3^x = 3 \quad \therefore x = 1 \quad \therefore y = 2$

or $3^x = 9 \quad \therefore 3^x = 3^2 \quad \therefore x = 2$

$\therefore y = 1$

$\therefore$ The S.S. $= \{(1, 2), (2, 1)\}$

9

$\because 9^{x+1} - (3^{x+3} + 3^x) + 3 = 0$

$\therefore 9^x \times 9 - 3^x(3^3 + 1) + 3 = 0$

$\therefore 9 \times 3^{2x} - 28 \times 3^x + 3 = 0$

$\therefore (9 \times 3^x - 1)(3^x - 3) = 0$

$\therefore 9 \times 3^x - 1 = 0$, then $3^x = \frac{1}{9}$

$\therefore 3^x = 3^{-2}$ $\quad \therefore x = -2$

or $3^x - 3 = 0$, then $3^x = 3$ $\quad \therefore x = 1$

$\therefore$ The S.S. $= \{-2 , 1\}$

10

$\because x^{n+1} = y^{n-1}$ $\quad \therefore (x^{n+1})^3 = (y^{n-1})^3$

$\therefore (x^3)^{n+1} = y^{3n-3}$ $\quad , \because x^3 = y^2$

$\therefore (y^2)^{n+1} = y^{3n-3}$ $\quad \therefore y^{2n+2} = y^{3n-3}$

$\therefore 2n + 2 = 3n - 3$ $\quad \therefore n = 5$

Third Higher skills

1

(1) (b)	(2) (d)	(3) (c)
(4) (c)	(5) (c)	(6) (c)
(7) (d)	(8) (d)	(9) (d)

Instructions to solve 1 :

(1) $\because \sqrt{x^2} = |x| = -x$ (because $x < 0$)

$, \sqrt[3]{x^3} = x$

$, \sqrt{x^2 - 2x + 1} = \sqrt{(x-1)^2} = |x - 1| = 1 - x$

(because $x < 0$)

$\therefore \sqrt{x^2} - \sqrt[3]{x^3} - \sqrt{x^2 - 2x + 1} + 1$

$= -x - x - (1 - x) + 1 = -x$

(2) $\because a = \sqrt[4]{\frac{\sqrt[3]{2}}{\sqrt{7}}}$ $\quad \therefore a^4 = \frac{\sqrt[3]{2}}{\sqrt{7}}$

$\therefore (a^4)^6 = \frac{4}{343}$ is a rational number

$\therefore a^{24}$ is a rational number.

(3) (c) is not always true because $\sqrt{x^2 y^6} = |x y^3|$

(4) $\because x^{\frac{2}{3}} = x^{\frac{3}{2}}$ $\quad \therefore x^{\frac{2}{3}} - x^{\frac{3}{2}} = 0$

$\therefore x^{\frac{2}{3}}\left(1 - x^{\frac{5}{6}}\right) = 0$

$\therefore x^{\frac{2}{3}} = 0$ and so $x = 0$ or $1 - x^{\frac{5}{6}} = 0$

and so $x^{\frac{5}{6}} = 1$

$\therefore x = 1$

$\therefore$ The solution set $= \{0 , 1\}$

(5) $\sqrt[5]{\frac{2}{27}} = \sqrt[5]{\frac{2 \times 3 \times 3}{27 \times 3 \times 3}} = \sqrt[5]{\frac{18}{243}} = \frac{\sqrt[5]{18}}{3}$

(6) $\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$ is true for all $\sqrt[n]{a} \in \mathbb{R}$, $n \in \mathbb{Z}^+ - \{1\}$

(7) $\because x^{\frac{2}{3}} = a$ $\quad \therefore x = a^{\frac{3}{2}} = \left(\sqrt{a}\right)^3$

$\therefore$ a must be greater than or equal to zero

i.e. $a \in \mathbb{R}^+ \cup \{0\}$

(8) $\because \sqrt[3]{2\sqrt{3\sqrt{2}}} = \sqrt[6]{m\sqrt{2}}$

raised both sides to the power of 6

$\therefore \left(2\sqrt{3\sqrt{2}}\right)^2 = m\sqrt{2}$

$\therefore 4 \times 3\sqrt{2} = m\sqrt{2}$ $\quad \therefore m = 12$

(9) $\sqrt[n]{a} \times \sqrt[m]{a} = a^{\frac{1}{n}} \times a^{\frac{1}{m}} = a^{\frac{1}{n} + \frac{1}{m}} = a^{\frac{m+n}{mn}}$

$= \sqrt[mn]{a^{m+n}}$

2

(1) $\frac{5^{x-1} \times x^{2-x}}{5} = 1$ $\quad \therefore 5^{x-2} \times x^{2-x} = 1$

$\therefore 5^{x-2} = x^{x-2}$ $\quad \therefore x = 5$ or $x = 2$

$\therefore$ The S.S. $= \{5 , 2\}$

(2) $\because (x - 3)^{x-5} = 1$ $\quad \therefore$ Either $x - 5 = 0$

, then $x = 5$ or $x - 3 = 1$, then $x = 4$

or $x - 3 = -1$ (If the exponent is even)

, then $x = 2$ [(refused) for the exponent in this case $(2 - 5)$ will be odd]

$\therefore$ The S.S. $= \{5 , 4\}$

(3) $(x - 3)^{(x-6)} = 1$

$\therefore$ Either $x - 6 = 0$, then $x = 6$

or $x - 3 = 1$, then $x = 4$

or $x - 3 = -1$ (If the exponent is even)

, then $x = 2$ (satisfies , for the exponent $(2 - 6)$ in this case will be even).

$\therefore$ The S.S. $= \{6 , 4 , 2\}$

Exercise 8

First Multiple choice questions

(1) d	(2) d	(3) a	(4) b	(5) c	(6) c
(7) c	(8) c	(9) a	(10) c	(11) a	(12) c
(13) b	(14) c	(15) c	(16) a	(17) d	(18) c
(19) d	(20) c	(21) c	(22) a	(23) a	(24) b
(25) b	(26) c	(27) c	(28) b	(29) d	(30) b
(31) b	(32) c	(33) d	(34) c	(35) a	(36) d
(37) b	(38) b	(39) b			

Second Essay questions

1

(1) Not exponential.

(2) Exponential function , its base = 5 , its power = x

(3) Not exponential.

(4) Not exponential.

(5) Exponential function , its base = $\frac{2}{3}$, its power = $x - 1$

(6) Not exponential.

2

$$\frac{f(x+4)-f(x+3)}{f(x+5)-f(x+4)}=\frac{5^{x+4}-5^{x+3}}{5^{x+5}-5^{x+4}}$$

$$=\frac{5^{x+3}(5-1)}{5^{x+4}(5-1)}=\frac{1}{5}$$

3

$\because f(2x-1)+f(x-2)=50$

$\therefore 7^{(2x-1)+1}+7^{(x-2)+1}=50$

$\therefore 7^{2x}+7^{x-1}=50$ (multiplying by 7)

$\therefore 7\times 7^{2x}+7^{x}-350=0$

$\therefore (7\times 7^{x}+50)(7^{x}-7)=0$

$\therefore 7\times 7^{x}+50=0$, then $7^{x}=\frac{-50}{7}$ (refused)

or $7^{x}-7=0$ $\therefore 7^{x}=7$

$\therefore x=1$

4

$\therefore f_1(2x-1)+f_2(x+1)=756$

$\therefore 3^{2x-1}+9^{x+1}=756$ $\therefore 3^{2x-1}+3^{2x+2}=756$

$\therefore 3^{2x}(3^{-1}+3^{2})=756$ $\therefore 3^{2x}\times\frac{28}{3}=756$

$\therefore 3^{2x}=756\times\frac{3}{28}$ $\therefore 3^{2x}=81$

$\therefore 3^{2x}=3^{4}$ $\therefore 2x=4$

$\therefore x=2$

5

$$\text{L.H.S.}=\frac{a^{x+1}+a^{x+2}}{a^{x+1}+a^{x}}=\frac{a^{x+1}(1+a)}{a^{x}(a+1)}=a$$

6

$$\text{L.H.S.}=\frac{3^{2x+2}+3^{2x-1}}{5\times 3^{2x}-7\times 3^{2x-1}}=\frac{3^{2x-1}(3^{3}+1)}{3^{2x-1}(5\times 3-7)}$$

$$=\frac{28}{8}=\frac{7}{2}$$

7

$$\text{L.H.S.}=\frac{3^{3(x+1)-1}\times 3^{3(x+2)-1}}{3^{3(x+3)-1}}$$

$$=\frac{3^{3x+2}\times 3^{3x+5}}{3^{3x+8}}=3^{3x+2+3x+5-3x-8}$$

$$=3^{3x-1}=f(x)$$

8

$$\text{L.H.S.}=\frac{2^{x+1}}{2^{x-1}}+\frac{2^{x-1}}{2^{x+1}}=2^{2}+\frac{1}{2^{2}}=\frac{17}{4}$$

9

$\because 7^{2x-1}+7^{2x+1}=\frac{50}{49}$ $\therefore 7^{2x-1}(1+49)=\frac{50}{49}$

$\therefore 7^{2x-1}=7^{-2}$ $\therefore 2x-1=-2$

$\therefore 2x=-1$ $\therefore x=\frac{-1}{2}$

10

$\because \frac{2^{x+1}-2^{x}}{2^{x}-2^{x-1}}=2^{x-2}$ $\therefore \frac{2^{x}(2-1)}{2^{x-1}(2-1)}=2^{x-2}$

$\therefore 2=2^{x-2}$ $\therefore x-2=1$

$\therefore x=3$

11

$\because \frac{3^{-2x-1}}{3^{-x+1}} = 729$ $\therefore 3^{-2x-1+x-1} = 3^6$

$\therefore -x-2=6$ $\therefore x=-8$

$\therefore$ The S.S. $= \{-8\}$

12

$\because 3^{x+1} + 3^{3-x} = 30$ (multiplying by 3^{x-1})

$\therefore 3^{2x} + 3^2 - 10 \times 3 \times 3^{x-1} = 0$

$\therefore 3^{2x} - 10 \times 3^x + 9 = 0$ $\therefore (3^x - 1)(3^x - 9) = 0$

$\therefore 3^x = 1$ $\therefore x = 0$

or $3^x = 9$ $\therefore x = 2$

13

$\because 2^{2x} - 6 \times 2^x + 8 = 0$ $\therefore (2^x - 2)(2^x - 4) = 0$

$\therefore 2^x = 2$

$\therefore x = 1$ or $2^x = 4$ $\therefore x = 2$

$\therefore$ The S.S. $= \{1 , 2\}$

14

$\because 3^x + 3^{\frac{x}{2}} = 12$

Put $3^{\frac{x}{2}} = y$ $\therefore y^2 + y - 12 = 0$

$\therefore (y+4)(y-3) = 0$ $\therefore y = -4$ (refused) or $y = 3$

$\therefore 3^{\frac{x}{2}} = 3$ $\therefore \frac{x}{2} = 1$

$\therefore x = 2$ $\therefore$ The S.S. $= \{2\}$

15

$\because f(x-2) = 3^x$ $\therefore f(x) = 3^{x+2}$

$\therefore f(3x+1) - f(3x-1) = 24$

$\therefore 3^{3x+3} - 3^{3x+1} = 24$ $\therefore 3^{3x+1}(3^2 - 1) = 24$

$\therefore 3^{3x+1} \times 8 = 24$ $\therefore 3^{3x+1} = 3$ $\therefore 3x + 1 = 1$

$\therefore 3x = 0$ $\therefore x = 0$

16

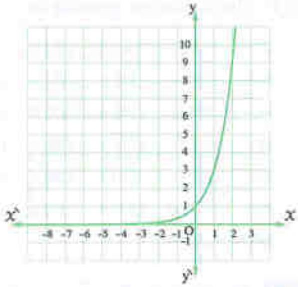
(1) Figure (7) (2) Figure (2) (3) Figure (5)

(4) Figure (1) (5) Figure (6) (6) Figure (3)

(7) Figure (4) (8) Figure (8)

17

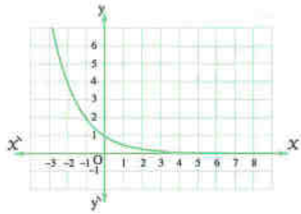
(1) $f(x) = 3^x$



The domain $= \mathbb{R}$, the range $= \mathbb{R}^+$

, the function is increasing on its domain.

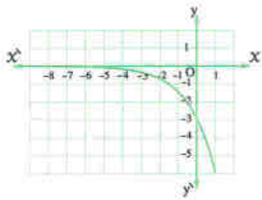
(2) $f(x) = \left(\frac{1}{2}\right)^x$



The domain $= \mathbb{R}$, the range $= \mathbb{R}^+$

, the function is decreasing on its domain.

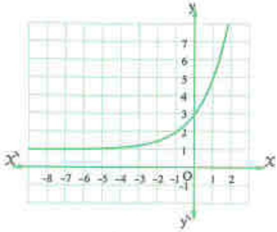
(3) $f(x) = -3(2)^x$



The domain $= \mathbb{R}$, the range $= \mathbb{R}^-$

, the function is decreasing on its domain.

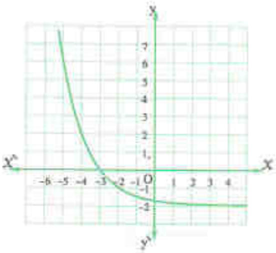
(4) $f(X) = 2^{X+1} + 1$



The domain $= \mathbb{R}$, the range $=]1 , \infty[$

, the function is increasing on its domain.

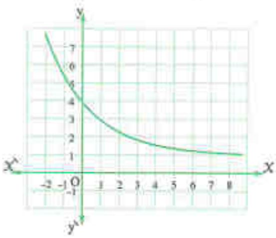
(5) $f(X) = \left(\frac{1}{2}\right)^{X+2} - 2$



The domain $= \mathbb{R}$, the range $=]-2 , \infty[$

, the function is decreasing on its domain.

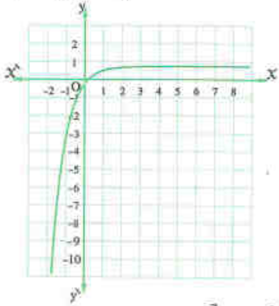
(6) $f(X) = 2\left(\frac{2}{3}\right)^{X-1} + 1$



The domain $= \mathbb{R}$, the range $=]1 , \infty[$

, the function is decreasing on its domain.

(7) $f(X) = -\left(\frac{1}{2}\right)^{2X} + \frac{3}{4}$

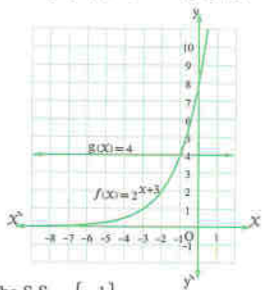


The domain $= \mathbb{R}$, the range $=]-\infty , \frac{3}{4}[$

, the function is increasing on its domain.

18

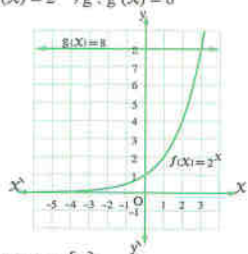
(1) From the graphical representation of the two functions : $f : f(X) = 2^{X+3}$, $g : g(X) = 4$



$\therefore$ The S.S. $= \{-1\}$

(2) From the graphical representation of the two functions :

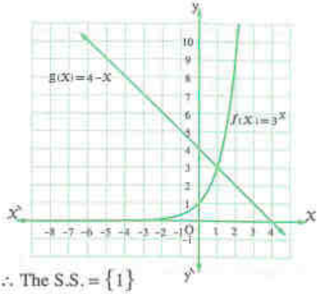
$f : f(X) = 2^X$, $g : g(X) = 8$



$\therefore$ The S.S. $= \{3\}$

(3) From the graphical representation of the two functions :

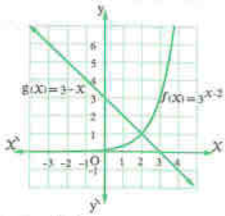
$f : f(X) = 3^X$, $g : g(X) = 4 - X$



$\therefore$ The S.S. $= \{1\}$

(4) From the graphical representation of the two functions :

$f : f(X) = 3^{X-2}$, $g : g(X) = 3 - X$

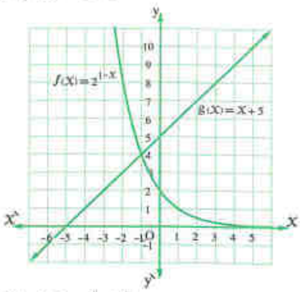


$\therefore$ The S.S. $= \{2\}$

(5) From the graphical representation of the two functions :

$f : f(X) = 2^{1-X} = 2 \times 2^{-X} = 2\left(\frac{1}{2}\right)^X$

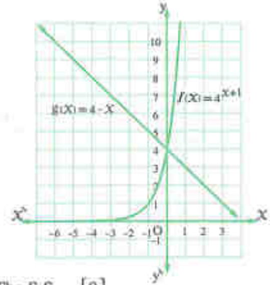
$, g : g(X) = X + 5$



$\therefore$ The S.S. $= \{-1\}$

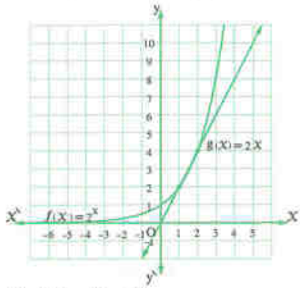
(6) From the graphical representation of the two functions :

$f : f(X) = 4^{X+1}$, $g : g(X) = 4 - X$



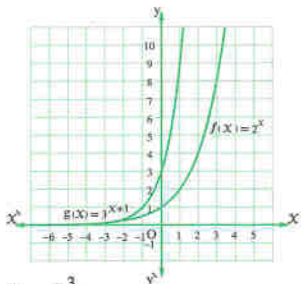
$\therefore$ The S.S. $= \{0\}$

(7) From the graphical representation of the two functions : $f : f(X) = 2^X$, $g : g(X) = 2X$



$\therefore$ The S.S. $= \{1 , 2\}$

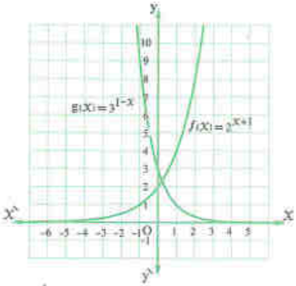
(8) From the graphical representation of the two functions : $f : f(X) = 2^X$, $g : g(X) = 3^{X+1}$



$\therefore X = -2\frac{3}{4}$

$\therefore$ The S.S. $= \left\{-2\frac{3}{4}\right\}$

(9) From the graphical representation of the two functions : $f : f(X) = 2^{X+1}$, $g : g(X) = 3^{1-X}$

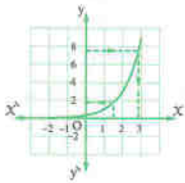


$\therefore X = \frac{1}{4}$

$\therefore$ The S.S. $= \left\{\frac{1}{4}\right\}$

19

X	−2	−1	0	1	2	3
$f(X)$	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

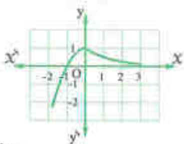


From the graph :

(1) $f\left(\frac{3}{2}\right) \simeq 1.7$

(2) When $3^{X-1} = 7\frac{1}{2}$, then $X \simeq 2.8$

20

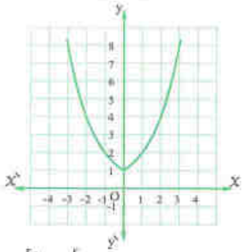


From the graph :

- The domain $= \mathbb{R}$, the range $=]-\infty , 1]$
- The function is increasing on $]-\infty , 0[$ and decreasing on $]0 , \infty[$

21 $f(X) = 2^X$, $X \geq 0$ $\qquad f(X) = 2^{-X}$ $X < 0$

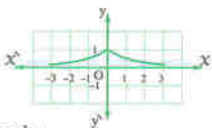
X	0	1	2	3	⓪	−1	−2	−3
$f(X)$	1	2	4	8	①	2	4	8



- The range $= [1 , \infty[$
- The function is decreasing on $]-\infty , 0[$, increasing on $]0 , \infty[$
- The function is even because it is symmetric about y-axis.

22 $f(X) = \left(\frac{1}{2}\right)^X$ when $X \geq 0$ $\qquad f(X) = \left(\frac{1}{2}\right)^{-X}$ when $X < 0$

X	0	1	2	3	⓪	−1	−2	−3
$f(X)$	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	①	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$



From the graph :

- The range $=]0 , 1]$
- The function is increasing on $]-\infty , 0]$ and decreasing on $[0 , \infty[$
- The function is even because it is symmetric about y-axis.

23

$5^{1+\sin X} + 5^{-1+\sin X} = 26 \quad \therefore 5^{\sin X}(5 + 5^{-1}) = 26$

$\therefore 5^{\sin X} = 5 \qquad \therefore \sin X = 1$

$\therefore X = \frac{\pi}{2} + 2\pi n$ where $n \in \mathbb{Z}$

24

The expression $= \frac{1}{2^x+1} + \frac{1}{2^{-x}+1} \times \frac{2^x}{2^x}$

$= \frac{1}{2^x+1} + \frac{2^x}{1+2^x} = \frac{1+2^x}{1+2^x} = 1$

$\therefore$ The value of the expression = 1

Whatever the value of x

25

$\because$ The acount function is $c(t) = a(1+r)^t$

$\therefore$ The total acount after 10 years

$= 80000\left(1 + \frac{10.5}{100}\right)^{10}$

$= 80000 (1.105)^{10} \simeq$ L.E. 217126

26

(1) The number of population

after n years since 2000 $= a(1+r)^n$

$= 43.3\left(1 + \frac{1.5}{100}\right)^n$

$= 43.3 (1.015)^n$

(2) In 2020 , the number of years

will be = 2020 – 2000 = 20 years

$\therefore$ The number of population

$= 43.3 (1.015)^{20} \simeq 58.3$ million people.

27

$a = 36400$, $r = \frac{4}{100} = 0.04$

t = the number of matches

, the numbers of fans $(y) = a(1-r)^t$

$= 36400 (1 - 0.04)^t$

$= 36400 (0.96)^t$

, in the 10th match :

The numbers of fans $(y) = 36400 (0.96)^{10}$

$\simeq 24200$ fans.

28

$a = 80$, $r = \frac{18}{100} = 0.18$

$t = 4$ years.

$\therefore$ The number of cows $= a(1+r)^t$

, the number of cows after 4 years

$= 80 (1 + 0.18)^4 \simeq 155$ cows.

29

$a = 4.6$, $r = \frac{4}{100} = 0.04$

(1) The exponential growth function after t years

$= a(1+r)^t = 4.6 (1 + 0.04)^t$

$= 4.6 (1.04)^t$ million people.

(2) After 5 years :

The number of population $= 4.6 (1.04)^5$

$\simeq 5.6$ million people.

30

$a = 254$, $r = \frac{5}{100} = 0.05$, $t = 9$ years

$\therefore$ The exponential decay function is $f(t) = a(1-r)^t$

$= 254 (1 - 0.05)^t = 254 (0.95)^t$

$\therefore$ The production of the mine in the 9th year

$= 254 (0.95)^9 \simeq 160$ kg.

31

$a = 2000$, $r = \frac{7}{100} = 0.07$

, the time interval = 11 hours

$\therefore$ The exponential growth function is

$f(t) = a(1+r)^t = 2000 (1 + 0.07)^t = 2000 (1.07)^t$

$\therefore$ The number of bacterium after 11 hours

$= 2000 (1.07)^{11} \simeq 4210$ bacterium.

32

By using the relation $A = P\left(1 + \frac{r}{n}\right)^{nt}$

(1) The interest is compounded annually $\therefore n = 1$

$C = 5000 (1 + 0.08)^{10} =$ L.E. 10794.62

(2) The interest is compounded quarter annually

$\therefore n = 4$

$C = 5000\left(1 + \frac{0.08}{4}\right)^{10 \times 4} =$ L.E. 11040.2

(3) The interest is compounded monthly $\therefore n = 12$

$C = 5000\left(1 + \frac{0.08}{12}\right)^{10 \times 12} =$ L.E. 11098.2

Third Higher skills

(1) (d)	(2) (c)	(3) (d)	(4) (c)
(5) (d)	(6) (a)	(7) (d)	(8) (a)

Instructions solving :

(1) $\because$ The function is decreasing if $0 < 2a < 1$

$\therefore 0 < a < \frac{1}{2}$ $\qquad \therefore a \in]0 , \frac{1}{2}[$

(2) $\because$ The function is increasing

$\therefore \frac{a}{3} > 1$ $\qquad \therefore a > 3$

(3) $\because f(X) = (a-2)^X$ is an exponential function

$\therefore a - 2 > 0 \quad , \quad a - 2 \neq 1$

$\therefore a > 2 \quad , \quad a \neq 3$

$\therefore a \in]2 , \infty[- \{3\}$

(4) The curve intersects the X-axis at $y = 0$

by substitute $f(X) = 0$ in the given functions

, no values can be obtained for X except in case (c)

$3^X - 1 = 0$

$\therefore 3^X = 1$ $\qquad \therefore X = 0$

i.e. The curve intersects X-axis at the point $(0 , 0)$

(5) By solving the two equations $y = 8$

, $y = 2^X$ together

$\therefore X = 3$ and by solving the two equations $y = 8$

, $y = \left(\frac{1}{2}\right)^X$ together $\qquad \therefore X = -3$

$\therefore A(3 , 8) , B(-3 , 8)$

$\therefore$ The length of $\overline{AB} = 6$ units.

(6) The reflection of the curve of the function $f(X) = 3^X$ on y-axis gives the curve of the function $f(X) = 3^{-X}$, then shift 5 units upwards gives the curve $f(X) = 5 + 3^{-X}$

(7) $\because f(X) = \frac{9^X}{9^X + 3}$

$\therefore f(1 - X) = \frac{9^{1-X}}{9^{1-X} + 3}$ (multiply by $\frac{9^X}{9^X}$)

$\therefore f(1 - X) = \frac{9}{9 + 3 \times 9^X} = \frac{9}{3(3 + 9^X)} = \frac{3}{3 + 9^X}$

$\therefore f(X) + f(1 - X) = \frac{9^X}{9^X + 3} + \frac{3}{3 + 9^X} = \frac{9^X + 3}{9^X + 3} = 1$

(8) $\because f(X) = \frac{4^X}{4^X + 2}$

$\therefore f(1 - X) = \frac{4^{1-X}}{4^{1-X} + 2}$ (multiply by $\frac{4^X}{4^X}$)

$\therefore f(1 - X) = \frac{4}{4 + 2 \times 4^X} = \frac{4}{2(2 + 4^X)} = \frac{2}{2 + 4^X}$

$\therefore f(X) + f(1 - X) = \frac{4^X}{4^X + 2} + \frac{2}{2 + 4^X} = \frac{4^X + 2}{4^X + 2} = 1$

$\therefore f\left(\frac{1}{11}\right) + f\left(\frac{10}{11}\right) = 1$

, $f\left(\frac{2}{11}\right) + f\left(\frac{9}{11}\right) = 1$

, $f\left(\frac{3}{11}\right) + f\left(\frac{8}{11}\right) = 1$

, $f\left(\frac{4}{11}\right) + f\left(\frac{7}{11}\right) = 1$

, $f\left(\frac{5}{11}\right) + f\left(\frac{6}{11}\right) = 1$

$\therefore$ The required expression $= 5$

Exercise 9

First Multiple choice questions

(1) c	(2) d	(3) d	(4) c	(5) c	(6) b
(7) c	(8) b	(9) c	(10) b	(11) c	(12) b
(13) b	(14) b	(15) b	(16) c	(17) c	(18) c
(19) b	(20) b	(21) c	(22) d	(23) c	(24) a
(25) a	(26) b	(27) c	(28) a	(29) a	(30) a

Second Essay questions

1

(1)

X	7	4	1	-1
$f^{-1}(X)$	-2	1	2	5

(2) $f^{-1} = \{(2 , 1) , (3 , 2) , (4 , 3)\}$

(3) Putting $X = 2y + 5$ $\qquad \therefore y = \frac{X - 5}{2}$

$\therefore f^{-1}(X) = \frac{X - 5}{2}$

(4) Putting $X = \frac{1}{2}y + 4$ $\qquad \therefore y = 2X - 8$

$\therefore f^{-1}(X) = 2X - 8$

(5) Putting $X = 5 + \frac{4}{y}$ $\qquad \therefore \frac{4}{y} = X - 5$

$\therefore y = \frac{4}{X - 5}$ $\qquad \therefore f^{-1}(X) = \frac{4}{X - 5}$

(6) Putting $X = 8y^3 - 1$ $\qquad \therefore 8y^3 = X + 1$

$\therefore y^3 = \frac{1}{8}(X + 1)$ $\qquad \therefore y = \frac{1}{2}\sqrt[3]{X + 1}$

$\therefore f^{-1}(X) = \frac{1}{2}\sqrt[3]{X + 1}$

(7) Putting $X = \sqrt[3]{y + 1}$ $\qquad \therefore y + 1 = X^3$

$\therefore y = X^3 - 1$ $\qquad \therefore f^{-1}(X) = X^3 - 1$

(8) Putting $X = \sqrt[3]{4 - y}$ $\qquad \therefore 4 - y = X^3$

$\therefore y = 4 - X^3$ $\qquad \therefore f^{-1}(X) = 4 - X^3$

(9) $\because y = 2 + \sqrt{3 - x}$ where $x \le 3$, $y \ge 2$

, exchanging the two variables

$\therefore x = 2 + \sqrt{3 - y}$

where $y \le 3$, $x \ge 2$ $\quad \therefore \sqrt{3-y} = x - 2$

$\therefore 3 - y = (x-2)^2$ $\quad \therefore y = 3 - (x-2)^3$

$\therefore f^{-1}(x) = 3 - (x-2)^2$ for every $x \ge 2$

(10) $\because y = x^2$, where $x \ge 0$, $y \ge 0$

, exchanging the two variables

$\therefore x = y^2$, where $y \ge 0$, $x \ge 0$

$\therefore y = \sqrt{x}$ $\quad \therefore f^{-1}(x) = \sqrt{x}$ for every $x \ge 0$

(11) $\because y = (x+2)^2$, where $x \le -2$, $y \ge 0$

, by exchanging the two variables

$\therefore x = (y+2)^2$, where $y \le -2$, $x \ge 0$

$\therefore y + 2 = -\sqrt{x}$ $\quad \therefore y = -\sqrt{x} - 2$

$\therefore f^{-1}(x) = -\sqrt{x} - 2$ for every $x \ge 0$

(12) $\because y = (x-1)^2 + 2$, where $x \ge 1$

, $y \ge 2$, by exchanging the two variables

$\therefore x = (y-1)^2 + 2$, where $y \ge 1$, $x \ge 2$

$\therefore (y-1)^2 = x - 2$ $\quad \therefore y - 1 = \sqrt{x-2}$

$\therefore y = \sqrt{x-2} + 1$ $\quad \therefore f^{-1}(x) = \sqrt{x-2} + 1$

for every $x \ge 2$

(13) $\because y = x^2 + 8x + 7 = (x+4)^2 - 9$

, where $x \ge -4$, $y \ge -9$, by exchanging the two variables

$\therefore x = y^2 + 8y + 7$, where $y \ge -4$, $x \ge -9$

$\therefore x = y^2 + 8y + 16 - 9$

$\therefore x = (y+4)^2 - 9$ $\quad \therefore (y+4)^2 = x + 9$

$\therefore y + 4 = \sqrt{x+9}$ $\quad \therefore y = \sqrt{x+9} - 4$

$\therefore f^{-1}(x) = \sqrt{x+9} - 4$ for every $x \ge -9$

(14) $\because y = \sqrt{9 - x^2}$, where $-3 \le x \le 0$

, $0 \le y \le 3$, by exchanging the two variables

$\therefore x = \sqrt{9 - y^2}$, $-3 \le y \le 0$

, $0 \le x \le 3$ $\quad \therefore x^2 = 9 - y^2$

$\therefore y^2 = 9 - x^2$ $\quad \therefore y = -\sqrt{9 - x^2}$

$\therefore f^{-1}(x) = -\sqrt{9 - x^2}$ for every $0 \le x \le 3$

(15) $\because y = \sqrt{9 - x^2}$, where $0 \le x \le 3$

, $0 \le y \le 3$, by exchanging the two variables

$\therefore x = \sqrt{9 - y^2}$, $0 \le y \le 3$

, $0 \le x \le 3$ $\quad \therefore x^2 = 9 - y^2$

$\therefore y^2 = 9 - x^2$ $\quad \therefore y = \sqrt{9 - x^2}$

$\therefore f^{-1}(x) = \sqrt{9 - x^2}$ for every $0 \le x \le 3$

(16) $\because y = \frac{1}{x^2 + 2}$, $x \in \mathbb{R}^+$, $y \in \left]0 , \frac{1}{2}\right[$

, by exchanging the two variables

$\therefore x = \frac{1}{y^2 + 2}$, $y \in \mathbb{R}^+$, $x \in \left]0 , \frac{1}{2}\right[$

$\therefore y^2 + 2 = \frac{1}{x}$ $\quad \therefore y^2 = \frac{1}{x} - 2$

$\therefore y = \sqrt{\frac{1}{x} - 2}$ $\quad \therefore f^{-1}(x) = \sqrt{\frac{1}{x} - 2}$

, $x \in \left]0 , \frac{1}{2}\right[$

2

(1) has an inverse function.

(2) has not an inverse function.

(3) has not an inverse function.

(4) has not an inverse function.

3

(1) $\because (f \circ g)(x) = f\left(g(x)\right)$

$= f\left(\frac{x+3}{2}\right) = 2\left(\frac{x+3}{2}\right) - 3$

$= x + 3 - 3 = x$

, $(g \circ f)(x) = g\left(f(x)\right) = g(2x - 3)$

$= \frac{(2x-3)+3}{2} = x$

$\therefore$ Each of f and g is an inverse function of the other.

(2) $\because (f \circ g)(x) = f\left(g(x)\right) = f\left(\sqrt{x-4}\right)$

$= \left(\sqrt{x-4}\right)^2 + 4$

$= x - 4 + 4 = x$

, $\because (g \circ f)(x) = g\left(f(x)\right) = g(x^2 + 4)$

$= \sqrt{x^2 + 4 - 4} = \sqrt{x^2} = |x|$

, $\because x \ge 0$ $\quad \therefore (g \circ f)(x) = x$

$\therefore$ Each of f and g is an inverse function of the other.

(3) $g(x) = \frac{5x-2}{x}$ $\quad \therefore g(x) = 5 - \frac{2}{x}$

$\therefore (f \circ g)(x) = f\left(g(x)\right)$

$= f\left(5 - \frac{2}{x}\right) = \frac{-2}{5 - \frac{2}{x} - 5} = x$

, $(g \circ f)(x) = g\left(f(x)\right) = g\left(\frac{-2}{x-5}\right)$

$= 5 - \frac{2}{\frac{-2}{x-5}}$

$= 5 + x - 5 = x$

$\therefore$ Each of f and g is an inverse function of the other.

(4) $(f \circ g)(x) = f\left(g(x)\right) = f\left(\frac{x^3}{4}\right) = \sqrt[3]{4 \times \frac{x^3}{4}} = x$

, $(g \circ f)(x) = g\left(f(x)\right) = g\left(\sqrt[3]{4x}\right)$

$= \frac{\left(\sqrt[3]{4x}\right)^3}{4} = \frac{4x}{4} = x$

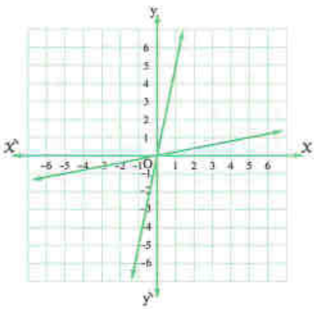
$\therefore$ Each of f and g is an inverse function of the other.

4

(1) $\because y = 5x$, by exchanging the two variables

$\therefore x = 5y$ $\quad \therefore y = \frac{1}{5}x$

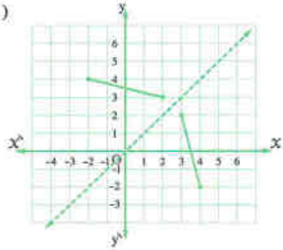
$\therefore f^{-1}(x) = \frac{1}{5}x$



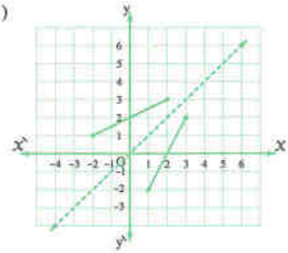
(2) $f^{-1}(b) + 2f^{-1}(c) = 3 + 2 \times 4 = 11$

5

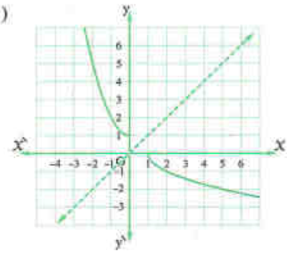
(1)



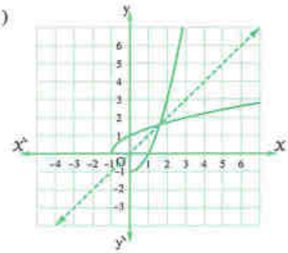
(2)



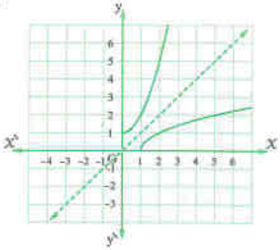
(3)



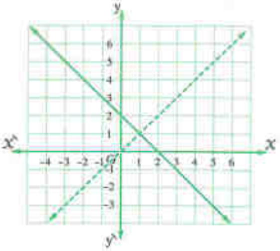
(4)



(5)



(6)



6

(1) $\because f(X) = 2X$ $\therefore y = 2X$

, by exchanging the two variables

$\therefore X = 2y$ $\therefore y = \frac{1}{2}X$

$\therefore f^{-1}(X) = \frac{1}{2}X \neq f(X)$

$\therefore$ The inverse function of f is not itself.

(2) $\because f(X) = -X$ $\therefore y = -X$

, by exchanging the two variables

$\therefore X = -y$ $\therefore y = -X$

$\therefore f^{-1}(X) = -X = f(X)$

$\therefore$ The inverse of f is itself.

(3) $\because f(X) = \frac{2}{X}$ $\therefore y = \frac{2}{X}$

, by exchanging the two variables

$\therefore X = \frac{2}{y}$ $\therefore y = \frac{2}{X}$

$\therefore f^{-1}(X) = \frac{2}{X} = f(X)$

$\therefore$ The inverse of f is itself.

(4) $\because f(X) = 7 - X$ $\therefore y = 7 - X$

, by exchanging the two variables

$\therefore X = 7 - y$ $\therefore y = 7 - X$

$\therefore f^{-1}(X) = 7 - X = f(X)$

$\therefore$ The inverse of f is itself.

(5) $\because f(X) = \frac{1}{X-3} + 5$ $\therefore y = \frac{1}{X-3} + 5$

, by exchanging the two variables

$\therefore X = \frac{1}{y-3} + 5$ $\therefore X - 5 = \frac{1}{y-3}$

$\therefore y - 3 = \frac{1}{X-5}$ $\therefore y = \frac{1}{X-5} + 3$

$\therefore f^{-1}(X) = \frac{1}{X-5} + 3 \neq f(X)$

$\therefore$ The inverse of f is not itself.

(6) $\because f(X) = \frac{1}{X-k} + k$ $\therefore y = \frac{1}{X-k} + k$

, by exchanging the two variables

$\therefore X = \frac{1}{y-k} + k$ $\therefore X - k = \frac{1}{y-k}$

$\therefore y - k = \frac{1}{X-k}$ $\therefore y = \frac{1}{X-k} + k$

$\therefore f^{-1}(X) = \frac{1}{X-k} + k = f(X)$

$\therefore$ The inverse of f is itself.

7

Rana's answer is the correct answer

for : $f^{-1}(X) \neq \frac{1}{f(X)}$ as in Wael's answer.

8

(1) $\because f(X) = X^2$

$\therefore f$ is not one-to-one function on its domain $\mathbb{R}$ and is one - to - one if $X \in [0, \infty[$ or $X \in]-\infty, 0]$

$\therefore$ The domain at which the function has an inverse function $= [0, \infty[$ or $]-\infty, 0]$

(2) $\because f(X) = X^3$

$\therefore f$ is one - to - one function on its domain $\mathbb{R}$

$\therefore$ The domain at which the function has an inverse $= \mathbb{R}$

(3) $\because f(X) = \frac{1}{2}X$

$\therefore f$ is one - to - one on its domain $\mathbb{R}$

$\therefore$ The domain at which the function has an inverse $= \mathbb{R}$

9

$\because y = \frac{X-1}{X+5}$

, by exchanging the two variables

$\therefore X = \frac{y-1}{y+5}$ $\therefore Xy + 5X = y - 1$

$\therefore Xy - y = -5X - 1$ $\therefore y(X-1) = -5X - 1$

$\therefore y = \frac{-5X-1}{X-1}$ $\therefore f^{-1}(X) = \frac{-5X-1}{X-1}$

10

$\because f(X) = 2X + a$ $\therefore y = 2X + a$

, by exchanging the two variables

$\therefore X = 2y + a$ $\therefore y = \frac{X-a}{2} = \frac{1}{2}X - \frac{a}{2}$

$\therefore f^{-1}(X) = \frac{1}{2}X - \frac{a}{2}$

$\because f^{-1}(X) = g(X)$ $\therefore \frac{1}{2}X - \frac{a}{2} = bX + 3$

$\therefore b = \frac{1}{2}$, $-\frac{a}{2} = 3$ $\therefore a = -6$

Third Higher skills

(1) (a) (2) (c) (3) (a) (4) (a)

(5) (c) (6) (d) (7) (b) (8) (a)

Instructions to solve :

(1) $\because f^{-1}(9) = 3$

$\therefore$ The point (3 , 9) $\in$ the curve of the function

, $\because f^{-1}(5) = 2$

$\therefore$ The point (2 , 5) $\in$ the curve of the function

, $\because f(X) = aX + b$ $\therefore 3a + b = 9$ (1)

, $2a + b = 5$ (2)

By solving the two equations (1) , (2) :

$\therefore a = 4$, $b = -3$

$\therefore a \times b = 4 \times -3 = -12$

(2) $\because g(X) = X - 3$ $\therefore g^{-1}(X) = X + 3$

, $\because g\left(f(X)\right) = g(X^2) = X^2 - 3$

$\therefore g\left(f(X)\right) = g^{-1}(X)$

$\therefore X^2 - 3 = X + 3$ $\therefore X^2 - X - 6 = 0$

$\therefore (X+2)(X-3) = 0$

$\therefore X = -2$ or $X = 3$

$\therefore$ S.S. $= \{3 , -2\}$

(3) Put $f(X) = 3$ and left find the value of X

$\therefore X = \frac{f(X)+1}{2-f(X)} = \frac{3+1}{2-3} = -4$

$f(-4) = 3$ $\therefore f^{-1}(3) = -4$

(4) $\because$ The domain $\mathbb{R} - \{1\}$ $\therefore b = 1$

$\therefore f(X) = \frac{aX+3}{X-1} = a + \frac{3+a}{X-1}$

From the rule of the function , the point of symmetry of the curve is (1 , a)

And from the domain and codomain , the point of symmetry of the curve is (1 , 3)

$\therefore a = 3$

$\therefore f(X) = \frac{3X+3}{X-1}$ by replacing the two variables

$\therefore X = \frac{3y+3}{y-1}$ $\therefore Xy - X = 3y + 3$

$(X-3)y = X + 3$

$\therefore y = \frac{X+3}{X-3}$ $\therefore f^{-1}(X) = \frac{X+3}{X-3}$

$\therefore f^{-1}(1) = -2$

(5) $\because f(X) = \sqrt[3]{X-5}$, by replacing the two variables

$\therefore X = \sqrt[3]{y-5}$ $\therefore y = X^3 + 5$

$\therefore f^{-1}(X) = X^3 + 5$

$\because (g \circ f^{-1})(X) = g\left(f^{-1}(X)\right) = g(X^3 + 5)$

$= 2(X^3 + 5) - 7$

$= 2X^3 + 3$

(6) $\because f(X) = 3X - 4$, by replacing the two variables

$\therefore X = 3y - 4$ $\therefore y = \frac{X+4}{3}$

$\therefore f^{-1}(X) = \frac{X+4}{3}$

$\therefore f^{-1}(X+2) = \frac{X+2+4}{3} = \frac{X+6}{3}$

(7) $\because f(X) = X^3 + 3X^2 + 3X + 1$

$= (X^3 + 1) + (3X^2 + 3X)$

$= (X+1)(X^2 - X + 1) + 3X(X+1)$

$= (X+1)(X^2 - X + 1 + 3X)$

$= (X+1)(X^2 + 2X + 1)$

$= (X+1)(X+1)^2 = (X+1)^3$

, by replacing the two variables

$\therefore X = (y+1)^3$ $\therefore y = \sqrt[3]{X} - 1$

$\therefore f^{-1}(X) = \sqrt[3]{X} - 1$

(8) ∵ The curve of f^{-1} passes through the two points $(-2, 0)$, $(-3, -2)$

$\therefore f^{-1}(-2)=0 \quad \therefore f(0)=-2$

$, f^{-1}(-3)=-2 \quad \therefore f(-2)=-3$

$\therefore (f \circ f)(\text{zero}) = f\big(f(\text{zero})\big) = f(-2) = -3$

Exercise 10

First Multiple choice questions

(1) b	(2) b	(3) c	(4) b	(5) d	(6) d
(7) d	(8) a	(9) a	(10) c	(11) a	(12) b
(13) d	(14) c	(15) a	(16) c	(17) c	(18) b
(19) b	(20) b	(21) b	(22) d	(23) b	(24) b
(25) b	(26) d	(27) a	(28) c	(29) a	(30) b
(31) c	(32) c	(33) b	(34) c	(35) a	(36) b
(37) a	(38) c	(39) a	(40) a	(41) d	(42) b
(43) a	(44) c	(45) d			

Second Essay questions

1

(1) $2^7 = 128$ (2) $\left(\frac{2}{5}\right)^2 = \frac{4}{25}$

(3) $2^{\frac{5}{2}} = 4\sqrt{2}$

2

(1) $\log_5 1 = \text{zero}$ (2) $\log_{10} 0.0001 = -4$

(3) $\log_5 \frac{1}{125} = -3$

3

(1) Let $\log_2 16 = x$ $\quad \therefore 2^x = 16$

$\therefore 2^x = 2^4 \quad \therefore x = 4$

$\therefore \log_2 16 = 4$

(2) Let $\log_8 1 = x$ $\quad \therefore 8^x = 1 = 8^0$

$\therefore x = 0 \quad \therefore \log_8 1 = 0$

(3) Let $\log 0.00001 = x$ $\quad \therefore 10^x = 0.00001$

$\therefore 10^x = \frac{1}{100000} \quad \therefore 10^x = 10^{-5}$

$\therefore x = -5 \quad \therefore \log 0.00001 = -5$

(4) Let $\log_{\frac{1}{2}} 128 = x$ $\quad \therefore \left(\frac{1}{2}\right)^x = 128 = 2^7$

$\therefore \left(\frac{1}{2}\right)^x = \left(\frac{1}{2}\right)^{-7} \quad \therefore x = -7$

$\therefore \log_{\frac{1}{2}} 128 = -7$

(5) Let $\log_2 \frac{1}{8} = x$ $\quad \therefore 2^x = \frac{1}{8}$

$\therefore 2^x = 2^{-3} \quad \therefore x = -3$

$\therefore \log_2 \frac{1}{8} = -3$

(6) Let $\log_{\sqrt{2}} 8\sqrt{2} = x$

$\therefore \left(\sqrt{2}\right)^x = 8\sqrt{2} = \left(\sqrt{2}\right)^7$

$\therefore x = 7 \quad \therefore \log_{\sqrt{2}} 8\sqrt{2} = 7$

(7) Let $\log_3 \sqrt[4]{27} = x$ $\quad \therefore 3^x = \sqrt[4]{27}$

$\therefore 3^x = 3^{\frac{3}{4}} \quad \therefore x = \frac{3}{4}$

$\therefore \log_3 \sqrt[4]{27} = \frac{3}{4}$

(8) Let $\log_2 \cos 45^\circ = \log_2 \frac{1}{\sqrt{2}} = x$

$\therefore 2^x = \frac{1}{\sqrt{2}} = 2^{-\frac{1}{2}} \quad \therefore x = -\frac{1}{2}$

$\therefore \log_2 \cos 45^\circ = \frac{-1}{2}$

4

(1) $2^7 = x$ $\quad \therefore x = 128$

(2) $5^{-2} = x$ $\quad \therefore x = \frac{1}{25}$

(3) $3^4 = x^2$ $\quad \therefore 81 = x^2 \quad \therefore x = \pm 9$

(4) $(81)^{\frac{3}{4}} = x$ $\quad \therefore x = 27$

(5) $2^{-5} = x^{-1}$ $\quad \therefore \frac{1}{2^5} = \frac{1}{x} \quad \therefore x = 32$

(6) $2 = \log_3 x$ $\quad \therefore x = 3^2 = 9$

(7) $3^0 = 2x - 5$ $\quad \therefore 1 = 2x - 5 \quad \therefore x = 3$

(8) $8^{\frac{2}{3}} = \sqrt[3]{x^2 + 48}$

$\therefore 8^2 = x^2 + 48 \quad \therefore x^2 = 16 \quad \therefore x = \pm 4$

(9) $\log_2 (x^2 - 2x) = 3$

$\therefore 2^3 = x^2 - 2x \quad \therefore x^2 - 2x - 8 = 0$

$\therefore (x-4)(x+2) = 0 \quad \therefore x = 4$ or $x = -2$

(10) $5 = |2x + 1|$

$\therefore 2x + 1 = 5$ or $2x + 1 = -5$

$\therefore x = 2$ or $x = -3$

(11) $2^4 = x(x+6)$ $\therefore x^2+6x-16=0$

$\therefore (x-2)(x+8)=0$ $\therefore x=2$ or $x=-8$

(12) $3 = x^2-2x$ $\therefore x^2-2x-3=0$

$\therefore (x-3)(x+1)=0$ $\therefore x=3$ or $x=-1$

(13) $(\log_3 x-4)(\log_3 x-5)=0$

$\therefore \log_3 x=4$ $\therefore x=3^4=81$

or $\log_3 x=5$ $\therefore x=3^5=243$

(14) $3^x-3^{x-2}=2^3$ $\therefore 3^{x-2}(3^2-1)=8$

$\therefore 3^{x-2}=1$ $\therefore x-2=0$

$\therefore x=2$

(15) $3^{\log_4(x+1.25)}=3^{-1}$ $\therefore \log_4(x+1.25)=-1$

$\therefore x+1.25=4^{-1}$ $\therefore x=-1$

(16) $\log_{10} x-2=\pm 2$ $\therefore \log_{10} x-2=2$

$\therefore \log_{10} x=4$ $\therefore x=10^4$

or $\log_{10} x-2=-2$ $\therefore \log_{10} x=0$

$\therefore x=1$

5

(1) $x^3=125=5^3$ $\therefore x=5$

$\therefore$ The S.S. $=\{5\}$

(2) $x^5=2$ $\therefore x=\sqrt[5]{2}$

$\therefore$ The S.S. $=\{\sqrt[5]{2}\}$

(3) $x^{-2}=3$ $\therefore x^2=\frac{1}{3}$

$\therefore x=\frac{1}{\sqrt{3}}$ (and negative solution is refused)

$\therefore$ The S.S. $=\left\{\frac{1}{\sqrt{3}}\right\}$

(4) $x^{\frac{-3}{4}}=\frac{1}{1000}$ $\therefore x=(10^{-3})^{\frac{-4}{3}}=10000$

$\therefore$ The S.S. $=\{10000\}$

(5) $(-x)^4=81=3^4$

$\therefore x=-3$ (and the positive solution is refused)

$\therefore$ The S.S. $=\{-3\}$

(6) $(x-1)^3=27=3^3$ $\therefore x-1=3$

$\therefore x=4$ $\therefore$ The S.S. $=\{4\}$

(7) $(x-1)^2=7-x$ $\therefore x^2-2x+1=7-x$

$\therefore x^2-x-6=0$ $\therefore (x+2)(x-3)=0$

$\therefore x=-2$ (refused) or $x=3$

$\therefore$ The S.S. $=\{3\}$

(8) $\left(\frac{2}{x}\right)^{-3}=64$ $\therefore \left(\frac{x}{2}\right)^3=4^3$

$\therefore \frac{x}{2}=4$ $\therefore x=8$

$\therefore$ The S.S. $=\{8\}$

(9) $x^2=5x$ $\therefore x^2-5x=0$

$\therefore x(x-5)=0$ $\therefore x=0$ (refused)

or $x=5$ $\therefore$ The S.S. $=\{5\}$

(10) $x^{5x}=x^6$ $\therefore 5x=6$

$\therefore x=\frac{6}{5}$ (noticing that $x \in \mathbb{R}^+-\{1\}$)

$\therefore$ The S.S. $=\left\{\frac{6}{5}\right\}$

(11) $\log_3 \log_x 27=9^0=1$ $\therefore \log_x 27=3$

$\therefore x^3=27=3^3$ $\therefore x=3$

$\therefore$ The S.S. $=\{3\}$

(12) $x=x^2-12$ $\therefore x^2-x-12=0$

$\therefore (x-4)(x+3)=0$

$\therefore x=4$ or $x=-3$ (refused)

$\therefore$ The S.S. $=\{4\}$

(13) $x=\sqrt{x-2}+2$

$\therefore \sqrt{x-2}=x-2$ (by squaring both sides)

$\therefore x-2=x^2-4x+4$ $\therefore x^2-5x+6=0$

$\therefore (x-2)(x-3)=0$ $\therefore x=2$ or $x=3$

$\therefore$ The S.S. $=\{2, 3\}$

(14) $(\log_3 x)^2-8\log_3 x+15=0$

$\therefore (\log_3 x-5)(\log_3 x-3)=0$

$\therefore \log_3 x=5$ or $\log_3 x=3$

$\therefore x=3^5=243$ or $x=3^3=27$

$\therefore$ The S.S. $=\{243, 27\}$

(15) Let $\log_3(x-2)=k$ $\therefore x-2=3^k$ (1)

, $\log_{27}(x-2)=y$

$\therefore x-2=27^y=3^{3y}$ (2)

From (1) and (2) : $\therefore 3y=k$ $\therefore y+3y=4$

$\therefore y=1$ $\therefore k=3$

i.e. $\log_3(x-2)=3$ $\therefore 3^3=x-2$

$\therefore x=29$ $\therefore$ The S.S. $=\{29\}$

6

(1) $4^x = 8\sqrt{2}$ $\therefore 2^{2x} = 2^{\frac{7}{2}}$

$\therefore 2x = \frac{7}{2}$ $\therefore x = \frac{7}{4}$

(2) $(\sqrt{5})^{x^2} = 625\sqrt{5} = (\sqrt{5})^9$

$\therefore x^2 = 9$ $\therefore x = \pm 3$

(3) $(0.3)^{x^{-2}} = 0.09 = (0.3)^2$ $\therefore x^{-2} = 2$

$\therefore x^2 = \frac{1}{2}$ $\therefore x = \pm \frac{1}{\sqrt{2}}$

(4) $\log_2 (2^x - 4) = 5 - x$

$\therefore 2^{5-x} = 2^x - 4$ (multiplying by 2^x)

$\therefore 2^5 = 2^{2x} - 4 \times 2^x$

$\therefore 2^{2x} - 4 \times 2^x - 32 = 0$

$\therefore (2^x - 8)(2^x + 4) = 0$

$\therefore 2^x = 8$ or $2^x = -4$ (refused)

$\therefore 2^x = 2^3$ $\therefore x = 3$

(5) $4^2 = 13 + \log_2 (x - 1)$ $\therefore \log_2 (x - 1) = 3$

$\therefore 2^3 = x - 1$ $\therefore x = 9$

(6) $3 = \frac{x^2}{2x - 3}$ $\therefore x^2 - 6x + 9 = 0$

$\therefore (x - 3)^2 = 0$ $\therefore x = 3$

7

(1) 0.4983 (2) 4.7549 (3) –2.1893

8

(1) 1.7159 (2) 25.8226 (3) 0.5012

9

(1) $\because \log_x 16 = y$ $\therefore x^y = 16$

, from first equation

$\therefore 16 = 5x - 4$ $\therefore x = 4$

$\therefore \log_4 16 = y$ $\therefore 4^y = 16 = 4^2$

$\therefore y = 2$ $\therefore$ The S.S. $= \{(4 , 2)\}$

(2) $\because \log_y 9 = 1$ $\therefore y = 9$

$\therefore \log_x \log_2 \log_x 9 = 0$

$\therefore \log_2 \log_x 9 = x^0 = 1$

$\therefore \log_x 9 = 2$ $\therefore x^2 = 9$

$\therefore x = 3$ (and the negative solution is refused)

$\therefore$ The S.S. $= \{(3 , 9)\}$

10

$\because \log_{16} 49 = a$ $\therefore (16)^a = 49$

$\therefore (4^2)^a = 7^2$ $\therefore (4^a)^2 = 7^2$ $\therefore 4^a = 7$

, $\because \log_7 2.5 = b$ $\therefore 7^b = 2.5$

$\therefore 4^{ab+1} = 4^{ab} \times 4 = (4^a)^b \times 4$

$= 7^b \times 4 = 2.5 \times 4 = 10$

11

(1) $2x + 1 > 0$ $\therefore x > -\frac{1}{2}$

$\therefore$ The domain $=]-\frac{1}{2} , \infty[$

(2) $x > 0$ $\therefore$ The domain $=]0 , \infty[$

(3) The function is defined for every x

satisfying $\begin{cases} x > 0 \\ x \neq 1 \end{cases}$

i.e. The domain $=]0 , \infty[- \{1\}$

(4) The function is defined for every x

satisfying $\begin{cases} x > 0 \\ x - 2 > 0 \\ x - 2 \neq 1 \end{cases}$ *i.e.* $\begin{cases} x > 0 \\ x > 2 \\ x \neq 3 \end{cases}$

i.e. The domain $=]2 , \infty[- \{3\}$

(5) The function is defined for every x

satisfying $\begin{cases} x > 0 \\ 2 - x > 0 \\ 2 - x \neq 1 \end{cases}$ *i.e.* $\begin{cases} x > 0 \\ x < 2 \\ x \neq 1 \end{cases}$

i.e. The domain $=]0 , 2[- \{1\}$

(6) The function is defined for every x

satisfying $\begin{cases} x - 3 > 0 \\ 5 - x > 0 \\ 5 - x \neq 1 \end{cases}$ *i.e.* $\begin{cases} x > 3 \\ x < 5 \\ x \neq 4 \end{cases}$

i.e. The domain $=]3 , 5[- \{4\}$

(7) The function is defined for every x

satisfying $\begin{cases} x^2 > 0 \\ x - 3 > 0 \\ x - 3 \neq 1 \end{cases}$ *i.e.* $\begin{cases} x^2 > 0 \\ x > 3 \\ x \neq 4 \end{cases}$

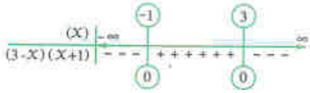
i.e. The domain $=]3 , \infty[- \{4\}$

(8) The function is defined for every X

satisfying $\begin{cases} 3X+1>0 \\ X+2>0 \\ X+2\neq 1 \end{cases}$ *i.e.* $\begin{cases} X>-\frac{1}{3} \\ X>-2 \\ X\neq -1 \end{cases}$

i.e. The domain $= \left]-\frac{1}{3}, \infty\right[$

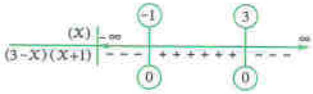
(9) The function is defined for every X satisfying $(3-X)(X+1)>0$



i.e. The domain $=]-1, 3[$

(10) The function is defined for every X

satisfying $\begin{cases} (3-X)(X+1)>0 \\ X>0 \\ X\neq 1 \end{cases}$



i.e. The domain $=]0, 3[- \{1\}$

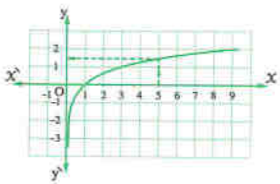
12

$\because$ The curve of the function passes through $(81, 4)$

$\therefore 4 = \log_a 81$ $\qquad \therefore a^4 = 81$

$\therefore a = 3$ (and the negative solution is refused)

X	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
$f(X)$	-2	-1	0	1	2



From the graph :

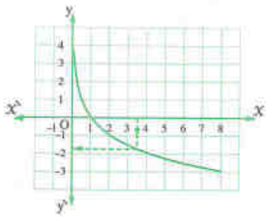
* The domain $= \mathbb{R}^+$, the range $= \mathbb{R}$
* The function is increasing on its domain
* The intersection point with the X-axis is $(1, 0)$
* $\log_3 5 \simeq 1.5$

13

$\because$ The curve of the function passes through $\left(\frac{1}{8}, 3\right)$

$\therefore 3 = \log_a \frac{1}{8}$ $\qquad \therefore a^3 = \frac{1}{8}$ $\qquad \therefore a = \frac{1}{2}$

X	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$f(X)$	2	1	0	-1	-2

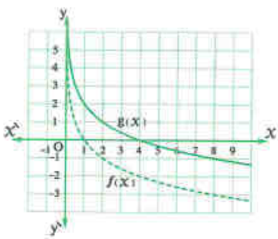


From the graph :

* The range $= \mathbb{R}$
* The function is decreasing on its domain.
* The intersection point with the X-axis is $(1, 0)$
* $\log_{\frac{1}{2}} 3\frac{1}{2} \simeq -1.8$

14

(1) The curve of the function g is the same curve of the function f with vertical translation 2 units in the direction of $\overrightarrow{Oy}$

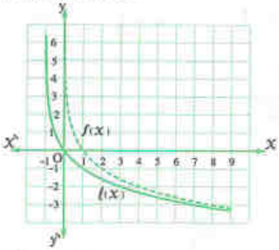


From the graph :

The domain $=]0, \infty[$, the range $= \mathbb{R}$

, the function is decreasing on its domain.

(2) The curve of the function ℓ is the same curve of the function f with horizontal translation 1 units in the direction of $\overrightarrow{OX}$

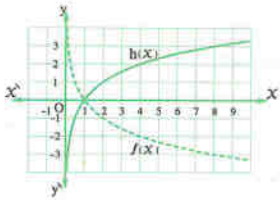


From the graph :

The domain = $]-1 , \infty[$, the range = $\mathbb{R}$

, the function is decreasing on its domain.

(3) The curve of the function h is the same curve of the function f with reflection in the X-axis.



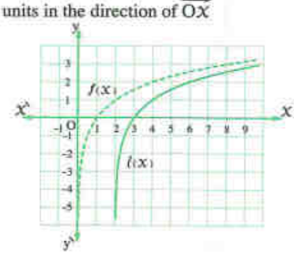
From the graph :

The domain = $]0 , \infty[$, the range = $\mathbb{R}$

, the function is increasing on its domain.

15

(1) The curve of the function ℓ is the same curve of the function f with horizontal translation 2 units in the direction of $\overrightarrow{OX}$

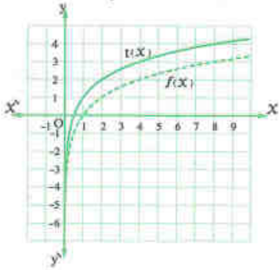


From the graph :

The domain = $]2 , \infty[$, the range = $\mathbb{R}$

, the function is increasing on its domain.

(2) The curve of the function t is the same curve of the function f with vertical translation 1 unit in the direction of $\overrightarrow{Oy}$

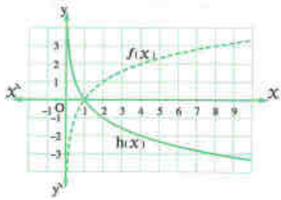


From the graph :

The domain = $]0 , \infty[$, the range = $\mathbb{R}$

, the function is increasing on its domain.

(3) The curve of the function h is the same curve of the function f with reflection in the X-axis.

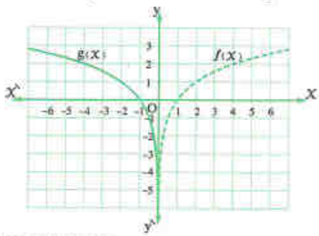


From the graph :

The domain = $]0 , \infty[$, the range = $\mathbb{R}$

, the function is decreasing on its domain.

(4) The curve of the function g is the same curve of the function f with reflection in the y-axis.

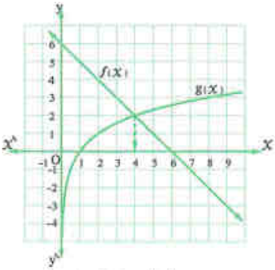


From the graph :

The domain = $]-\infty , 0[$, the range = $\mathbb{R}$

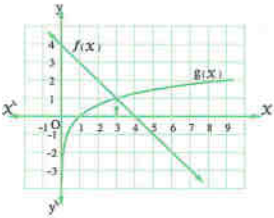
, the function is decreasing on its domain.

16



From the graph , the S.S. = $\{4\}$

17



From the graph , the S.S. = $\{3\}$

Third Higher skills

(1) (b)	(2) (d)	(3) (a)	(4) (b)
(5) (c)	(6) (a)	(7) (b)	(8) (d)

Instructions solving :

(1) $\because f^{-1}(5) = 14$ $\therefore f(14) = 5$

$\therefore \log_a (2 \times 14 + 4) = 5$

$\therefore a^5 = 32$ $\therefore a = 2$

(2) $\because f^{-1}(a+3) = 32$ $\therefore f(32) = a + 3$

$\therefore \log_4 32 = a + 3$ $\therefore 4^{a+3} = 32$

$\therefore 2^{2a+6} = 2^5$ $\therefore 2a + 6 = 5$

$\therefore a = -\frac{1}{2}$

(3) $(f \circ g)(10) = f\left(g(10)\right) = f(7) = \log_2 8 = 3$

(4) $\because \log(x-5) > \text{zero}$

$\therefore x - 5 > 1$ $\therefore x > 6$

(5) $\because \log_3 \log_2 \left|\,|x-1| + 5\,\right| = 1$

$\therefore \log_2 \left|\,|x-1| + 5\,\right| = 3$

$\therefore \left|\,|x-1| + 5\,\right| = 8$ $\therefore |x-1| + 5 = \pm 8$

$\therefore |x-1| = -13$ (Refused) or $|x-1| = 3$

$\therefore x - 1 = \pm 3$ $\therefore x = 1 \pm 3$

$x = 4$ or $x = -2$

$\therefore$ The solution set $= \{-2 , 4\}$

(6) $\because$ The curve of the function passes through $(11 , 3) , (4 , 0)$

$\therefore f(4) = 0$ $\therefore \log_a (4 + b) = \text{zero}$

$\therefore 4 + b = 1$ $\therefore b = -3$

$, f(11) = 3$ $\therefore \log_a (11 + b) = 3$

$\therefore 11 + b = a^3$ $\therefore a^3 = 8$ $\therefore a = 2$

$\therefore f(x) = \log_2 (x - 3)$ $\therefore f(7) = \log_2 (7-3) = 2$

$, f^{-1}(2) = 7$ $\therefore f(7) + f^{-1}(2) = 9$

(7) To identify the domain of the function

$f(x) = \log|x^2 - 9|$, put $|x^2 - 9| > 0$

i.e. $|(x-3)(x+3)| > 0$

$\therefore x \in \mathbb{R} - \{3 , -3\}$

$\therefore$ The domain of the function $f = \mathbb{R} - \{3 , -3\}$

(8) To identify the domain of the function

$$f(x) = \frac{\log_2 (x+3)}{x^2 + 3x + 2} = \frac{\log_2 (x+3)}{(x+1)(x+2)}$$

$\therefore x + 3 > 0$ $\therefore x > -3$

and $x \neq -1$, $x \neq -2$

$\therefore$ The domain $=]-3 , \infty[- \{-1 , -2\}$

Exercise 11

First Multiple choice questions

(1) a	(2) c	(3) b	(4) a	(5) c	(6) d
(7) c	(8) b	(9) c	(10) a	(11) b	(12) c
(13) c	(14) c	(15) c	(16) c	(17) b	(18) b
(19) d	(20) b	(21) c	(22) a	(23) b	(24) b
(25) c	(26) b	(27) c	(28) a	(29) d	(30) c
(31) a	(32) d	(33) a	(34) a	(35) d	(36) a
(37) c	(38) d	(39) d	(40) b		

UNIT 2

Second Essay questions

1

(1) $\log 2 + \log 5 = \log (2 \times 5) = \log 10 = 1$

(2) $\log_5 15 - \log_5 3 = \log_5 \frac{15}{3} = \log_5 5 = 1$

(3) $\log_2 5 \times \log_5 2 = \frac{\log 5}{\log 2} \times \frac{\log 2}{\log 5} = 1$

(4) $\log_3 81 \times \log_9 3 = \log_9 3^4 \times \log_9 9^{\frac{1}{2}} = 4 \times \frac{1}{2} = 2$

(5) $\log_2 \frac{15 \times 14}{105} = \log_2 2 = 1$

(6) $\log 54 - 3 \log 3 - \log 2$

$= \log 54 - \log 27 - \log 2$

$= \log \frac{54}{27 \times 2} = \log 1 = 0$

(7) $\log_4 \log_2 \log_2 2^2 = \log_4 \log_2 2 = \log_4 1 = 0$

(8) $1 + \log 3 - \log 2 - \log 15$

$= 1 + \log \left(\frac{3}{2 \times 15}\right) = 1 + \log \frac{1}{10}$

$= 1 + \log (10)^{-1} = 1 - 1 = 0$

(9) $\log 0.009 - \log \frac{27}{16} + \log 15 \frac{5}{8} - \log \frac{1}{12}$

$= \log \left(0.009 \times \frac{16}{27} \times 15 \frac{5}{8} \times 12\right) = \log 1 = 0$

(10) $\log_{100} \frac{7}{170} - \log_{100} \frac{35}{18} + \log_{100} \frac{17}{36}$

$= \log_{100} \left(\frac{7}{170} \times \frac{18}{35} \times \frac{17}{36}\right)$

$= \log_{100} \frac{1}{100} = \log_{100} (100)^{-1} = -1$

(11) $2 \log 25 + \log \left(\frac{1}{3} + \frac{1}{5}\right) + 2 \log 3 - \log 30$

$= \log 625 + \log \frac{8}{15} + \log 9 - \log 30$

$= \log \left(625 \times \frac{8}{15} \times 9 \times \frac{1}{30}\right) = \log 100 = 2$

(12) $\log 25 + \frac{\log 8 \times \log 16}{\log 64}$

$= \log 25 + \frac{3 \log 2 \times 4 \log 2}{6 \log 2}$

$= \log 25 + 2 \log 2 = \log 25 + \log 4$

$= \log (25 \times 4) = \log 100 = 2$

(13) $\frac{\log_2 25 + \log_2 4}{\log_2 30 - \log_2 3} = \frac{\log_2 (25 \times 4)}{\log_2 \frac{30}{3}}$

$= \frac{\log_2 100}{\log_2 10} = \frac{2 \log_2 10}{\log_2 10} = 2$

(14) $\frac{(\log 3)^2 - \log 3^2}{\log 3 - \log 100} = \frac{(\log 3)^2 - 2 \log 3}{\log 3 - 2 \log 10}$

$= \frac{\log 3 (\log 3 - 2)}{(\log 3 - 2)} = \log 3$

(15) $\log_5 \frac{\log_3 3 - \log_3 2}{\log_3 \left(\frac{3}{2}\right)^5} = \log_5 \frac{\log_3 \frac{3}{2}}{5 \log_3 \frac{3}{2}} = \log_5 \frac{1}{5} = -1$

(16) $\log_3 \frac{\log x^{18}}{\log x^2} = \log_3 \frac{18 \log x}{2 \log x} = \log_3 9 = 2$

(17) $\log_{abc} a + \log_{abc} b + \log_{abc} c = \log_{abc} abc = 1$

(18) $\frac{1}{2} \log_3 a + \frac{1}{2} \log_3 b + 2 \log_3 c - \log_3 \sqrt{ab} - \log_3 3c^2$

$= \log_3 a^{\frac{1}{2}} + \log_3 b^{\frac{1}{2}} + \log_3 c^2 - \log_3 \sqrt{ab}$

$- (\log_3 3 + \log_3 c^2)$

$= \log_3 \sqrt{a} + \log_3 \sqrt{b} + \log_3 c^2 - \log_3 \sqrt{ab}$

$- \log_3 3 - \log_3 c^2$

$= \log_3 \sqrt{ab} - \log_3 \sqrt{ab} - 1 = -1$

(19) $\frac{1}{\log_2 12} + \frac{1}{\log_8 12} + \frac{1}{\log_9 12}$

$= \log_{12} 2 + \log_{12} 8 + \log_{12} 9 = \log_{12} (2 \times 8 \times 9)$

$= \log_{12} 144 = \log_{12} (12)^2 = 2 \log_{12} 12 = 2$

2

(1) L.H.S. $= \log \frac{(15)^2 \times 7}{3 \times 175} = \log_3 3 = \log_3 \left(\sqrt{3}\right)^2$

$= 2 \log_3 \sqrt{3}$

(2) L.H.S. $= \log_2 \frac{3 \times 297}{11 \times 98} - 2 \log_2 = \frac{9}{7}$

$= \log_2 \frac{3 \times 297 \times 7^2}{11 \times 98 \times 9^2} = \log_2 \frac{1}{2} = -1$

(3) L.H.S. $= (\log 10 - \log 5)(\log 100 - \log 25)$

$= (\log \frac{10}{5})(\log \frac{100}{25}) = (\log 2)(\log 4)$

$= (\log 2)(2 \log 2) = 2 (\log 2)^2$

(4) L.H.S. $= \frac{\log 3^6 - \log 2^6}{\log 3^2 - \log 2^2}$

$= \frac{6 \log 3 - 6 \log 2}{2 \log 3 - 2 \log 2} = \frac{6 (\log 3 - \log 2)}{2 (\log 3 - \log 2)} = 3$

(5) L.H.S. $= \frac{(\log 5)^2 - 2 \log 5}{\log 5 - 2}$

$= \frac{(\log 5)(\log 5 - 2)}{\log 5 - 2} = \log 5$

, R.H.S. $= \log 10 - \log 2 = \log \frac{10}{2} = \log 5$

$\therefore$ L.H.S. = R.H.S.

(6) L.H.S. $= \frac{\log_2 \log_5 \log_3 3^5}{\log_7 \log_3 \log_2 2^9}$

$= \frac{\log_2 \log_5 5}{\log_7 \log_3 9} = \frac{\log_2 1}{\log_7 2} = 0$

3

(1) $x \log 5 = \log 13$

$\therefore x = \frac{\log 13}{\log 5} \simeq 1.59$

(2) $(x+2) \log 3 = \log 6$

$\therefore x = \frac{\log 6 - 2\log 3}{\log 3} \simeq -0.37$

(3) $(7 - 2x) \log 3 = \log 13.4$

$\therefore x = \frac{7\log 3 - \log 13.4}{2\log 3} \simeq 2.32$

(4) $(x+1) \log 7 = (x-2) \log 3$

$\therefore x(\log 7 - \log 3) = -2\log 3 - \log 7$

$\therefore x = \frac{-2\log 3 - \log 7}{\log 7 - \log 3} \simeq -4.89$

(5) $(2x - 3) \log 3 = (1 - x) \log 11$

$\therefore x(2\log 3 + \log 11) = 3\log 3 + \log 11$

$\therefore x = \frac{3\log 3 + \log 11}{2\log 3 + \log 11} \simeq 1.24$

(6) $x^{\frac{8}{5}} = 94.5$ $\quad \therefore \frac{8}{5} \log |x| = \log 94.5$

$\therefore \log |x| = \frac{\log 94.5}{\frac{8}{5}}$

$\therefore |x| = 10^{\frac{5\log 94.5}{8}} \simeq 17.17$

$\therefore x = \pm 17.17$

(7) $10^{2x} = \frac{5}{7}$ $\quad \therefore 2x \log 10 = \log \frac{5}{7}$

$\therefore 2x = \log \frac{5}{7}$ $\quad \therefore x = \frac{1}{2} \log \frac{5}{7} \simeq -0.07$

(8) $7^{x-1}(7^2+1) = 300$ $\quad \therefore 7^{x-1} = 6$

$\therefore (x-1) \log 7 = \log 6$

$\therefore x = \frac{\log 6 + \log 7}{\log 7} \simeq 1.92$

(9) By dividing on 3^{x+1} $\quad \therefore \left(\frac{2}{3}\right)^x = 2 \times 3$

$\therefore x \log \frac{2}{3} = \log 6$ $\quad \therefore x = \frac{\log 6}{\log \frac{2}{3}} \simeq -4.42$

(10) $\log 3 + (x - 11) \log 2 = \log 8 + (5x - 1) \log 6$

$\therefore x(\log 2 - 5\log 6)$

$= \log 8 - \log 3 + 11\log 2 - \log 6$

$\therefore x = \frac{\log 8 - \log 3 + 11\log 2 - \log 6}{\log 2 - 5\log 6} \simeq -0.82$

(11) $5^{2x} - 27 \times 5^x + 50 = 0$ $\quad \therefore (5^x - 25)(5^x - 2) = 0$

$\therefore 5^x = 5^2$ $\quad \therefore x = 2$

or $5^x = 2$ $\quad \therefore x \log 5 = \log 2$

$\therefore x = \frac{\log 2}{\log 5} \simeq 0.43$

$\therefore x = 2$ or $x = 0.43$

(12) $(3 - 5x) \log 8 - (4 + x) \log 7 = \log 2$

$\therefore x(-5\log 8 - \log 7) = \log 2 - 3\log 8 + 4\log 7$

$\therefore x = \frac{\log 2 - 3\log 8 + 4\log 7}{-5\log 8 - \log 7} \simeq -0.18$

(13) $\log_5(4^{x+1}) = x - 2$ $\quad \therefore 4^{x+1} = 5^{x-2}$

$\therefore (x+1) \log 4 = (x-2) \log 5$

$\therefore x \log 4 + \log 4 = x \log 5 - 2\log 5$

$\therefore x(\log 4 - \log 5) = -\log 4 - 2\log 5$

$\therefore x = \frac{\log 4 + 2\log 5}{\log 5 - \log 4} \simeq 20.64$

4

(1) $\log_3 15 = \log_3 5 + \log_3 3 = 1.465 + 1 = 2.465$

(2) $\log_3 135 = \log_3 27 + \log_3 5 = 3\log_3 3 + \log_3 5$

$= 3 + 1.465 = 4.465$

(3) $\log_3 \frac{5}{9} = \log_3 5 - \log_3 9 = \log_3 5 - 2\log_3 3$

$= 1.465 - 2 = -0.535$

5

(1) $\log 6 = \log 2 + \log 3 = x + y$

(2) $\log_{18} 12 = \frac{\log 12}{\log 18} = \frac{\log 4 + \log 3}{\log 2 + \log 9}$

$= \frac{2\log 2 + \log 3}{\log 2 + 2\log 3} = \frac{2x + y}{x + 2y}$

6

(1) (False) Correction : $\log_x x^x = x \ \forall \ x \in \mathbb{R}^+ - \{1\}$

(2) (False) Correction : $\log x^n = n \log x \ \forall \ x \in \mathbb{R}^+$

(3) (False) Correction : $\log_a xy = \log_a x + \log_a y$

(4) (False) Correction : $\log_a xy = \log_a x + \log_a y$

(5) (True)

(6) (True) (7) (True)

(8) (False) Correction : $\log_a x^4 = 4 \log_a |x|$

7

(1) $\log_3 (x+6) = \log_3 x^2 \quad \therefore x+6 = x^2$

$\therefore x^2 - x - 6 = 0 \quad \therefore (x+2)(x-3) = 0$

$\therefore x = -2$ (refused) or $x = 3$

$\therefore$ The S.S. $= \{3\}$

(2) $\log_4 x(x+6) = 3 \quad \therefore 4^3 = x^2 + 6x$

$\therefore x^2 + 6x - 64 = 0$

$\therefore x = -3 + \sqrt{73}$ or $x = -3 - \sqrt{73}$ (refused)

$\therefore$ The S.S. $= \{-3 + \sqrt{73}\}$

(3) $\log_3 x^3 = 3 \quad \therefore 3 \log_3 x = 3$

$\therefore \log_3 x = 1 \quad \therefore x = 3$

$\therefore$ The S.S. $= \{3\}$

(4) $\log_3 (x-1)(x+1) = \log_3 8$

$\therefore x^2 - 1 = 8 \quad \therefore x^2 = 9$

$\therefore x = 3$ or $x = -3$ (refused)

$\therefore$ S.S. $= \{3\}$

(5) $\log \frac{x+8}{x-1} = 1 \quad \therefore \frac{x+8}{x-1} = 10$

$\therefore 10x - 10 = x + 8 \quad \therefore 9x = 18 \quad \therefore x = 2$

$\therefore$ The S.S. $= \{2\}$

(6) $\log_4 x = \log_4 4 - \log_4 (x-3)$

$\therefore \log_4 x = \log_4 \frac{4}{x-3} \quad \therefore x = \frac{4}{x-3}$

$\therefore x^2 - 3x - 4 = 0 \quad \therefore (x+1)(x-4) = 0$

$\therefore x = -1$ (refused) or $x = 4$

$\therefore$ The S.S. $= \{4\}$

(7) $\log_5 2x^2 = \log_5 18 \quad \therefore 2x^2 = 18$

$\therefore x^2 = 9 \quad \therefore x = \pm 3$

$\therefore$ The S.S. $= \{3, -3\}$

(8) $\log_3 (7x^2 - 4) = \log_3 (x^2 \times 3)$

$\therefore 7x^2 - 4 = 3x^2 \quad \therefore 4x^2 = 4$

$\therefore x = 1$ or $x = -1$ (refused)

$\therefore$ The S.S. $= \{1\}$

(9) $\log_3 \frac{(x+2)^5}{(x-1)^5} = \log_3 2^5 \quad \therefore \frac{x+2}{x-1} = 2$

$\therefore x + 2 = 2x - 2 \quad \therefore x = 4$

$\therefore$ The S.S. $= \{4\}$

(10) $\log (8-x)(x-6) = 0 \quad \therefore 14x - x^2 - 48 = 1$

$\therefore x^2 - 14x + 49 = 0 \quad \therefore (x-7)^2 = 0$

$\therefore x = 7$

$\therefore$ The S.S. $= \{7\}$

(11) $\log (x+2)(x-2) = \log 10 - \log 2$

$\therefore \log (x^2 - 4) = \log \frac{10}{2} = \log 5$

$\therefore x^2 - 4 = 5 \quad \therefore x^2 = 9$

$\therefore x = 3$ or $x = -3$ (refused)

$\therefore$ The S.S. $= \{3\}$

(12) $\log 7 \times \log 3^6 = \log 7^2 \times \log x^3$

$\therefore 6 \log 7 \log 3 = 6 \log 7 \log x$

$\therefore \log 3 = \log x \quad \therefore x = 3$

$\therefore$ The solution set $= \{3\}$

(13) $\log_2 \frac{x^2 + 6x + 9}{x-1} = \log_5 5^4 = 4$

$\therefore \frac{x^2 + 6x + 9}{x-1} = 2^4 = 16$

$\therefore x^2 - 10x + 25 = 0 \quad \therefore (x-5)^2 =$ zero

$\therefore x = 5$

$\therefore$ The solution set $= \{5\}$

(14) $\log x = \frac{(\log 3)^2 - 3 \log 3}{\log 3 - \log 1000} = \frac{\log 3 [\log 3 - 3]}{\log 3 - 3} = \log 3$

$\therefore x = 3 \quad \therefore$ The S.S. $= \{3\}$

(15) $(\log x)^2 - \log x^2 - 3 = 0$

$\therefore (\log x)^2 - 2 \log x - 3 = 0$

$\therefore (\log x + 1)(\log x - 3) = 0$

$\therefore \log x = -1$ or $\log x = 3$

$\therefore x = 0.1$ or $x = 1000$

$\therefore$ The S.S. $= \{0.1, 1000\}$

(16) $(\log x)^2 + 2 \log x + 1 = (\log 5)^2$

$\therefore (\log x + 1)^2 = (\log 5)^2$

$\therefore \log x + 1 = \pm \log 5$

$\therefore \log x = \log 5 - 1 = \log 5 - \log 10 = \log \frac{1}{2}$

$\therefore x = \frac{1}{2}$

or $\log x = -\log 5 - 1 = -\log 5 - \log 10$

$= -\log 50 = \log (50)^{-1}$

$\therefore x = \frac{1}{50}$

$\therefore$ The S.S. $= \{\frac{1}{2}, \frac{1}{50}\}$

(17) $\log_x \frac{4 \times 14}{27} = 3 \log_x x + \log_x 7 = \log_x (x^3 \times 7)$

$\therefore 7x^3 = \frac{56}{27}$ $\qquad \therefore x^3 = \frac{8}{27}$

$\therefore x = \frac{2}{3}$ $\qquad \therefore$ The S.S. $= \{\frac{2}{3}\}$

(18) $\log (\sqrt[3]{3x-1} \times \sqrt[3]{x-2}) = \log 20 - \log 10$

$\therefore \log \sqrt[3]{(3x-1)(x-2)} = \log 2$

$\therefore \sqrt[3]{(3x-1)(x-2)} = 2$

$\therefore (3x-1)(x-2) = 2^3$

$\therefore 3x^2 - 7x - 6 = 0$

$\therefore (3x+2)(x-3) = 0$

$\therefore x = \frac{-2}{3}$ (refused) or $x = 3$

$\therefore$ The S.S. $= \{3\}$

8

(1) $\frac{\log x}{\log 2} = \frac{\log 9}{\log 4}$ $\qquad \therefore \frac{\log x}{\log 2} = \frac{2 \log 3}{2 \log 2}$

$\therefore \log x = \log 3$ $\qquad \therefore x = 3$

$\therefore$ The S.S. $= \{3\}$

(2) Taking logarithms of the two sides

$\therefore \log 3^{\log x} = \log 2^{\log 3}$

$\therefore \log x \log 3 = \log 3 \log 2$

$\therefore \log x = \log 2$

$\therefore x = 2$

$\therefore$ The S.S. $= \{2\}$

(3) Taking logarithms of the two sides

$\therefore \log x^{\log x} = \log 10$ $\quad \therefore \log x \log x = 1$

$\therefore (\log x)^2 = 1$

$\therefore \log x = 1$ or $\log x = -1$

$\therefore x = 10$ or $x = 0.1$

$\therefore$ The S.S. $= \{10, 0.1\}$

(4) $\frac{\log x}{\log 3} = \frac{\log 3}{\log x}$ $\qquad \therefore (\log x)^2 = (\log 3)^2$

$\therefore \log x = \log 3$ then $x = 3$

or $\log x = -\log 3 = \log 3^{-1}$ then $x = \frac{1}{3}$

$\therefore$ The S.S. $= \{3, \frac{1}{3}\}$

(5) $x^{2 \log_x x} = 9 \log 10$ $\qquad \therefore x^2 = 9$

$\therefore x = 3$ or $x = -3$ (refused)

$\therefore$ The S.S. $= \{3\}$

(6) $4^{\log x} \times 5^{\log x} = 8000$ $\qquad \therefore 20^{\log x} = 8000$

$\therefore 20^{\log x} = 20^3$ $\qquad \therefore \log x = 3$

$\therefore x = 1000$ $\qquad \therefore$ The S.S. $= \{1000\}$

(7) $2^{(\log x)^2} \times 2^6 = 2^{\log x^5}$

$\therefore (\log x)^2 + 6 = 5 \log x$

$\therefore (\log x)^2 - 5 \log x + 6 = 0$

$\therefore (\log x - 2)(\log x - 3) = 0$

$\therefore \log x = 2$, then $x = 100$

or $\log x = 3$, then $x = 1000$

$\therefore$ The S.S. $= \{100, 1000\}$

(8) Multiplying by $2 \log x$

$\therefore 2 (\log x)^2 - 3 \log x - 2 = 0$

$\therefore (2 \log x + 1)(\log x - 2) = 0$ $\qquad \therefore \log x = \frac{-1}{2}$

$\therefore x = \frac{1}{\sqrt{10}}$ or $\log x = 2$ $\qquad \therefore x = 100$

$\therefore$ The S.S. $= \{\frac{1}{\sqrt{10}}, 100\}$

(9) Let $\log_2 x = k$ $\qquad \therefore \log_x 2 = \frac{1}{k}$

$\therefore k + \frac{1}{k} = 2$ $\qquad \therefore k^2 - 2k + 1 = 0$

$\therefore (k-1)^2 = 0$ $\qquad \therefore k = 1$

$\therefore \log_2 x = 1$ $\qquad \therefore x = 2$

$\therefore$ The S.S. $= \{2\}$

(10) Let $\log x = k$ $\qquad \therefore \log_x 10 = \frac{1}{k}$

$\therefore \log x - 2 \log_x 10 = 1$

$\therefore k - \frac{2}{k} = 1$ $\qquad \therefore k^2 - k - 2 = 0$

$\therefore (k+1)(k-2) = 0$ $\qquad \therefore k = -1$ or $k = 2$

$\therefore \log x = -1$ or $\log x = 2$

$\therefore x = 10^{-1} = \frac{1}{10}$ or $x = 10^2 = 100$

$\therefore$ The S.S. $= \{\frac{1}{10}, 100\}$

(11) $\log \frac{x}{2} \times \log \left(\frac{x}{2}\right)^{-1} = -1$

$\therefore -\left(\log \frac{x}{2}\right)^2 = -1 \qquad \therefore \left(\log \frac{x}{2}\right)^2 = 1$

$\therefore \log \frac{x}{2} = 1 \qquad \therefore \frac{x}{2} = 10 \qquad \therefore x = 20$

or $\log \frac{x}{2} = -1 \qquad \therefore \frac{x}{2} = \frac{1}{10} \qquad \therefore x = 0.2$

$\therefore$ The S.S. $= \{20 , 0.2\}$

(12) $(\log x)^3 = 9 \log x$

$\therefore (\log x)^3 - 9 \log x = 0$

$\therefore \log x \left((\log x)^2 - 9\right) = 0$

$\therefore \log x (\log x - 3) (\log x + 3) = 0$

$\therefore \log x = 0 \qquad \therefore x = 1$

or $\log x = 3 \qquad \therefore x = 1000$

or $\log x = -3 \qquad \therefore x = 0.001$

$\therefore$ The S.S. $= \{1 , 1000 , 0.001\}$

(13) Taking logarithms of two sides

$\therefore \log x \log x = \log (100\, x)$

$\therefore (\log x)^2 = \log 100 + \log x$

$\therefore (\log x)^2 - \log x - 2 = 0$

$\therefore (\log x - 2) (\log x + 1) = 0$

$\therefore \log x = 2$ or $\log x = -1$

$\therefore x = 100$ or $x = \frac{1}{10}$

$\therefore$ S.S. $= \left\{100 , \frac{1}{10}\right\}$

(14) $(\log x + 2) (\log x - \log 100) = 5$

$\therefore (\log x + 2) (\log x - 2) = 5$

$\therefore (\log x)^2 - 4 = 5 \qquad \therefore (\log x)^2 = 9$

$\therefore \log x = 3 \qquad \therefore x = 1000$

or $\log x = -3 \qquad \therefore x = 0.001$

$\therefore$ The S.S. $= \{1000 , 0.001\}$

(15) $\frac{\log x}{\log 3} + \frac{\log x^2}{\log 9} + 3 = 0$

$\therefore \frac{\log x}{\log 3} + \frac{2 \log x}{2 \log 3} = -3$

$\therefore \frac{2 \log x}{\log 3} = -3 \qquad \therefore \log x = \frac{-3}{2} \log 3$

$\therefore \log x = \log 3^{-\frac{3}{2}} = \log \frac{1}{3\sqrt{3}}$

$\therefore x = \frac{1}{3\sqrt{3}} \qquad \therefore$ The S.S. $= \left\{\frac{1}{3\sqrt{3}}\right\}$

(16) $\frac{\log x}{\log 2} = \frac{\log (x + 6)}{\log 4} \qquad \therefore \frac{\log x}{\log 2} = \frac{\log (x + 6)}{2 \log 2}$

$\therefore 2 \log x = \log (x + 6)$

$\therefore x^2 = x + 6 \qquad \therefore x^2 - x - 6 = 0$

$\therefore (x - 3) (x + 2) = 0$

$\therefore x = 3$ or $x = -2$ (refused)

$\therefore$ The S.S. $= \{3\}$

(17) $\frac{\log x}{\log 2} + \frac{\log x}{\log 4} = \frac{-3}{2} \qquad \therefore \frac{\log x}{\log 2} + \frac{\log x}{2 \log 2} = \frac{-3}{2}$

$\therefore \frac{3 \log x}{2 \log 2} = \frac{-3}{2} \qquad \therefore \frac{\log x}{\log 2} = -1$

$\therefore \log x = \log 2^{-1} \qquad \therefore x = \frac{1}{2}$

$\therefore$ The S.S. $= \left\{\frac{1}{2}\right\}$

(18) $\because \sqrt{\log_2 x} = \log_2 \sqrt{x}$ by squaring

$\therefore \log_2 x = \log_2 \sqrt{x} \times \log_2 \sqrt{x}$

$= \log_2 x^{\frac{1}{2}} \times \log_2 x^{\frac{1}{2}}$

$\therefore \log_2 x = \frac{1}{4} (\log_2 x)^2$

$\therefore (\log_2 x)^2 - 4 \log_2 x = 0$

$\therefore \log_2 x (\log_2 x - 4) = 0$

$\therefore \log_2 x = 0$ or $\log_2 x = 4$

$\therefore x = 2^0 = 1$ or $x = 2^4 = 16$

$\therefore$ The S.S. $= \{1 , 16\}$

(19) By squaring $x^{\log \sqrt{x}} = 100$ taking the logarithms of the two sides

$\therefore \log \sqrt{x} \log x = 2$

$\therefore \frac{1}{2} (\log x)^2 = 2 \qquad \therefore (\log x)^2 = 4$

$\therefore \log x = 2$ or $\log x = -2$

$\therefore x = 100$ or $x = \frac{1}{100}$

$\therefore$ The S.S. $= \left\{100 , \frac{1}{100}\right\}$

(20) $\sqrt{\log \left(91 + 3^{\sqrt{\frac{x}{2}}}\right)} = \sqrt{2}$

$\therefore \log \left(91 + 3^{\sqrt{\frac{x}{2}}}\right) = 2$

$\therefore 91 + 3^{\sqrt{\frac{x}{2}}} = 10^2 = 100$

$\therefore 3^{\sqrt{\frac{x}{2}}} = 9 = 3^2 \qquad \therefore \sqrt{\frac{x}{2}} = 2$

$\therefore \frac{x}{2} = 4 \qquad \therefore x = 8$

$\therefore$ The S.S. $= \{8\}$

(21) $\log 1 - \log (2^x + x - 1) = x (\log 5 - \log 10)$

$\therefore -\log (2^x + x - 1) = x \log \frac{1}{2}$

$\therefore -\log (2^x + x - 1) = -x \log 2$

$\therefore \log (2^x + x - 1) = \log 2^x$

$\therefore 2^x + x - 1 = 2^x \qquad \therefore x - 1 = 0$

$\therefore x = 1 \qquad \therefore$ The S.S. $= \{1\}$

9

(1) From the figure geometry :

$\frac{\log x}{\log 6}=\frac{\log 49}{\log 7}$ $\quad \therefore \frac{\log x}{\log 6}=\frac{2\log 7}{\log 7}$

$\therefore 2\log 6=\log x \quad \therefore x=6^2=36$

(2) $MD=\log_{81}243=\log_{81}(81)^{\frac{5}{4}}=\frac{5}{4}\log_{81}81=\frac{5}{4}$

$, \because MD=\frac{1}{3}AD \quad \therefore AD=\frac{15}{4}\ , AD=\frac{1}{2}BC$

$\therefore BC=2\times\frac{15}{4}=7.5$ cm.

(3) $m(\angle A)=90^\circ\ , \overline{AD}\perp\overline{BC}$

$\therefore (AC)^2=CD\times CB$

$\therefore \left(5\sqrt{3}\right)^2=\log_x 27\times(\log_x 27+\log_x 3)$

$\therefore 75=\log_x 27\times\log_x 81$

$\therefore 75=3\log_x 3\times 4\log_x 3$

$\therefore 75=12(\log_x 3)^2$

$\therefore (\log_x 3)^2=\frac{75}{12}=\frac{25}{4}$

$\therefore \log_x 3=\frac{5}{2}$, then $x^{\frac{5}{2}}=3$

$\therefore x=3^{\frac{2}{5}}=1.6$

or $\log_x 3=\frac{-5}{2}$

$\therefore x^{-\frac{5}{2}}=3$

$\therefore x=\left(\frac{1}{3}\right)^{\frac{2}{5}}=0.6$

(4) $\because \overrightarrow{AD}$ bisects $\angle A$ $\quad \therefore \frac{CA}{AB}=\frac{CD}{DB}$

$\therefore \frac{\log_2 5}{\log_4 x}=\frac{\log b}{\log b^2}$ $\quad \therefore \frac{\log_2 5}{\log_4 x}=\frac{\log b}{2\log b}$

$\therefore 2\log_2 5=\log_4 x$ $\quad \therefore \log_2 25=\log_4 x$

$\therefore \frac{\log 25}{\log 2}=\frac{\log x}{\log 4}$ $\quad \therefore \frac{\log 25}{\log 2}=\frac{\log x}{2\log 2}$

$\therefore \log x=2\log 25=\log 625$

$\therefore x=625$

10

(1) $\because \log y^3=3-\log 125$

$\therefore \log y^3=\log 1000-\log 125=\log\frac{1000}{125}=\log 8$

$\therefore y^3=8 \quad \therefore y=2$

$, \because \log xy=1-\log 5$

By substituting by $y=2$

$\therefore \log 2x=1-\log 5=\log 10-\log 5=\log 2$

$\therefore 2x=2 \quad \therefore x=1$

$\therefore S.S.=\{(1\ ,2)\}$

(2) $\because \log_2(xy)+2=2+\log_2 9$

$\therefore \log_2(xy)=\log_2 9 \quad \therefore xy=9$

$, \because x=10-y \quad \therefore (10-y)y=9$

$\therefore y^2-10y+9=0 \quad \therefore (y-9)(y-1)=0$

$\therefore y=9$, then $x=1$

or $y=1$, then $x=9$

$\therefore S.S.=\{(1\ ,9)\ ,(9\ ,1)\}$

(3) $\therefore 3\log x+2\log y=11$ (1)

$, 2\log x-3\log y=3$ (2)

Multiply (1) by 3 and (2) by 2 , then adding

$\therefore 13\log x=39 \quad \therefore \log x=3$

$\therefore x=1000$, from (1)

$\therefore 3\log 1000+2\log y=11$

$\therefore 9+2\log y=11 \quad \therefore \log y=1$

$\therefore y=10$

$\therefore S.S.=\{(1000\ ,10)\}$

11

(1) L.H.S. $=\log_3\frac{x^3\cdot y^4}{x^3 y^2}$

$=\log_3 x^2y^2=\log_3(xy)^2$

$=2\log_3 xy=2\log_3 9\sqrt{3}$

$=2\log_3 3^{\frac{5}{2}}=5$

(2) $\because \frac{1}{3}\log a=\frac{1}{5}\log b=\log c$

$\therefore c=a^{\frac{1}{3}} \quad \therefore a=c^3$

$, c=b^{\frac{1}{5}} \quad \therefore b=c^5$

$\therefore ab=c^3\times c^5=c^8$

(3) $\because \log\frac{x+y}{3}=\frac{1}{2}\log xy=\log(xy)^{\frac{1}{2}}$

$\therefore \frac{x+y}{3}=(xy)^{\frac{1}{2}}$ (Squaring the two sides)

$\therefore x^2+y^2+2xy=9xy$

$\therefore x^2+y^2=7xy$ (Dividing by xy)

$\therefore \frac{x}{y}+\frac{y}{x}=7$

(4) $\because 3^{\log_a x}=a$ taking the logarithms of the two sides to the base a

$\therefore \log_a x\log_a 3=1$

$\therefore \log_a x$ is the multiplicative inverse of the number $\log_a 3$, when $a=9$

$\therefore \log_9 x \log_9 3 = 1 \quad \therefore \log_9 x \log_9 9^{\frac{1}{2}} = 1$

$\therefore \frac{1}{2} \log_9 x = 1 \quad \therefore \log_9 x = 2$

(5) $\log_{a^2} b^2 = \frac{\log b^2}{\log a^2} = \frac{2 \log b}{2 \log a} = \frac{\log b}{\log a} = \log_a b$

$\therefore \log_a b^2 + \log_{a^2} b^4 = \log_a b^2 + \log_a b^2 = \log_a b^4$

$= 4 \log_a b$

(6) $\log \frac{x^3 \times y^4}{x y^2} = 2 \log 6 \quad \therefore \log x^2 y^2 = \log 36$

$\therefore x^2 y^2 = 36 \quad \therefore x y = 6$

$\therefore x = \frac{6}{y}$

(7) $2 \log (x + y) = \log (x + y)^2$

$= \log (x^2 + 2 x y + y^2)$

$= \log (10 x y)$

$= \log 10 + \log x + \log y$

$= 1 + \log x + \log y$

(8) $\log (x + y) = \frac{1}{2} \log (x y) + \log 2$

$\therefore \log (x + y) - \log 2 = \frac{1}{2} \log (x y)$

$\therefore \log \frac{x + y}{2} = \log (x y)^{\frac{1}{2}}$

$\therefore \frac{x + y}{2} = x^{\frac{1}{2}} y^{\frac{1}{2}}$ «By squaring»

$\therefore x^2 + y^2 + 2 x y = 4 x y$

$\therefore x^2 - 2 x y + y^2 = 0 \quad \therefore (x - y)^2 = 0$

$\therefore x - y = 0 \quad \therefore x = y$

(9) $\because \log x y^3 = 1 \quad \therefore x y^3 = 10$

$\therefore x = \frac{10}{y^3}$ (1)

$, \because \log x^2 y = 1 \quad \therefore x^2 y = 10$

, from (1) : $\therefore \frac{100}{y^6} \times y = 10$

$\therefore \frac{10}{y^5} = 1 \quad \therefore y^5 = 10$

$\therefore y = 10^{\frac{1}{5}}$

, from (1) : $x = 10^{\frac{2}{5}}$

$\therefore \log x y = \log (10^{\frac{1}{5}} \times 10^{\frac{2}{5}})$

$= \log 10^{\frac{3}{5}} = \frac{3}{5} \log 10 = \frac{3}{5}$

(10) $\frac{1}{\log_x y} = \frac{1}{\frac{\log y}{\log x}} = \frac{\log x}{\log y} = \log_y x$

$, \because \log_3 x + 3 \log_x 3 = 4$

Let $\log_3 x = k \quad \therefore k + \frac{3}{k} = 4$

$\therefore k^2 + 3 = 4 k \quad \therefore k^2 - 4 k + 3 = 0$

$\therefore (k - 1)(k - 3) = 0 \quad \therefore k = 1$ or $k = 3$

$\therefore \log_3 x = 1$ or $\log_3 x = 3$

$\therefore x = 3$ or $x = 27$

$\therefore$ S.S. $= \{3 , 27\}$

(11) $\because \log_x y = \frac{1}{\log_y x}$

$, \because \frac{4}{\log_x y} + \frac{7}{\log_z y} = 3 \log_y z$

$\therefore 4 \log_y x + 7 \log_y z = 3 \log_y z$

$\therefore x^4 \times z^7 = z^3 \quad \therefore x^4 \times z^4 = 1$

$\therefore x z = 1$

(12) $\because y = a^{\log_a x}$

Taking the logarithms of base a to two sides.

$\therefore \log_a y = \log_a x \log_a a$

$\therefore \log_a y = \log_a x \quad \therefore y = x$

$\therefore 3^{\frac{1}{2} \log_3 49} = 3^{\log_3 \sqrt{49}} = 3^{\log_3 7} = 7$

(13) Let $\log_{a^y} a^x = \frac{\log a^x}{\log a^y} = \frac{x \log a}{y \log a} = \frac{x}{y}$

$\therefore \log_{\sqrt[3]{5}} 125 = \log_{(5)^{\frac{1}{3}}} (5)^3 = \frac{3}{\frac{1}{3}} = 9$

$, \log_8 \sqrt{2} = \log_{2^3} (2)^{\frac{1}{2}} = \frac{\frac{1}{2}}{3} = \frac{1}{6}$

$\therefore \log_{\sqrt[3]{5}} 125 \times \log_8 \sqrt{2} = 9 \times \frac{1}{6} = \frac{3}{2}$

(14) Let $\log_b a = x \quad \therefore b^x = a$

$\therefore x \log 6 = \log a \quad \therefore x = \frac{\log a}{\log b}$

$\therefore \log_b a = \frac{\log a}{\log b}$

$\therefore$ L.H.S. $= \frac{1}{\log_a a + \log_a b} + \frac{1}{\log_b a + \log_b b}$

$= \frac{1}{1 + \frac{\log b}{\log a}} + \frac{1}{1 + \frac{\log a}{\log b}}$

$= \frac{\log a}{\log a + \log b} + \frac{\log b}{\log a + \log b}$

$= \frac{\log a + \log b}{\log a + \log b} = 1$

(15) $\log_c \frac{a \times b \times 4}{(a + b)^2} = 0 \quad \therefore \frac{4 a b}{(a + b)^2} = 1$

$\therefore (a + b)^2 = 4 a b$

$\therefore a^2 + 2ab + b^2 = 4ab \quad \therefore a^2 - 2ab + b^2 = 0$

$\therefore (a-b)^2 = 0 \quad \therefore a - b = 0$

(16) $x^{2\log x} \times x^3 = 10^5$

$\therefore \log (x^{2\log x} \times x^3) = 5 \log 10$

$\therefore \log x^{2\log x} + \log x^3 = 5$

$\therefore 2 \log x \log x + 3 \log x = 5$

$\therefore 2 (\log x)^2 + 3 \log x - 5 = 0$

$\therefore (2 \log x + 5)(\log x - 1) = 0$

$\therefore \log x = \frac{-5}{2} \quad \therefore x = 10^{\frac{-5}{2}} = \frac{1}{100\sqrt{10}}$

or $\log x = 1 \quad \therefore x = 10$

(17) $y - \frac{1}{y} = 1 \quad \therefore y^2 - y - 1 = 0$

$\therefore y = \frac{1 \pm \sqrt{1 - 4 \times 1 \times (-1)}}{2 \times 1} = \frac{1 \pm \sqrt{5}}{2}$

$\therefore a^x = \frac{1 \pm \sqrt{5}}{2} \quad \therefore x \log_a a = \log_a \frac{1+\sqrt{5}}{2}$

$\therefore x = \log_a \frac{1+\sqrt{5}}{2}$

(The negative solution is refused)

(18) $f(x+1) + f(x-1) = f(3)$

$\therefore \log (x+1)^2 + \log (x-1)^2 = \log 3^2$

$\therefore \log (x+1)^2 (x-1)^2 = \log 9$

$\therefore (x^2 - 1)^2 = 9$

$\therefore x^2 - 1 = 3$ or $x^2 - 1 = -3$ (refused)

$\therefore x^2 = 4 \quad \therefore x = \pm 2$

$\therefore$ S.S. $= \{2, -2\}$

(19) $\because f(x) = \log (\sqrt{x^2+1} - x)$

, $f(-x) = \log (\sqrt{x^2+1} + x)$

$= \log (\sqrt{x^2+1} + x \times \frac{\sqrt{x^2+1} - x}{\sqrt{x^2+1} - x})$

$= \log (\frac{x^2 + 1 - x^2}{\sqrt{x^2+1} - x})$

$= \log (\frac{1}{\sqrt{x^2+1} - x}) = \log (\sqrt{x^2+1} - x)^{-1}$

$= -\log (\sqrt{x^2+1} - x) = -f(x)$

$\therefore$ The function is odd.

(20) $\because \log 4^{47} \simeq 28.3 \quad \therefore 28 < \log 4^{47} < 29$

$\therefore \log 10^{28} < \log 4^{47} < \log 10^{29}$

$\therefore 10^{28} < 4^{47} < 10^{29}$

$\therefore$ The number of digits of the number 4^{47} is 29 digits.

Third Higher skills

1

(1) (a)	(2) (d)	(3) (c)	(4) (d)
(5) (b)	(6) (c)	(7) (c)	(8) (c)
(9) (a)	(10) (d)	(11) (d)	(12) (d)
(13) (c)	(14) (c)	(15) (c)	

Instructions to solve 1 :

(1) $\because \log x = z + \log y$

$\therefore \log x - \log y = z \quad \therefore \log \frac{x}{y} = z$

$\therefore \frac{x}{y} = (10)^z \quad \therefore x = y \times (10)^z$

(2) The expression $= \frac{1}{\log_b b + \log_b a + \log_b c}$

$+ \frac{1}{\log_c c + \log_c a + \log_c b}$

$+ \frac{1}{\log_a a + \log_a b + \log_a c}$

$= \frac{1}{\log_b abc} + \frac{1}{\log_c abc} + \frac{1}{\log_a abc}$

$= \log_{abc} b + \log_{abc} c + \log_{abc} a$

$= \log_{abc} abc = 1$

(3) $\because (25)^x - 12 \times (5)^x + 27 = 0$

$\therefore (5^x - 9)(5^x - 3) = 0$

$\therefore 5^x - 9 = 0$ and so $5^x = 9$

$\therefore x = \log_5 9$

or $5^x - 3 = 0$ and so $5^x = 3$

$\therefore x = \log_5 3$

$\therefore$ The sum of the two roots $= \log_5 9 + \log_5 3$

$= \log_5 27$

(4) Put $3^{\log x} = y$ (Take log to both sides)

$\therefore \log x \log 3 = \log y$

i.e. $\log 3 \log x = \log y$

$\therefore \log x^{\log 3} = \log y \quad \therefore x^{\log 3} = y$

i.e. $3^{\log x} = x^{\log 3}$

(5) $\because \log_c \log_b \log_a a^{b^c} = \log_c \log_b (b^c \log_a a)$

$= \log_c \log_b b^c$

$= \log_c (c \log_b b) = \log_c c = 1$

(6) $\because b^x - 2b^{-x} = 1 \quad \therefore b^{2x} - b^x - 2 = 0$

$\therefore (b^x + 1)(b^x - 2) = 0$

$\therefore b^x = -1$ (Refused) or $b^x = 2$ and so $x = \log_b 2$

(7) $\because \frac{1}{\log_2 x} + \frac{1}{\log_4 x} + \frac{1}{\log_8 x} + \frac{1}{\log_{16} x} = 5$

$\therefore \log_x 2 + \log_x 4 + \log_x 8 + \log_x 16 = 5$

$\therefore \log_x (2 \times 4 \times 8 \times 16) = 5$

$\therefore \log_x 1024 = 5 \qquad \therefore x^5 = 1024 = 4^5$

$\therefore x = 4$

(8) $\because \log 1 = \text{zero}$, $\log 10 = 1$

, $\because \log a \in]0 , 1[\qquad \therefore a \in]1 , 10[$

(9) Draw the curve of the function $f(a) = \log_3 a$ you can find that if $a \in]0 , 9]$

, then $\log_3 a \in]-\infty , 2]$

(10) $\because 2^{\log_2 (x+4)} + 3^{\log_3 (x+5)} = 25^{\log_5 7}$

$\therefore (x+4) + (x+5) = 5^{2\log_5 7} = 5^{\log_5 49} = 49$

$\therefore 2x + 9 = 49 \qquad \therefore 2x = 40$

$\therefore x = 20 \qquad \therefore \text{S.S.} = \{20\}$

(11) $\because \log_y x = \log_x y \qquad \therefore \log_y x = \frac{1}{\log_y x}$

$\therefore (\log_y x)^2 = 1 \qquad \therefore \log_y x = \pm 1$

$\therefore x = y$ or $x = y^{-1} = \frac{1}{y}$

$\therefore$ The two answers (a) and (b) are true

$\therefore$ (d) is the answer.

(12) $\because 3^a = 5^b \qquad \therefore a \log 3 = b \log 5$

$\therefore \frac{a}{b} = \frac{\log 5}{\log 3} = \log_3 5 \qquad , \because 3^a = 7^c$

$\therefore a \log 3 = c \log 7 \qquad \therefore \frac{a}{c} = \frac{\log 7}{\log 3} = \log_3 7$

$\therefore \frac{a}{b} + \frac{a}{c} = \log_3 5 + \log_3 7 = \log_3 35$

(13) $\because \log_2 x + \log_4 x + \log_8 x = 11$

$\therefore \frac{\log x}{\log 2} + \frac{\log x}{\log 4} + \frac{\log x}{\log 8} = 11$

$\therefore \frac{\log x}{\log 2} + \frac{\log x}{2 \log 2} + \frac{\log x}{3 \log 2} = 11$

$\therefore \frac{6 \log x + 3 \log x + 2 \log x}{6 \log 2} = 11$

$\therefore 11 \log x = 11 \times 6 \log 2$

$\therefore \log x = 6 \log 2 = \log 2^6 = \log 64$

$\therefore x = 64$

(14) $\because \log_2 3 \times \log_3 4 \times \log_4 5 \times ... \times \log_n (n+1) = 10$

$\therefore \frac{\log 3}{\log 2} \times \frac{\log 4}{\log 3} \times \frac{\log 5}{\log 4} \times ... \times \frac{\log (n+1)}{\log n} = 10$

$\therefore \log (n+1) = 10 \log 2 \therefore \log (n+1) = \log 2^{10}$

$\therefore n + 1 = 1024 \qquad \therefore n = 1023$

(15) $\because f(x) = \log 2^x \qquad \therefore f(x) = x \log 2$

$\therefore f(x)$ is a linear function represented by a straight line whose slope log 2

$\therefore$ The answer is (c)

2

(1) The expression

$= \log_3 \left(\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times ..... \times \frac{80}{81}\right)$

$\therefore \log_3 \left(\frac{1}{81}\right) = \log_3 3^{-4} = -4$

(2) The expression

$= \log \tan 1° + \log \tan 2° + \cdots + \log \tan 45° + \cdots + \log \tan 88° + \log \tan 89°$

$= \log (\tan 1° \tan 2° ... \tan 45° ... \tan 88° \tan 89°)$

$= \log (\tan 1° \tan 2° ... \tan 45° ... \cot 2° \cot 1°)$

$= \log \tan 45° = \log 1 = 0$

(3) The expression

$= \log \tan 1° \times \log \tan 2° \times \cdots \times \log \tan 45° \times \cdots \times \log \tan 73°$

$= \log \tan 1° \times \log \tan 2° \times \cdots \times \log 1 \times \cdots \times \log \tan 73° = 0$

3

$\because \log_{b^2} x^2 = \frac{\log x^2}{\log b^2} = \frac{2 \log x}{2 \log b} = \frac{\log x}{\log b} = \log_b x$

$, \because \log_{b^3} x^3 = \frac{\log x^3}{\log b^3} = \frac{3 \log x}{3 \log b} = \frac{\log x}{\log b} = \log_b x$

and so on , ...

$\log_{b^n} x^n = \frac{\log x^n}{\log b^n} = \frac{n \log x}{n \log b} = \frac{\log x}{\log b} = \log_b x$

$\therefore \log_b x + \log_{b^2} x^2 + \log_{b^3} x^3 + ... + \log_{b^n} x^n$

$= \log_b x + \log_b x + \log_b x + ... + \log_b x$

$= n \log_b x = \log_b x^n$

Answers of Life Applications on Unit Two

1

The increase in the length of the radius

$= \left(\frac{3 \times 36 \times \pi}{4\pi}\right)^{\frac{1}{3}} - \left(\frac{3 \times \frac{32}{3} \times \pi}{4\pi}\right)^{\frac{1}{3}}$

$= (27)^{\frac{1}{3}} - (8)^{\frac{1}{3}} = 1$ length unit.

2

(1) S_{10} (The sum of the first 10 numbers)

$= 2(2^{10} - 1) = 2^{11} - 2 = 2046$

(2) $S_n = 131070$ $\therefore 2(2^n - 1) = 131070$

$\therefore 2^n - 1 = 65535$ $\therefore 2^n = 65536$

$\therefore 2^n = 2^{16}$ $\therefore n = 16$

$\therefore$ Number of terms = 16 terms.

3

(1) $f(0) = 70 - 4 \log_2 1 = 70$ marks.

(2) $f(7) = 70 - 4 \log_2 8 = 58$ marks.

4

(1) $f(0) = 85 - 25 \log 1 = 85\%$

(2) $f(3) = 85 - 25 \log 4 \simeq 69.95\%$

(3) $f(12) = 85 - 25 \log 13 \simeq 57.15\%$

5

(1) $f(3600) = \frac{10}{100} \times 3600 = 360$ pounds.

(2) $f(8000) = \frac{10}{100} \times 8000 + 100 \log (8000 - 4999)$

$= 1147.7266$ pounds.

6

(1) $(PH) = -\log 10^{-3} = 3 \log 10 = 3$

(2) When the (PH) = 9

$\therefore 9 = -\log (H^+)$ $\therefore \log (H^+) = -9$

$\therefore H^+ = 10^{-9}$

i.e. The concentration of the hydrogen = 10^{-9}

7

(1) a is the initial number of the population , n is the number of years.

$\therefore$ The number of the population after n years

$= a\left(1 + \frac{7}{100}\right)^n = a(1.07)^n$

$\therefore$ The number of the population

After 1 year = a (1.07)

(2) When the population is doubled

$\therefore 2a = a(1.07)^n$ $\therefore 2 = (1.07)^n$

Taking the logarithms to the both sides.

$\therefore n \log 1.07 = \log 2$

$\therefore n \simeq 10$ years.

8

$N = 10^5 (1.3)^{t-2010}$

(1) In 2015 :

The number of the population $N = 10^5 (1.3)^{2015-2010}$

$= 371293$ people.

(2) When the number of the population 1.4 million people

$\therefore 1.4 \times 10^6 = 10^5 (1.3)^{t-2010}$

$\therefore (1.3)^{t-2010} = 14$

Taking the logarithms to the both sides.

$\therefore (t - 2010) \log 1.3 = \log 14$

$\therefore t - 2010 \simeq 10$ $\therefore t \simeq 2020$

9

(1) $M = \log \frac{I}{I_0}$

$\therefore M = \log \frac{1.5 \times 10^6 I_0}{I_0} = \log (1.5 \times 10^6) \simeq 6.2$

i.e. The earthquake scale on richter scale = 6.2

(2) When the magnitude of the earthquake (M) = 8

$\therefore 8 = \log \frac{I}{I_0}$ $\therefore \frac{I}{I_0} = 10^8$

$\therefore I = 10^8 I_0$

i.e. The earthquake intensity = 10^8 times its reference intensity.

10

$k = k_o (0.9)^n$ $\therefore \frac{40}{100} k_o = k_o (0 - 9)^n$

$\therefore 0.4 = (0.9)^n$

Taking the logarithms to the both sides.

$\therefore n \log 0.9 = \log 0.4$ $\therefore n = \frac{\log 0.4}{\log 0.9} \simeq 9$ years.

11

Let the primary efficiency of the machine A_o

$\therefore A = A_o (1 - \frac{5}{100})^n$

When the machine stops working

$\therefore \frac{60}{100} A_o = A_o (0.95)^n$

$\therefore 0.6 = (0.95)^n$

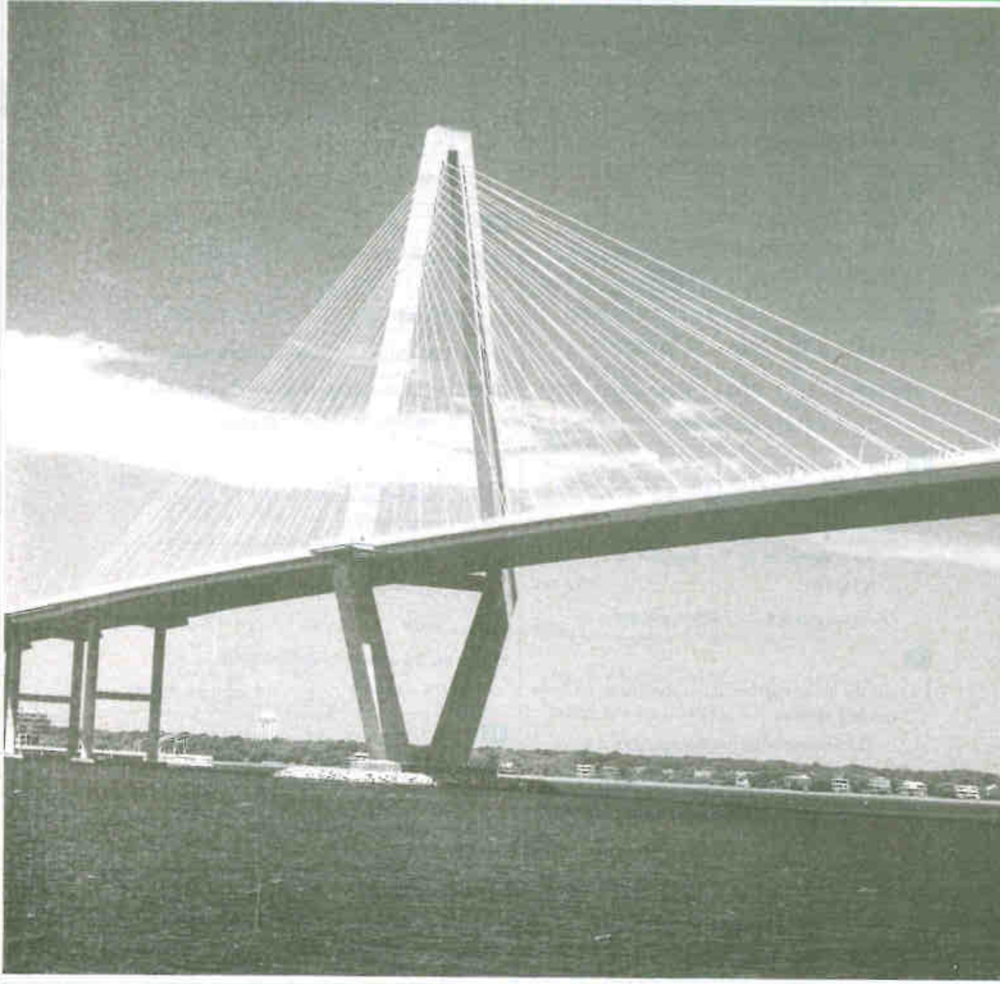
Taking the logarithms to the both sides

$\therefore n \log 0.95 = \log 0.6$

$\therefore n \simeq 10$ years.

SECOND

Answers of Calculus and Trigonometry



Answers of "Unit Three"

Exercise 12

First Multiple choice questions

(1) c (2) a (3) d (4) c

(5) First : a Second : c Third : d Fourth : d

(6) First : a Second : b Third : d Fourth : b
Fifth : d Sixth : b Eeventh : a

(7) First : c Second : d Third : d Fourth : c
Fifth : d Sixth : b Seventh : b Eighth : b
Ninth : c Tenth : d

Second Essay questions

1

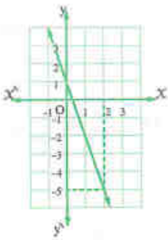
x	1.9	1.99	1.999	⟶	2	⟵	2.001	2.01	2.1
$f(x)$	13.5	13.95	13.995	⟶	14	⟵	14.005	14.05	14.5

$\therefore \lim\limits_{x \to 2} f(x) = 14$

2

(1) Graphically :

$\therefore \lim\limits_{x \to 2} (1 - 3x) = -5$



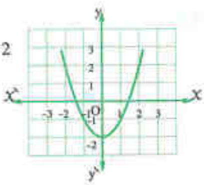
Numerically :

x	1.9	1.99	1.999	⟶	2	⟵	2.001	2.01	2.1
$f(x)$	−4.7	−4.97	−4.997	⟶	−5	⟵	−5.003	−5.03	−5.3

$\therefore \lim\limits_{x \to 2} (1 - 3x) = -5$

(2) Graphically :

$\therefore \lim\limits_{x \to 0} (x^2 - 2) = -2$



Numerically :

x	−0.1	−0.01	−0.001	⟶	0	⟵	0.001	0.01	0.1
$f(x)$	−1.99	−1.9999	−1.999999	⟶	−2	⟵	−1.999999	−1.9999	−1.99

$\therefore \lim\limits_{x \to 0} (x^2 - 2) = -2$

3

(1) undefined (2) 3
(3) 2 (4) does not exist

4

(1) 1 (2) 2
(3) 1 (4) does not exist

5

(1) 2 (2) 1 (3) 1
(4) 1 (5) 1.5 (6) 1.5

6

(1) 1 (2) −1
(3) 3 (4) does not exist

7

(1) 2 (2) 1 (3) 1 (4) 1

8

(1) undefined (2) ∞ (3) ∞ (4) ∞

9

(1) undefined (2) ∞
(3) −∞ (4) does not exist

10

(1) 1 (2) 1 (3) 4
(4) does not exist (5) undefined (6) 3
(7) 3 (8) 3

Third Higher skills

(1) (c) (2) (c) (3) (d) (4) (a)
(5) (d) (6) (a) (7) (c)

Instructions solving :

(1) Notice that each of the figures (a) , (b) and (d) contains a jump at $x = 3$ therefore the limit of the function at $x = 3$ is not exist
But at figure (c) there is an open dot at $x = 3$ therefore the limit of the function at $x = 3$ is exist.

(2) At $x \longrightarrow 0$

$\therefore$ The chord $\overline{AB} \longrightarrow$ length of the diameter (2 r)

i.e. $y \longrightarrow 10$

(3) At $\theta \longrightarrow \frac{\pi}{2}$ $\therefore y \longrightarrow \sqrt{(10)^2 + (10)^2}$

i.e. $y \longrightarrow 10\sqrt{2}$

(4) $\because$ The curve intersects the x-axis at $x = 3$

$\therefore$ The curve passes through the point (3 , 0)

, $\because$ the function is polynomial

$\therefore \lim_{x \to 3} f(x) = \text{zero}$

(5) $\because$ The curve intersects the y-axis at y = 3

$\therefore$ The curve passes through the point (0 , 3)

, $\because$ the function is polynomial

$\therefore \lim_{x \to 0} f(x) = 3$

(6) $\because \lim_{x \to 0} f(x) = \lim_{x \to -1^+} f(x) + \lim_{x \to 3^-} f(x)$

$\therefore 1 = -2 + a$ $\therefore a = 3$

(7) $\because \lim_{x \to a^-} f(x) = \lim_{x \to 1^-} f(x) + \lim_{x \to 3^-} f(x)$

$\therefore \lim_{x \to a^-} f(x) = -3 + 1 = -2$

$\therefore$ All values of $a \in]-1 , 3]$

satisfies $\lim_{x \to a^-} f(x) = -2$

$\therefore$ The greatest value of a is 3

Exercise 13

First Multiple choice questions

(1) c	(2) b	(3) c	(4) a	(5) c	(6) b
(7) b	(8) b	(9) b	(10) b	(11) a	(12) d
(13) b	(14) c	(15) d	(16) b	(17) c	(18) b
(19) c	(20) c	(21) b	(22) b	(23) d	(24) c
(25) a	(26) d	(27) a	(28) c	(29) b	(30) d
(31) a	(32) a	(33) b	(34) c	(35) d	

Second Essay questions

1

(1) $\lim_{x \to 5} \frac{(x-5)(x+5)}{x-5} = \lim_{x \to 5} (x+5) = 10$

(2) $\lim_{x \to 3} \frac{(x-3)(x-5)}{x-3} = \lim_{x \to 3} (x-5) = -2$

(3) $\lim_{x \to 0} \frac{x^2}{x^2(3x-2)} = \lim_{x \to 0} \frac{1}{3x-2} = -\frac{1}{2}$

(4) $\lim_{x \to 2} \frac{5(x-2)}{4(x-2)} = \frac{5}{4}$

(5) $\lim_{x \to 4} \frac{4(x-4)(x+4)}{(x-4)} = \lim_{x \to 4} 4(x+4) = 32$

(6) $\lim_{x \to 4} \frac{2(x-4)}{(x-4)(x+3)} = \lim_{x \to 4} \frac{2}{x+3} = \frac{2}{7}$

(7) $\lim_{x \to -3} \frac{(x+3)(x+1)}{(x-3)(x+3)} = \lim_{x \to -3} \frac{x+1}{x-3} = \frac{-2}{-6} = \frac{1}{3}$

(8) $\lim_{x \to 2} \frac{x(x-2)}{(x-2)(x+1)} = \lim_{x \to 2} \frac{x}{x+1} = \frac{2}{3}$

(9) $\lim_{x \to -1} \frac{(x-4)(x+1)}{(x-2)(x+1)} = \lim_{x \to -1} \frac{x-4}{x-2} = \frac{5}{3}$

(10) $\lim_{x \to \frac{1}{2}} \frac{(2x-1)(x-2)}{(2x-1)} = \lim_{x \to \frac{1}{2}} (x-2) = -\frac{3}{2}$

(11) $\lim_{x \to 9} \frac{-(x-9)}{(x-9)(x+9)} = \lim_{x \to 9} \frac{-1}{x+9} = -\frac{1}{18}$

(12) $\lim_{x \to 1} \frac{(3x+4)(x-1)}{(x-1)(x+1)} = \lim_{x \to 1} \frac{3x+4}{x+1} = \frac{7}{2}$

(13) $\lim_{x \to \frac{3}{4}} \frac{(4x-3)(4x+3)}{2(4x-3)} = \lim_{x \to \frac{3}{4}} \frac{4x+3}{2} = 3$

(14) $\lim_{x \to 3} \frac{(2x+1)(x-3)}{(2x+3)(x-3)} = \lim_{x \to 3} \frac{2x+1}{2x+3} = \frac{7}{9}$

(15) $\lim_{x \to 3} \frac{(x-3)(x+5)}{5(x-3)(x+3)} = \lim_{x \to 3} \frac{x+5}{5(x+3)} = \frac{4}{15}$

(16) $\lim_{x \to 2} \frac{(x-2)(x^2+2x+4)}{3(x-2)(x+2)}$

$= \lim_{x \to 2} \frac{x^2+2x+4}{3(x+2)} = \frac{12}{12} = 1$

2

(1) $\lim_{x \to 0} \frac{(x+2-2)(x+2+2)}{x(x+1)} = \lim_{x \to 0} \frac{x+4}{x+1} = 4$

(2) $\lim_{x \to 0} \frac{4x^2-4x}{5x} = \lim_{x \to 0} \frac{4x(x-1)}{5x}$

$= \lim_{x \to 0} \frac{4(x-1)}{5} = \frac{-4}{5}$

(3) $\lim\limits_{x \to 2} \frac{(x-3-1)(x-3+1)}{(x-2)(2x+1)}$

$= \lim\limits_{x \to 2} \frac{x-4}{2x+1} = \frac{-2}{5}$

(4) $\lim\limits_{x \to -3} \frac{(x+2+1)\left[(x+2)^2-(x+2)+1\right]}{2x(x+3)}$

$= \lim\limits_{x \to -3} \frac{(x+2)^2-(x+2)+1}{2x} = \frac{-1}{2}$

(5) $\lim\limits_{x \to 2} \frac{(x^2-4)(x^2+5)}{x-2}$

$= \lim\limits_{x \to 2} \frac{(x-2)(x+2)(x^2+5)}{x-2}$

$= \lim\limits_{x \to 2} (x+2)(x^2+5) = 36$

(6) $\lim\limits_{x \to 2} \frac{(x-2)^2(x+2)^2}{(x-2)}$

$= \lim\limits_{x \to 2} (x-2)(x+2)^2 = \text{zero}$

(7) $\lim\limits_{x \to 1} \frac{2(x-1)}{(x-1)^5+2(x-1)}$

$= \lim\limits_{x \to 1} \frac{2(x-1)}{(x-1)\left[(x-1)^4+2\right]}$

$= \lim\limits_{x \to 1} \frac{2}{(x-1)^4+2} = 1$

(8) $\lim\limits_{x \to -2} \frac{(x+2)}{(x+2)(x-2)(x^2+4)}$

$= \lim\limits_{x \to -2} \frac{1}{(x-2)(x^2+4)} = -\frac{1}{32}$

(9) $\lim\limits_{x \to 9} \frac{\left(\sqrt{x}-3\right)\left(\sqrt{x}+4\right)}{\left(\sqrt{x}-3\right)\left(\sqrt{x}+3\right)} = \lim\limits_{x \to 9} \frac{\sqrt{x}+4}{\sqrt{x}+3} = \frac{7}{6}$

(10) $\lim\limits_{x \to 0} \frac{1}{x} \times \frac{2-(2+x)}{2(2+x)} = \lim\limits_{x \to 0} \frac{-x}{2x(2+x)}$

$= \lim\limits_{x \to 0} \frac{-1}{2(x+2)} = -\frac{1}{4}$

(11) $\lim\limits_{x \to 2} \frac{x^3(x-2)-6(x-2)}{(x-2)(x+3)}$

$= \lim\limits_{x \to 2} \frac{(x-2)(x^3-6)}{(x-2)(x+3)} = \lim\limits_{x \to 2} \frac{x^3-6}{x+3} = \frac{2}{5}$

(12) $\lim\limits_{x \to 3} \frac{5}{x} + \lim\limits_{x \to 3} \frac{x(x-3)}{x-3} = \frac{5}{3} + 3 = \frac{14}{3}$

(13) $\lim\limits_{x \to -1} \frac{x^2-3x-4}{x^2-1} = \lim\limits_{x \to -1} \frac{(x+1)(x-4)}{(x+1)(x-1)}$

$= \lim\limits_{x \to -1} \frac{x-4}{x-1} = \frac{5}{2}$

(14) $\lim\limits_{x \to 1} \left(\frac{1}{x-1} - \frac{3}{(x-1)(x^2+x+1)}\right)$

$= \lim\limits_{x \to 1} \left(\frac{x^2+x+1-3}{(x-1)(x^2+x+1)}\right)$

$= \lim\limits_{x \to 1} \left(\frac{(x-1)(x+2)}{(x-1)(x^2+x+1)}\right)$

$= \lim\limits_{x \to 1} \frac{x+2}{x^2+x+1} = 1$

3

* We use the long division or the synthetic division in each of the following. We divide each of the numerator and denominator by the factor which gives zero , we get :

(1) $\lim\limits_{x \to 4} \frac{(x-4)(x^2+4x+1)}{x-4}$

$= \lim\limits_{x \to 4} (x^2+4x+1) = 33$

(2) $\lim\limits_{x \to 1} \frac{(x-1)(x^2+x-1)}{(x-1)(x+2)}$

$= \lim\limits_{x \to 1} \frac{x^2+x-1}{x+2} = \frac{1}{3}$

(3) $\lim\limits_{x \to 4} \frac{x(x-4)(x^2+4x-5)}{x-4}$

$= \lim\limits_{x \to 4} x(x^2+4x-5) = 108$

(4) $\lim\limits_{x \to -2} \frac{(x+2)(2x^2-x+2)}{(x+2)(x^2-2x+4)}$

$= \lim\limits_{x \to -2} \frac{2x^2-x+2}{x^2-2x+4} = \frac{12}{12} = 1$

(5) $\lim\limits_{x \to -2} \frac{(x+2)^2}{(x-3)(x+2)^2} = \lim\limits_{x \to -2} \frac{1}{x-3} = -\frac{1}{5}$

(6) $\lim\limits_{x \to 1} \frac{(x-1)^2(x-3)}{x(x-1)^2} = \lim\limits_{x \to 1} \frac{x-3}{x} = -2$

4

(1) $\lim\limits_{x \to 9} \frac{\left(\sqrt{x}-3\right)\left(\sqrt{x}+3\right)}{(x-9)\left(\sqrt{x}+3\right)}$

$= \lim\limits_{x \to 9} \frac{x-9}{(x-9)\left(\sqrt{x}+3\right)} = \lim\limits_{x \to 9} \frac{1}{\sqrt{x}+3} = \frac{1}{6}$

(2) $\lim\limits_{x \to 5} \frac{\left(\sqrt{x-1}-2\right)\left(\sqrt{x-1}+2\right)}{(x-5)\left(\sqrt{x-1}+2\right)}$

$= \lim\limits_{x \to 5} \frac{x-1-4}{(x-5)\left(\sqrt{x-1}+2\right)}$

$= \lim\limits_{x \to 5} \frac{1}{\sqrt{x-1}+2} = \frac{1}{4}$

(3) $\lim\limits_{x\to -1} \frac{x+1}{\sqrt{x+5}-2} \times \frac{\sqrt{x+5}+2}{\sqrt{x+5}+2}$

$= \lim\limits_{x\to -1} \frac{(x+1)\left(\sqrt{x+5}+2\right)}{x+5-4}$

$= \lim\limits_{x\to -1} \left(\sqrt{x+5}+2\right) = 4$

(4) $\lim\limits_{x\to 0} \frac{\left(\sqrt{3-2x}-\sqrt{3}\right)\left(\sqrt{3-2x}+\sqrt{3}\right)}{x\left(\sqrt{3-2x}+\sqrt{3}\right)}$

$= \lim\limits_{x\to 0} \frac{(3-2x)-3}{x\left[\sqrt{3-2x}+\sqrt{3}\right]}$

$= \lim\limits_{x\to 0} \frac{-2}{\sqrt{3-2x}+\sqrt{3}} = \frac{-1}{\sqrt{3}}$

(5) $\lim\limits_{x\to 0} \frac{\left(\sqrt{2x+9}-3\right)\left(\sqrt{2x+9}+3\right)}{x(x+1)\left(\sqrt{2x+9}+3\right)}$

$= \lim\limits_{x\to 0} \frac{2x+9-9}{x(x+1)\left(\sqrt{2x+9}+3\right)}$

$= \lim\limits_{x\to 0} \frac{2}{(x+1)\left(\sqrt{2x+9}+3\right)} = \frac{1}{3}$

(6) $\lim\limits_{x\to 5} \frac{x(x-5)}{\left(\sqrt{x+4}-3\right)} \times \frac{\left(\sqrt{x+4}+3\right)}{\left(\sqrt{x+4}+3\right)}$

$= \lim\limits_{x\to 5} \frac{x(x-5)\left(\sqrt{x+4}+3\right)}{x+4-9}$

$= \lim\limits_{x\to 5} x\left(\sqrt{x+4}+3\right) = 30$

(7) $\lim\limits_{x\to 3} \frac{(x-3)(x+2)\left(\sqrt{5x-6}+3\right)}{\left(\sqrt{5x-6}-3\right)\left(\sqrt{5x-6}+3\right)}$

$= \lim\limits_{x\to 3} \frac{(x-3)(x+2)\left(\sqrt{5x-6}+3\right)}{5x-6-9}$

$= \lim\limits_{x\to 3} \frac{(x-3)(x+2)\left(\sqrt{5x-6}+3\right)}{5(x-3)}$

$= \lim\limits_{x\to 3} \frac{(x+2)\left(\sqrt{5x-6}+3\right)}{5} = 6$

(8) $\lim\limits_{x\to 0} \frac{\left(\sqrt{1+x}-\sqrt{1-x}\right)\left(\sqrt{1+x}+\sqrt{1-x}\right)}{2x\left(\sqrt{1+x}+\sqrt{1-x}\right)}$

$= \lim\limits_{x\to 0} \frac{(1+x)-(1-x)}{2x\left(\sqrt{1+x}+\sqrt{1-x}\right)}$

$= \lim\limits_{x\to 0} \frac{1}{\sqrt{1+x}+\sqrt{1-x}} = \frac{1}{2}$

(9) $\lim\limits_{x\to 1} \frac{(x-1)\left(\sqrt{3x-2}+\sqrt{x}\right)}{\left(\sqrt{3x-2}-\sqrt{x}\right)\left(\sqrt{3x-2}+\sqrt{x}\right)}$

$= \lim\limits_{x\to 1} \frac{(x-1)\left(\sqrt{3x-2}+\sqrt{x}\right)}{3x-2-x}$

$= \lim\limits_{x\to 1} \frac{(x-1)\left(\sqrt{3x-2}+\sqrt{x}\right)}{2(x-1)}$

$= \lim\limits_{x\to 1} \frac{\sqrt{3x-2}+\sqrt{x}}{2} = 1$

(10) $\lim\limits_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1} \times \frac{\sqrt{x^2+8}+3}{\sqrt{x^2+8}+3}$

$= \lim\limits_{x\to -1} \frac{x^2+8-9}{(x+1)\left(\sqrt{x^2+8}+3\right)}$

$= \lim\limits_{x\to -1} \frac{(x+1)(x-1)}{(x+1)\left(\sqrt{x^2+8}+3\right)}$

$= \lim\limits_{x\to -1} \frac{x-1}{\sqrt{x^2+8}+3} = -\frac{1}{3}$

(11) $\lim\limits_{x\to 1} \frac{(1-x)\left(2+\sqrt{x^2+3}\right)}{\left(2-\sqrt{x^2+3}\right)\left(2+\sqrt{x^2+3}\right)}$

$= \lim\limits_{x\to 1} \frac{(1-x)\left(2+\sqrt{x^2+3}\right)}{4-x^2-3}$

$= \lim\limits_{x\to 1} \frac{(1-x)\left(2+\sqrt{x^2+3}\right)}{(1-x)(1+x)}$

$= \lim\limits_{x\to 1} \frac{2+\sqrt{x^2+3}}{1+x} = 2$

(12) $\lim\limits_{x\to 3} \frac{\left(\sqrt{x+1}-2\right)\left(\sqrt{x+1}+2\right)\left(\sqrt{x-2}+1\right)}{\left(\sqrt{x-2}-1\right)\left(\sqrt{x+1}+2\right)\left(\sqrt{x-2}+1\right)}$

$= \lim\limits_{x\to 3} \frac{(x+1-4)\left(\sqrt{x-2}+1\right)}{(x-2-1)\left(\sqrt{x+1}+2\right)}$

$= \lim\limits_{x\to 3} \frac{\sqrt{x-2}+1}{\sqrt{x+1}+2} = \frac{2}{4} = \frac{1}{2}$

(13) $\lim\limits_{x\to 4} \frac{\left(\sqrt{24-5x}-2\right)\left(\sqrt{24-5x}+2\right)\left(3+\sqrt{x+5}\right)}{\left(3-\sqrt{x+5}\right)\left(3+\sqrt{x+5}\right)\left(\sqrt{24-5x}+2\right)}$

$= \lim\limits_{x\to 4} \frac{(24-5x-4)\left(3+\sqrt{x+5}\right)}{(9-x-5)\left(\sqrt{24-5x}+2\right)}$

$= \lim\limits_{x\to 4} \frac{5(4-x)\left(3+\sqrt{x+5}\right)}{(4-x)\left(\sqrt{24-5x}+2\right)}$

$= \lim\limits_{x\to 4} \frac{5\left(3+\sqrt{x+5}\right)}{\sqrt{24-5x}+2} = 7.5$

(14) $\lim\limits_{x \to 3} \frac{(x-3)\left(x+\sqrt{x+6}\right)}{\left(x-\sqrt{x+6}\right)\left(x+\sqrt{x+6}\right)}$

$= \lim\limits_{x \to 3} \frac{(x-3)\left(x+\sqrt{x+6}\right)}{x^2-x-6}$

$= \lim\limits_{x \to 3} \frac{(x-3)\left(x+\sqrt{x+6}\right)}{(x-3)(x+2)}$

$= \lim\limits_{x \to 3} \frac{x+\sqrt{x+6}}{x+2} = \frac{6}{5}$

5

$\lim\limits_{x \to 2} \frac{f(x)-5}{x-2} = 1$ (has an existance)

$\therefore f(2)-5=0$ $\qquad \therefore f(2)=5$

$\therefore \lim\limits_{x \to 2} f(x) = f(2) = 5$

6

$\lim\limits_{x \to 0} \frac{f(x)}{x^2} = 5$ $\qquad \therefore f(x) = x^2\left(g(x)\right)$

(1) $\lim\limits_{x \to 0} f(x) = 0$

(2) $\lim\limits_{x \to 0} \frac{f(x)}{x} = \lim\limits_{x \to 0} \frac{x^2\left(g(x)\right)}{x}$

$= \lim\limits_{x \to 0} x\left(g(x)\right)$

$= \lim\limits_{x \to 0} x \times \lim\limits_{x \to 0} g(x) = 0 \times 5 = 0$

7

$\lim\limits_{x \to -1} \frac{(x+1)(x-a)}{x+1} = 4$ $\qquad \therefore \lim\limits_{x \to -1} (x-a) = 4$

$\therefore -1-a=4$ $\qquad \therefore a=-5$

8

$\because \lim\limits_{x \to 1} \frac{x^2+ax+b}{x-1}$ exists and equals 5

, $\because$ the denominator = zero at $x=1$

$\therefore$ The numerator = zero at $x=1$

$\therefore 1+a+b=0$ $\qquad \therefore b=-a-1$

$\therefore \lim\limits_{x \to 1} \frac{x^2+ax-a-1}{x-1} = 5$

$\therefore \lim\limits_{x \to 1} \frac{(x^2-1)+(ax-a)}{x-1} = 5$

$\therefore \lim\limits_{x \to 1} \frac{(x-1)(x+1)+a(x-1)}{(x-1)} = 5$

$\therefore \lim\limits_{x \to 1} \frac{(x-1)(x+1+a)}{(x-1)} = 5$

$\therefore 2+a=5$ $\qquad \therefore a=3$ and hence $b=-4$

9 $\lim\limits_{x \to 100} (0.2x^2+40x+150) = 6150$ pounds.

Third Higher skills

1

(1) (a) (2) (d) (3) (d) (4) (a)

Instructions to solve 1 :

(1) $\because x\left(f(x)+1\right) = f(x)+x^2$

$\therefore x f(x) + x = f(x) + x^2$

$\therefore x f(x) - f(x) = x^2 - x$

$\therefore (x-1) f(x) = x(x-1)$

$\therefore f(x) = \frac{x(x-1)}{x-1} = x$ where $x \neq 1$

$\therefore \lim\limits_{x \to 1} f(x) = 1$

(2) $\because \lim\limits_{x \to 0} \frac{\sqrt{mx+k}-3}{x}$ is exist and equals 2

, $\because$ the denominator = 0 at $x=0$

$\therefore$ The numerator = 0 at $x=0$ $\therefore \sqrt{k}=3 \therefore k=9$

, $\because \lim\limits_{x \to 0} \frac{\sqrt{mx+9}-3}{x} = 2$

$\therefore \lim\limits_{x \to 0} \frac{\left(\sqrt{mx+9}-3\right)}{x} \times \frac{\left(\sqrt{mx+9}+3\right)}{\left(\sqrt{mx+9}+3\right)} = 2$

$\therefore \lim\limits_{x \to 0} \frac{mx+9-9}{x\left(\sqrt{mx+9}+3\right)} = 2$

$\therefore \lim\limits_{x \to 0} \frac{m}{\sqrt{mx+9}+3} = 2$ $\qquad \therefore \frac{m}{6} = 2$

$\therefore m=12$ $\qquad \therefore \frac{m}{k} = \frac{12}{9} = \frac{4}{3}$

(3) $\because 2\lim\limits_{x \to m} f(x) - 5\lim\limits_{x \to m} g(x) = 10$ (1)

, $\lim\limits_{x \to m} f(x) + \lim\limits_{x \to m} g(x) = 6$ (2)

, by multiplying (2) by 5

$\therefore 5\lim\limits_{x \to m} f(x) + 5\lim\limits_{x \to m} g(x) = 30$ (3)

, solving (1) and (3) : $\therefore 7\lim\limits_{x \to m} f(x) = 40$

$\therefore 7\lim\limits_{x \to m} f(x) = \frac{40}{7}$

, substituting in (2) : $\lim\limits_{x \to m} g(x) = 6 - \frac{40}{7} = \frac{2}{7}$

$\therefore \lim\limits_{x \to m} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to m} f(x)}{\lim\limits_{x \to m} g(x)} = \frac{\frac{40}{7}}{\frac{2}{7}} = 20$

(4) $\lim\limits_{x \to 2} \frac{2x^2 - n(x)}{x-2} = \lim\limits_{x \to 2} \frac{2x^2 - n(x) + 8 - 8}{x-2}$

$= \lim\limits_{x \to 2} \frac{2x^2 - 8}{x-2} - \lim\limits_{x \to 2} \frac{n(x) - 8}{x-2}$

$= \lim\limits_{x \to 2} \frac{2(x-2)(x+2)}{(x-2)} - \lim\limits_{x \to 2} \frac{n(x) - 8}{x-2}$

$= \lim\limits_{x \to 2} 2(x+2) - \lim\limits_{x \to 2} \frac{n(x)-8}{x-2} = 8 - 7 = 1$

2

(1) $\lim\limits_{x \to 5} \left[\frac{(x-5)(x+1)}{(x-5)(x-1)}\right]^2$

$= \lim\limits_{x \to 5} \left[\frac{x+1}{x-1}\right]^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$

(2) $\lim\limits_{x \to 1} \sqrt{\frac{(x-1)(x+3)}{x(x-1)}}$

$= \lim\limits_{x \to 1} \sqrt{\frac{x+3}{x}} = \sqrt{4} = 2$

(3) $\lim\limits_{x \to \sqrt{2}} \frac{\left(x-\sqrt{2}\right)\left(x+\sqrt{2}\right)}{\left(x-\sqrt{2}\right)\left(x+2\sqrt{2}\right)}$

$= \lim\limits_{x \to \sqrt{2}} \frac{x+\sqrt{2}}{x+2\sqrt{2}} = \frac{2\sqrt{2}}{3\sqrt{2}} = \frac{2}{3}$

(4) Multiply each of the numerator and denominator by $(-x^3)$, we get :

$\lim\limits_{x \to 1} \frac{7x^3 - 2x^2 - 5}{4x^3 - 3x - 1}$

$= \lim\limits_{x \to 1} \frac{(x-1)(7x^2+5x+5)}{(x-1)(4x^2+4x+1)}$

$= \lim\limits_{x \to 1} \frac{7x^2+5x+5}{4x^2+4x+1} = \frac{17}{9}$

(5) $\lim\limits_{\tan x \to 3} \frac{(\tan x - 3)(\tan x + 1)}{(\tan x - 3)(\tan x - 1)}$

$= \lim\limits_{\tan x \to 3} \frac{\tan x + 1}{\tan x - 1} = 2$

(6) $\lim\limits_{x \to 0} \frac{(\cos x - 1)(\cos x - 4)}{-(\cos x + 1)(\cos x - 1)}$

$= \lim\limits_{x \to 0} \frac{\cos x - 4}{-(\cos x + 1)} = \frac{3}{2}$

(7) $\lim\limits_{(x-4) \to 4} \frac{x-8}{x^2-64} = \lim\limits_{x \to 8} \frac{x-8}{(x-8)(x+8)} = \frac{1}{16}$

(8) $\lim\limits_{x \to 1} \frac{x^2+3-4-2(x^2-x)}{(x^2-x)\left(\sqrt{x^2+3}-2\right)}$

$= \lim\limits_{x \to 1} \frac{-(x-1)^2}{x(x-1)\left(\sqrt{x^2+3}-2\right)}$

$= \lim\limits_{x \to 1} \frac{-(x-1)}{x\left(\sqrt{x^2+3}-2\right)} \times \frac{\sqrt{x^2+3}+2}{\sqrt{x^2+3}+2}$

$= \lim\limits_{x \to 1} \frac{-(x-1)\left(\sqrt{x^2+3}+2\right)}{x(x^2-1)}$

$= \lim\limits_{x \to 1} \frac{-\left(\sqrt{x^2+3}+2\right)}{x(x+1)} = \frac{-4}{2} = -2$

Exercise 14

First Multiple choice questions

(1) d	(2) d	(3) d	(4) a	(5) a	(6) d
(7) a	(8) c	(9) b	(10) b	(11) b	(12) c
(13) d	(14) c	(15) c	(16) c	(17) d	(18) d
(19) c	(20) d	(21) b	(22) b	(23) b	(24) a
(25) b	(26) a	(27) b	(28) d	(29) a	(30) a
(31) b	(32) b				

Second Essay questions

1

(1) $\lim\limits_{x \to 2} \frac{x^3 - 2^3}{x-2} = 3(2)^2 = 12$

(2) $\lim\limits_{x \to -5} \frac{x^4 - (-5)^4}{x-(-5)} = \frac{4}{1}(-5)^{4-1} = -500$

(3) $\lim\limits_{x \to a} \frac{x^5 - a^5}{x-a} = 5a^4$

(4) $\lim\limits_{x \to -3} \frac{x^4 - (-3)^4}{x^5 - (-3)^5} = \frac{4}{5}(-3)^{4-5} = -\frac{4}{15}$

(5) $\lim\limits_{x \to 2} \frac{x^6 - (2)^6}{x^7 - (2)^7} = \frac{6}{7}(2)^{6-7} = \frac{3}{7}$

(6) $\lim\limits_{x \to 2} \frac{2(x^6 - 64)}{x^2 - 4} = 2 \lim\limits_{x \to 2} \frac{x^6 - 2^6}{x^2 - 2^2}$

$= 2\left(\frac{6}{2}\right) \times 2^{6-2} = 96$

(7) $\lim\limits_{x \to -2} \frac{x^6 - 64}{3(x+2)} = \frac{1}{3} \lim\limits_{x \to -2} \frac{x^6 - (-2)^6}{x-(-2)}$

$= \frac{1}{3} \times 6 \times (-2)^5 = -64$

(8) $\lim_{x\to -1} \frac{x(x^9+1)}{x(x^6-1)} = \lim_{x\to -1} \frac{x^9-(-1)^9}{x^6-(-1)^6}$

$= \frac{9}{6}(-1)^3 = \frac{-9}{6} = \frac{-3}{2}$

(9) $-1 \times \lim_{x\to 1} \frac{x^9-1}{x^7-1} = (-1)\times \frac{9}{7}(1)^{9-7} = \frac{-9}{7}$

(10) $\lim_{2x\to -1} \frac{(2x)^5-(-1)^5}{(2x)^6-(-1)^6} = \frac{5}{6}(-1)^{-1} = -\frac{5}{6}$

(11) $\lim_{3x\to -2} \frac{(3x)^5-(-2)^5}{(3x)^3-(-2)^3} = \frac{5}{3}(-2)^{5-3} = \frac{20}{3}$

(12) $\lim_{2x\to 3} \frac{x(16x^4-81)}{x^2(2x-3)}$

$= \lim_{x\to \frac{3}{2}} \frac{1}{x} \times \lim_{2x\to 3} \frac{(2x)^4-(3)^4}{2x-3}$

$= \frac{2}{3}\times 4(3)^3 = 72$

2

(1) $\lim_{x\to 2} \frac{x^{-7}-(2)^{-7}}{x-2} = -7(2)^{-8} = \frac{-7}{256}$

(2) $\lim_{x\to -1} \frac{x^{-4}-(-1)^{-4}}{x^{-18}-(-1)^{-18}} = \frac{-4}{-18} = \frac{2}{9}$

(3) $\lim_{x\to \sqrt{2}} \frac{x^7-(\sqrt{2})^7}{x^2-(\sqrt{2})^2} = \frac{7}{2}\left(\sqrt{2}\right)^5 = 14\sqrt{2}$

(4) $\lim_{x\to -\sqrt{2}} \frac{x^8-(-\sqrt{2})^8}{x^5-(-\sqrt{2})^5} = \frac{8}{5}\left(-\sqrt{2}\right)^3 = \frac{-16\sqrt{2}}{5}$

(5) $\lim_{\sqrt{2}x\to -3} \frac{(\sqrt{2}x)^6-(-3)^6}{\sqrt{2}x-(-3)} = 6(-3)^5 = -1458$

(6) $\lim_{\sin x\to \frac{1}{2}} \frac{\sin^3 x-\left(\frac{1}{2}\right)^3}{\sin x-\frac{1}{2}} = 3\left(\frac{1}{2}\right)^2 = \frac{3}{4}$

(7) $\lim_{x\to 2} \frac{x^{-5}-(2)^{-5}}{x^{-7}-(2)^{-7}} = \frac{-5}{-7}(2)^2 = \frac{20}{7}$

(8) $8\lim_{x\to -2} \frac{x^{-3}-\left(-\frac{1}{8}\right)}{x^{-8}-\left(\frac{1}{256}\right)} = 8\lim_{x\to -2} \frac{x^{-3}-(-2)^{-3}}{x^{-8}-(-2)^{-8}}$

$= 8\times\frac{-3}{-8}(-2)^5 = -96$

(9) $\lim_{x\to 2} \frac{x^{-8}-2^{-8}}{x-2} = -8\times 2^{-9} = \frac{-1}{64}$

(10) $\lim_{x\to 4} \frac{x^{\frac{7}{2}}-(4)^{\frac{7}{2}}}{x^{\frac{5}{2}}-(4)^{\frac{5}{2}}} = \frac{7}{5}\times 4 = \frac{28}{5}$

(11) $\lim_{x\to 1} \frac{x^{\frac{1}{7}}-1^{\frac{1}{7}}}{x-1} = \frac{1}{7}$

(12) $\lim_{x\to 4} \frac{x^{\frac{3}{2}}-(4)^{\frac{3}{2}}}{x^2-(4)^2} = \frac{3}{4}(4)^{-\frac{1}{2}} = \frac{3}{8}$

(13) $\lim_{x\to 16} \frac{x^{\frac{7}{4}}-16^{\frac{7}{4}}}{x-16} = \frac{\frac{7}{4}}{1}\times(16)^{\frac{7}{4}-1} = 14$

(14) $\lim_{x\to 1} \frac{x^{\frac{6}{5}}-(1)^{\frac{6}{5}}}{x^2-(1)^2} = \frac{3}{5}$

(15) $\lim_{x\to 1} \frac{x^{\frac{1}{2}}(x^{10}-1)}{x^{\frac{2}{3}}(x^4-1)}$

$= \lim_{x\to 1} x^{-\frac{1}{6}} \times \lim_{x\to 1} \frac{x^{10}-(1)^{10}}{x^4-(1)^4} = \frac{5}{2}$

(16) $\frac{1}{2}\lim_{x\to 4} \frac{x^4-256}{x-4} = \frac{1}{2}\lim_{x\to 4} \frac{x^4-4^4}{x-4}$

$= \frac{1}{2}\times 4(4)^3 = 128$

(17) $\lim_{x\to -1} \frac{\frac{1}{x^4}(x^8-1)}{\frac{1}{x^3}(x^6-1)}$

$= \lim_{x\to -1} \frac{1}{x}\times \lim_{x\to -1} \frac{x^8-(-1)^8}{x^6-(-1)^6}$

$= -1\times\frac{8}{6}(-1)^2 = \frac{-4}{3}$

(18) $\lim_{2x\to 1} \frac{\frac{5}{16}\left[16x^4-1\right]}{2x-1} = \frac{5}{16}\lim_{2x\to 1} \frac{(2x)^4-(1)^4}{2x-1}$

$= \frac{5}{16}\times 4(1)^3 = \frac{5}{4}$

(19) $\lim_{x\to 2} \frac{x^{10}-2^{10}}{(x-2)(x-1)}$

$= \lim_{x\to 2} \frac{1}{x-1}\times \lim_{x\to 2} \frac{x^{10}-2^{10}}{x-2}$

$= 1\times 10(2)^9 = 5120$

(20) $\lim_{x\to 1} \frac{x^{17}-(1)^{17}}{(3x+5)(x-1)}$

$= \lim_{x\to 1} \frac{1}{3x+5}\times \lim_{x\to 1} \frac{x^{17}-(1)^{17}}{x-1}$

$= \frac{1}{8}\times 17(1)^{16} = \frac{17}{8}$

3

(1) $\lim_{(1+x)\to 1} \frac{(1+x)^{10}-(1)^{10}}{(1+x)^7-(1)^7} = \frac{10}{7}(1)^3 = \frac{10}{7}$

(2) $\lim_{(x-5)\to 1} \frac{(x-5)^7-(1)^7}{(x-5)-(1)} = 7(1)^6 = 7$

(3) $\lim\limits_{(x+3)\to 1} \frac{(x+3)^5-(1)^5}{(x+3)-(1)} = 5\,(1)^4 = 5$

(4) $\lim\limits_{x\to 0} \frac{(x+2)^5-2^5}{x} = 5\,(2)^4 = 80$

(5) $\frac{1}{6}\lim\limits_{h\to 0} \frac{(3+h)^4-3^4}{h} = \frac{1}{6}\times 4\,(3)^3 = 18$

(6) $4\lim\limits_{h\to 0} \frac{(1+4h)^8-(1)^8}{4h}$

$= 4\lim\limits_{(1+4h)\to 1} \frac{(1+4h)^8-(1)^8}{(1+4h)-(1)} = 4\times 8\,(1)^7 = 32$

(7) $\frac{-2}{5}\lim\limits_{x\to 0} \frac{(1-2x)^5-1^5}{-2x}$

$= \frac{-2}{5}\lim\limits_{(1-2x)\to 1} \frac{(1-2x)^5-1^5}{(1-2x)-1} = \frac{-2}{5}\times 5 = -2$

(8) $3\lim\limits_{h\to 0} \frac{(x+3h)^5-x^5}{3h}$

$= 3\lim\limits_{(x+3h)\to x} \frac{(x+3h)^5-x^5}{(x+3h)-x}$

$= 3\times 5x^4 = 15x^4$

(9) $\frac{-2}{51}\lim\limits_{(x-2h)\to x} \frac{(x-2h)^{17}-x^{17}}{(x-2h)-x} = \frac{-2}{51}\times 17x^{16}$

$= -\frac{2}{3}x^{16}$

(10) $3\lim\limits_{(3x+2)\to -1} \frac{(3x+2)^9-(-1)^9}{(3x+2)-(-1)} = 3\times 9\,(-1)^8 = 27$

(11) $\frac{3}{2}\lim\limits_{x\to 0} \frac{\sqrt[3]{1+3x}-1}{3x}$

$= \frac{3}{2}\lim\limits_{(1+3x)\to 1} \frac{(1+3x)^{\frac{1}{3}}-(1)^{\frac{1}{3}}}{(1+3x)-(1)}$

$= \frac{3}{2}\times\frac{1}{3}\times(1)^{-\frac{2}{3}} = \frac{1}{2}$

(12) $\lim\limits_{x\to 1} \frac{\sqrt[3]{26+x}-3}{26+x-27}$

$= \lim\limits_{(26+x)\to 27} \frac{(26+x)^{\frac{1}{3}}-(27)^{\frac{1}{3}}}{(26+x)-(27)}$

$= \frac{1}{3}(27)^{-\frac{2}{3}} = \frac{1}{27}$

(13) $\lim\limits_{x\to 7} \frac{(x+25)^{\frac{1}{5}}-(32)^{\frac{1}{5}}}{(x+25)-32} = \frac{\frac{1}{5}}{1}(32)^{\frac{1}{5}-1} = \frac{1}{80}$

(14) $\lim\limits_{(x-4)\to -2} \frac{(x-4)^5-(-2)^5}{(x-4)-(-2)} = \left(\frac{5}{1}\right)\times(-2)^{5-1} = 80$

(15) $\lim\limits_{x\to -2} \frac{(x+3)^5-1}{(x-2)(x+2)}$

$= \lim\limits_{x\to -2}\frac{1}{x-2}\times\lim\limits_{(x+3)\to 1}\frac{(x+3)^5-1^5}{(x+3)-1}$

$= -\frac{1}{4}\times 5\times(1)^4 = -\frac{5}{4}$

(16) $\lim\limits_{x\to 1} \frac{x^{19}-1+x^8-1}{x-1}$

$= \lim\limits_{x\to 1}\frac{x^{19}-(1)^{19}}{x-1} + \lim\limits_{x\to 1}\frac{x^8-(1)^8}{x-1}$

$= 19\,(1)^{18} + 8\,(1)^7 = 27$

(17) $\lim\limits_{x\to -1} \frac{x^7+1+x^9+1}{x+1}$

$= \lim\limits_{x\to -1}\frac{x^7-(-1)^7}{x-(-1)} + \lim\limits_{x\to -1}\frac{x^9-(-1)^9}{x-(-1)}$

$= 7\,(-1)^6 + 9\,(-1)^8 = 16$

(18) $\lim\limits_{x\to 1} \frac{\sqrt{x}-1+\sqrt[3]{x}-1}{x-1}$

$= \lim\limits_{x\to 1}\frac{(x)^{\frac{1}{2}}-(1)^{\frac{1}{2}}}{x-1} + \lim\limits_{x\to 1}\frac{(x)^{\frac{1}{3}}-(1)^{\frac{1}{3}}}{x-1}$

$= \frac{1}{2}+\frac{1}{3} = \frac{5}{6}$

(19) $\lim\limits_{x\to 2} \frac{x^5-32+x^2-4}{x-2}$

$= \lim\limits_{x\to 2}\frac{x^5-2^5}{x-2} + \lim\limits_{x\to 2}\frac{x^2-2^2}{x-2}$

$= 5\,(2)^4 + 2\,(2) = 84$

(20) $\lim\limits_{x\to 5} \frac{(x-3)^5-2^5}{x(x-5)}$

$= \lim\limits_{x\to 5}\frac{1}{x}\times\lim\limits_{(x-3)\to 2}\frac{(x-3)^5-2^5}{(x-3)-2}$

$= \frac{1}{5}\times\frac{5}{1}(2)^{5-1} = 16$

(21) $\lim\limits_{x\to 0} \frac{1}{3(x-2)}$

$\times 2\lim\limits_{(2x-3)\to -3} \frac{(2x-3)^5-(-3)^5}{(2x-3)-(-3)}$

$= -\frac{1}{6}\times 2\times 5\,(-3)^4 = -135$

4 $\because \lim\limits_{x\to a} \frac{x^{12}-a^{12}}{x^{10}-a^{10}} = 30$ $\quad\therefore \frac{12}{10}a^2 = 30$

$\therefore a^2 = 25$ $\quad\therefore a = \pm 5$

5 $\lim_{x \to -1} \frac{x^{15}-(-1)^{15}}{x-(-1)} = 15(-1)^{14} = 15$

$, \because \lim_{x \to k} \frac{x^5-k^5}{x^3-k^3} = \frac{5}{3}(k)^2$

$\therefore \frac{5}{3}k^2 = 15 \quad \therefore k^2 = 9 \quad \therefore k = \pm 3$

6 $\because$ The limit is exist $\quad \therefore 64 = 2^n$

$\therefore n = 6$

$\therefore$ The limit $= \frac{6}{1} \times 2^{6-1} \quad \therefore \ell = 192$

7 $\lim_{x \to 2} \frac{f(x)-f(2)}{x-2} = \lim_{x \to 2} \frac{\frac{1}{x^4}-\frac{1}{2^4}}{x-2}$

$= \lim_{x \to 2} \frac{x^{-4}-2^{-4}}{x-2} = -4(2)^{-5} = \frac{-4}{32} = \frac{-1}{8}$

8

(1) $\lim_{x \to 2} \frac{x^3-2^3}{x^5-2^5} + \lim_{x \to 2} \frac{x^4-2^4}{x^7-2^7}$

$= \frac{3}{5}(2)^{-2} + \frac{4}{7}(2)^{-3} = \frac{31}{140}$

(2) $\lim_{x \to 3} \left(\frac{x-2}{x^2-4} \times \frac{x^5-243}{x-3}\right)$

$= \lim_{x \to 3} \frac{1}{x+2} \times \lim_{x \to 3} \frac{x^5-3^5}{x-3} = \frac{1}{5} \times 5 \times 3^4 = 81$

(3) $\left[\lim_{x \to -3} \frac{x^4-(-3)^4}{x^3-(-3)^3}\right]^3 = \left[\frac{4}{3}(-3)\right]^3 = -64$

(4) $\lim_{x \to 1} \frac{(x^{25}-1)^2}{(x-1)^2} = \lim_{x \to 1} \left(\frac{x^{25}-1}{x-1}\right)^2$

$= \left[\lim_{x \to 1} \frac{x^{25}-1^{25}}{x-1}\right]^2 = \left[25(1)^{24}\right]^2 = 625$

(5) $\lim_{x \to 1} \frac{(x^6-1)^2}{(x-1)^2} = \left[\lim_{x \to 1} \frac{x^6-(1)^6}{x-1}\right]^2$

$= \left[6(1)^5\right]^2 = 36$

(6) $\lim_{x \to 2} \left[\frac{(x^4-16)^3}{(x^3-8)^3} \times \frac{x^4-16}{x^8-256}\right]$

$= \lim_{x \to 2} \left(\frac{x^4-16}{x^3-8}\right)^3 \times \lim_{x \to 2} \frac{x^4-16}{x^8-256}$

$= \left(\lim_{x \to 2} \frac{x^4-2^4}{x^3-2^3}\right)^3 \times \lim_{x \to 2} \frac{x^4-2^4}{x^8-2^8}$

$= \left[\frac{4}{3}(2)\right]^3 \times \frac{4}{8}(2)^{-4} = \frac{16}{27}$

(7) $\lim_{x \to 1} \frac{x^{\frac{1}{3}}-(1)^{\frac{1}{3}}}{x-1} \times \lim_{x \to 1} \frac{x^{\frac{2}{5}}-(1)^{\frac{2}{5}}}{x-1}$

$= \frac{1}{3} \times \frac{2}{5} = \frac{2}{15}$

(8) $\lim_{x \to 0} \frac{(1-x)^{15}\left[\left(\frac{1+x}{1-x}\right)^{15}-1\right]}{(1-x)^9\left[\left(\frac{1+x}{1-x}\right)^9-1\right]}$

$= \lim_{x \to 0} (1-x)^6 \times \lim_{x \to 0} \frac{\left(\frac{1+x}{1-x}\right)^{15}-1}{\left(\frac{1+x}{1-x}\right)^9-1}$

$= 1 \times \lim_{x \to 0} \frac{\left(\frac{1+x}{1-x}\right)^{15}-1}{\left(\frac{1+x}{1-x}\right)^9-1}$

$\because x \to 0 \quad \therefore \frac{1+x}{1-x} \to 1$

$\therefore \lim_{\left(\frac{1+x}{1-x}\right) \to 1} \frac{\left(\frac{1+x}{1-x}\right)^{15}-(1)^{15}}{\left(\frac{1+x}{1-x}\right)^9-(1)^9} = \frac{15}{9} = \frac{5}{3}$

(9) $\lim_{x \to 1} \frac{x^{12}-1}{x^2(x-1)+(x-1)}$

$= \lim_{x \to 1} \frac{x^{12}-1}{(x-1)(x^2+1)}$

$= \lim_{x \to 1} \frac{1}{x^2+1} \times \lim_{x \to 1} \frac{x^{12}-(1)^{12}}{x-1}$

$= \frac{1}{2} \times 12(1)^{11} = 6$

Third Higher skills

(1) $\lim_{x \to 1} \frac{x^{\frac{1}{2}}(x-1)}{x^{\frac{1}{2}}(x^5-1)} = \lim_{x \to 1} \frac{x-1}{x^5-1^5} = \frac{1}{5}(1)^{-4} = \frac{1}{5}$

(2) $\frac{5}{7} \lim_{x \to 0} \frac{(1+5x)^{10}-1}{5x} \times \lim_{x \to 0} \frac{7x}{(1+7x)^8-1}$

$= \frac{5}{7} \lim_{(1+5x) \to 1} \frac{(1+5x)^{10}-(1)^{10}}{(1+5x)-1}$

$\times \lim_{(1+7x) \to 1} \frac{(1+7x)-1}{(1+7x)^8-(1)^8}$

$= \frac{5}{7}(10)(1)^9 \times \frac{1}{8}(1)^{-7} = \frac{25}{28}$

(3) $\lim_{x\to 0} \frac{\sqrt[3]{x+8}-2}{x} \times \lim_{x\to 0} \frac{x}{(x+3)^5-243}$

$= \lim_{(x+8)\to 8} \frac{(x+8)^{\frac{1}{3}}-(8)^{\frac{1}{3}}}{(x+8)-8}$

$\times \lim_{(x+3)\to 3} \frac{(x+3)-3}{(x+3)^5-3^5}$

$= \frac{1}{3}(8)^{-\frac{2}{3}} \times \frac{1}{5} \times 3^{-4} = \frac{1}{4860}$

(4) $\lim_{x\to 2} \frac{\sqrt[3]{2x-3}-1}{2x-4} \times \lim_{x\to 2} \frac{2x-4}{\sqrt[5]{3x-5}-1}$

$= \lim_{(2x-3)\to 1} \frac{(2x-3)^{\frac{1}{3}}-(1)^{\frac{1}{3}}}{(2x-3)-1}$

$\times \frac{2}{3} \lim_{(3x-5)\to 1} \frac{(3x-5)-1}{(3x-5)^{\frac{1}{5}}-(1)^{\frac{1}{5}}}$

$= \frac{1}{3}(1)^{\frac{-2}{3}} \times \frac{2}{3} \times 5(1)^{\frac{4}{5}} = \frac{10}{9}$

(5) $\lim_{x\to 2} \frac{(x^2-3)^5-1}{(x-2)(x+2)} \times (x+2)$

$= \lim_{(x^2-3)\to 1} \frac{(x^2-3)^5-(1)^5}{(x^2-3)-1} \times \lim_{x\to 2}(x+2)$

$= 5(1)^4 \times 4 = 20$

(6) $\lim_{x\to -1} \frac{(4x+5)^6-1+7x+7}{x+1}$

$= \lim_{x\to -1} \frac{(4x+5)^6-1}{x+1} + \lim_{x\to -1} \frac{7x+7}{x+1}$

$= 4 \lim_{x\to -1} \frac{(4x+5)^6-1}{4x+4} + \lim_{x\to -1} 7\left(\frac{x+1}{x+1}\right)$

$= 4 \lim_{(4x+5)\to 1} \frac{(4x+5)^6-(1)^6}{(4x+5)-(1)} + 7$

$= 4 \times 6(1)^5 + 7 = 31$

(7) $\lim_{x\to 1} \frac{(5x-4)^{10}-1}{x-1} + \lim_{x\to 1} \frac{4x-4}{x-1}$

$= 5 \lim_{(5x-4)\to 1} \frac{(5x-4)^{10}-(1)^{10}}{(5x-4)-(1)} + \lim_{x\to 1} \frac{4(x-1)}{(x-1)}$

$= 5 \times 10(1)^9 + 4 = 54$

(8) $\lim_{x\to 0} \frac{\sqrt[5]{x+1}-1-\sqrt[4]{x+1}+1}{x}$

$= \lim_{x\to 0} \frac{\sqrt[5]{x+1}-1}{x} - \lim_{x\to 0} \frac{\sqrt[4]{x+1}-1}{x}$

$= \lim_{x\to 0} \frac{(x+1)^{\frac{1}{5}}-(1)^{\frac{1}{5}}}{x}$

$- \lim_{x\to 0} \frac{(x+1)^{\frac{1}{4}}-(1)^{\frac{1}{4}}}{x}$

$= \frac{1}{5}(1)^{-\frac{4}{5}} - \frac{1}{4}(1)^{-\frac{3}{4}} = -\frac{1}{20}$

(9) $\lim_{x\to 0} \frac{(x+1)^{\frac{1}{5}}-1-(x+1)^7+1}{x}$

$= \lim_{x\to 0} \frac{(x+1)^{\frac{1}{5}}-(1)^{\frac{1}{5}}}{x}$

$- \lim_{x\to 0} \frac{(x+1)^7-(1)^7}{x} = \frac{1}{5} - 7 = -6\frac{4}{5}$

(10) $\lim_{x\to 2} \frac{5\sqrt{x-1}-5}{x-2} + \lim_{x\to 2} \frac{\sqrt[3]{x-1}-1}{x-2}$

$= 5 \lim_{(x-1)\to 1} \frac{(x-1)^{\frac{1}{2}}-(1)^{\frac{1}{2}}}{(x-1)-(1)}$

$+ \lim_{(x-1)\to 1} \frac{(x-1)^{\frac{1}{3}}-(1)^{\frac{1}{3}}}{(x-1)-(1)} = \frac{5}{2} + \frac{1}{3} = \frac{17}{6}$

(11) $\lim_{x\to 1} \frac{x^7-(1)^7}{x-1} + \lim_{x\to 1} \frac{x^9-(1)^9}{x-1}$

$+ \lim_{x\to 1} \frac{x^{11}-(1)^{11}}{x-1} = 7 + 9 + 11 = 27$

(12) $\lim_{x\to 2} \frac{x^4(x-1)^6 - x^4 + x^4 - 16}{x-2}$

$= \lim_{x\to 2} x^4\left[\frac{(x-1)^6-1}{x-2}\right] + \lim_{x\to 2} \frac{x^4-16}{x-2}$

$= \left(\lim_{x\to 2} x^4\right)\left(\lim_{x-1\to 1} \frac{(x-1)^6-(1)^6}{(x-1)-1}\right)$

$+ \lim_{x\to 2} \frac{x^4-2^4}{x-2}$

$= 16 \times 6(1)^5 + 4(2)^3 = 128$

(13) $\lim_{x\to 2} \frac{x^5\sqrt[3]{x+6} - 2x^5 + 2x^5 - 64}{x-2}$

$= \lim_{x\to 2} \frac{x^5\left[\sqrt[3]{x+6}-2\right]}{x-2} + \lim_{x\to 2} \frac{2(x^5-32)}{x-2}$

$= \left(\lim_{x\to 2} x^5\right)\left(\lim_{x+6\to 8} \frac{(x+6)^{\frac{1}{3}}-(8)^{\frac{1}{3}}}{(x+6)-(8)}\right)$

$+ 2 \lim_{x\to 2} \frac{x^5-2^5}{x-2}$

$= 2^5 \times \frac{1}{3} \times (8)^{\frac{1}{3}-1} + 2 \times 5 \times 2^{5-1} = 162\frac{2}{3}$

Exercise 15

First Multiple choice questions

(1) d	(2) b	(3) d	(4) d	(5) a	(6) c
(7) d	(8) c	(9) a	(10) c	(11) c	(12) c
(13) b	(14) d	(15) a	(16) d	(17) a	(18) c
(19) d	(20) b	(21) b	(22) b	(23) d	(24) c
(25) c	(26) c	(27) b	(28) a	(29) a	(30) d
(31) b	(32) b	(33) c	(34) c		

Second Essay questions

1

(1) By dividing both of numerator and denominator by x

$$\lim_{x \to \infty} \frac{2 - \frac{5}{x}}{3 + \frac{8}{x}} = \frac{2}{3}$$

(2) By dividing both of numerator and denominator by x^2

$$\lim_{x \to \infty} \frac{\frac{2}{x} - \frac{5}{x^2}}{3 + \frac{8}{x^2}} = \text{zero}$$

(3) By dividing both of numerator and denominator by x

$$\lim_{x \to \infty} \frac{2x - \frac{5}{x}}{3 + \frac{8}{x}} = \infty$$

(4) By dividing both of numerator and denominator by x, we get: $\lim_{x \to \infty} \frac{\frac{4}{x} - 2}{3 - \frac{1}{x}} = \frac{-2}{3}$

(5) By dividing both of numerator and denominator by x^2, we get: $\lim_{x \to \infty} \frac{\frac{5}{x^2} - \frac{6}{x} - 3}{2 + \frac{1}{x} + \frac{4}{x^2}} = -\frac{3}{2}$

(6) At $x \to \infty$, then $|x| \to x$

$$\therefore \lim_{x \to \infty} \frac{x^3 - 2}{x^3 + 1}$$

By dividing both of numerator and denominator by x^3, we get: $\lim_{x \to \infty} \frac{1 - \frac{2}{x^3}}{1 + \frac{1}{x^3}} = 1$

(7) By dividing both of numerator and denominator by x^3, we get: $\lim_{x \to \infty} \frac{\frac{6}{x^2} - 4}{\frac{2}{x^3} - 7} = \frac{-4}{-7} = \frac{4}{7}$

(8) By dividing both of numerator and denominator by x^5, we get: $\lim_{x \to \infty} \frac{4 + \frac{5}{x^5}}{8 + \frac{1}{x} - \frac{2}{x^5}} = \frac{4}{8} = \frac{1}{2}$

(9) By dividing both of numerator and denominator by x^3, we get: $\lim_{x \to \infty} \frac{\frac{7}{x} + \frac{1}{x^3}}{4 - \frac{8}{x^2} + \frac{1}{x^3}} = \text{zero}$

(10) By dividing both of numerator and denominator by x^4, we get: $\lim_{x \to \infty} \frac{5x^3 + \frac{2}{x^3} - \frac{1}{x^4}}{6 + \frac{13}{x^4}} = \infty$

(11) By dividing both of numerator and denominator by x^{14}, we get: $\lim_{x \to \infty} \frac{\frac{5}{x^{14}} - \frac{7}{x^6} + 3}{\frac{7}{x^{14}} - 6 + \frac{2}{x^8}} = -\frac{1}{2}$

(12) $\lim_{x \to \infty} \left(\frac{7}{x^2} + \frac{2}{x} - 3\right) = -3$

(13) $\lim_{x \to \infty} \frac{\frac{5}{x^3} + \frac{4}{x^2} - 3}{\frac{7}{x^3} - \frac{2}{x^2} + 8} = \frac{-3}{8}$

(14) At $x \to \infty$, then $|2x|^3 \to (2x)^3$

$$\therefore \lim_{x \to \infty} \frac{5x^3 - 4x^2 + 2}{7 - x + 8x^3}$$

By dividing both of numerator and denominator by x^3, we get: $\lim_{x \to \infty} \frac{5 - \frac{4}{x} + \frac{2}{x^3}}{\frac{7}{x^3} - \frac{1}{x^2} + 8} = \frac{5}{8}$

(15) $\lim_{x \to \infty} (x^3 + 5x^2 + 1) = \infty + \infty + 1 = \infty$

(16) $\lim_{x \to \infty} \left[\sqrt{x^2 + 5x + 7} + x\right] = \sqrt{\infty + \infty + 7} + \infty = \infty$

(17) $\lim_{x \to \infty} (5 + x - x^2) = 5 + \infty - \infty$

= unspecified quantity

$$\therefore \lim_{x \to \infty} x^2\left(\frac{5}{x^2} + \frac{1}{x} - 1\right)$$

$$= \lim_{x \to \infty} x^2 \times \lim_{x \to \infty} \left(\frac{5}{x^2} + \frac{1}{x} - 1\right)$$

$$= \infty \times -1 = -\infty$$

2

(1) By dividing both of numerator and denominator by x^2, we get :

$$\lim_{x\to\infty} \frac{\left(3+\frac{9}{x}\right)\left(2+\frac{7}{x}\right)}{\left(1+\frac{1}{x}\right)\left(5+\frac{4}{x}\right)} = \frac{3\times 2}{1\times 5} = \frac{6}{5}$$

(2) By dividing both of numerator and denominator by x^2, we get : $\lim_{x\to\infty} \frac{3-\frac{4}{x}+\frac{5}{x^2}}{\left(1+\frac{2}{x}\right)^2} = \frac{3}{1} = 3$

(3) By dividing both of numerator and denominator by x^2, we get : $\lim_{x\to\infty} \frac{6-\frac{5}{x}}{\left(\frac{3}{x}-1\right)\left(\frac{2}{x}+1\right)} = \frac{6}{-1} = -6$

(4) By dividing both of numerator and denominator by x^3, we get : $\lim_{x\to\infty} \frac{8-\frac{1}{x^2}+\frac{1}{x^3}}{\left(1+\frac{1}{x}\right)\left(2-\frac{3}{x^2}\right)} = \frac{8}{2} = 4$

(5) By dividing both of numerator and denominator by x^3, we get : $\lim_{x\to\infty} \frac{1-\frac{4}{x^2}+\frac{5}{x^3}}{\left(2-\frac{1}{x}\right)^3} = \frac{1}{8}$

(6) By dividing both of numerator and denominator by x^3, we get : $\lim_{x\to\infty} \frac{\left(2+\frac{3}{x}\right)\left(4-\frac{5}{x^2}\right)}{\left(3-\frac{8}{x^2}\right)\left(5-\frac{3}{x}\right)} = \frac{8}{15}$

(7) By dividing both of numerator and denominator by x^3,

we get : $\lim_{x\to\infty} \frac{\left(2+\frac{3}{x}\right)\left(5-\frac{1}{x}\right)\left(1-\frac{2}{x}\right)}{1\times\left(1+\frac{1}{x}\right)\left(3-\frac{1}{x}\right)} = \frac{10}{3}$

(8) By dividing both of numerator and denominator by $x = \sqrt{x^2}$

we get : $\lim_{x\to\infty} \frac{\left(\frac{7}{\sqrt{x}}+1\right)\left(\frac{3}{\sqrt{x}}+1\right)}{4-\frac{3}{x}} = \frac{1}{4}$

(9) By dividing both of numerator and denominator by x^3,

we get : $\lim_{x\to\infty} \frac{\left(6-\frac{5}{x}\right)\left(3-\frac{4}{x}\right)^2}{\left(3-\frac{2}{x}\right)^3} = \frac{6\times 9}{27} = 2$

(10) By dividing both of numerator and denominator by x^5,

we get : $\lim_{x\to\infty} \frac{\left(1+\frac{2}{x}\right)^3\left(\frac{3}{x^2}-2\right)}{3\left(1+\frac{7}{x^2}\right)}$

$= \frac{1^3\times -2}{3\times 1} = -\frac{2}{3}$

(11) By dividing both of numerator and denominator by $x^6 = (x^2)^3$,

we get : $\lim_{x\to\infty} \frac{\left(3-\frac{1}{x^2}\right)^3}{18+\frac{1}{x^4}+\frac{1}{x^6}} = \frac{27}{18} = \frac{3}{2}$

3

(1) By dividing both of numerator and denominator by $x = \sqrt{x^2}$, we get : $\lim_{x\to\infty} \frac{2+\frac{3}{x}}{\sqrt{9+\frac{25}{x^2}}} = \frac{2}{3}$

(2) By dividing both of numerator and denominator by x we get : $\lim_{x\to\infty} \frac{\sqrt{\frac{3}{x^2}+4}}{1} = \sqrt{4} = 2$

(3) By dividing both of numerator and denominator by $x = \sqrt{x^2}$, we get : $\lim_{x\to\infty} \frac{2+\frac{1}{x}}{\sqrt{4+\frac{3}{x}-\frac{4}{x^2}}} = \frac{2}{2} = 1$

(4) By dividing both of numerator and denominator by $x = \sqrt[3]{x^3}$, we get : $\lim_{x\to\infty} \frac{2-\frac{3}{x}}{\sqrt[3]{125+\frac{5}{x^3}}} = \frac{2}{5}$

(5) By dividing both of numerator and denominator by $x = \sqrt[3]{x^3}$, we get : $\lim_{x\to\infty} \frac{\sqrt[3]{8+\frac{5}{x^2}-\frac{2}{x^3}}}{3+\frac{2}{x}} = \frac{2}{3}$

(6) By dividing both of numerator and denominator by $x^3 = \sqrt{x^6}$, we get : $\lim_{x\to\infty} \frac{\frac{4}{x^3}-3}{\sqrt{1+\frac{9}{x^6}}} = -3$

(7) By dividing both of numerator and denominator by $x = \sqrt{x^2} = \sqrt[3]{x^3}$,

we get : $\lim_{x\to\infty} \frac{\sqrt{9-\frac{3}{x}+\frac{8}{x^2}}}{\sqrt[3]{\frac{3}{x}+125+\frac{2}{x^3}}} = \frac{3}{5}$

(8) By dividing both of numerator and denominator by $\sqrt{x^2}=\sqrt[4]{x^4}$,

we get: $\lim\limits_{x\to\infty} \dfrac{\sqrt{1+\frac{1}{x^2}}}{\sqrt[4]{1+\frac{2}{x^4}}} = \frac{1}{1} = 1$

(9) By dividing both of numerator and denominator by $x=\sqrt[3]{x^3}=\sqrt[5]{x^5}$,

we get: $\lim\limits_{x\to\infty} \dfrac{\sqrt[3]{8+\frac{1}{x^3}}}{\sqrt[5]{32+\frac{1}{x^4}}} = \frac{2}{2} = 1$

(10) By dividing both of numerator and denominator by $\sqrt[3]{x^2}=\sqrt[6]{x^4}$,

we get: $\lim\limits_{x\to\infty} \dfrac{\sqrt[3]{8+\frac{1}{x^2}}}{\sqrt[6]{1-\frac{3}{x^4}}} = 2$

(11) By dividing both of numerator and denominator by $\sqrt{x}$,

we get: $\lim\limits_{x\to\infty} \dfrac{\sqrt{3-\frac{2}{x}}-\sqrt{12+\frac{7}{x}}}{\sqrt{27-\frac{5}{x}}}$

$= \dfrac{\sqrt{3}-2\sqrt{3}}{3\sqrt{3}} = \dfrac{-1}{3}$

(12) By dividing both of numerator and denominator by $x=\sqrt{x^2}$,

we get: $\lim\limits_{x\to\infty} \dfrac{\sqrt{4+\frac{7}{x^2}}+3}{2+\frac{9}{x}} = \frac{5}{2}$

(13) By dividing both of numerator and denominator by $\sqrt[4]{x^6}=\sqrt[6]{x^9}=\sqrt{x^3}$

we get: $\lim\limits_{x\to\infty} \dfrac{\sqrt[4]{1+\frac{8}{x^6}}-\sqrt[6]{\frac{1}{x^7}+\frac{4}{x^9}}}{\sqrt{1+\frac{9}{x^3}}} = 1$

(14) By dividing both of numerator and denominator by $x^2=x\sqrt{x^2}$,

we get: $\lim\limits_{x\to\infty} \dfrac{3-\frac{1}{x}-\frac{7}{x^2}}{\left(1+\frac{1}{x}\right)\sqrt{25+\frac{1}{x^2}}} = \frac{3}{5}$

4

(1) $\lim\limits_{x\to\infty} 7 + \lim\limits_{x\to\infty} \dfrac{2x^2}{(x+3)^2} = 7 + \lim\limits_{x\to\infty} \dfrac{2}{\left(1+\frac{3}{x}\right)^2}$

$= 7 + 2 = 9$

(2) $\lim\limits_{x\to\infty} \dfrac{x}{2x+1} + \lim\limits_{x\to\infty} \dfrac{3x^2}{(x-3)^2}$

$= \lim\limits_{x\to\infty} \dfrac{1}{2+\frac{1}{x}} + \lim\limits_{x\to\infty} \dfrac{3}{\left(1-\frac{3}{x}\right)^2} = \frac{1}{2}+3 = \frac{7}{2}$

(3) $\lim\limits_{x\to\infty} \frac{2}{3} - \lim\limits_{x\to\infty} \dfrac{3x}{2x+7}$

$= \frac{2}{3} - \lim\limits_{x\to\infty} \dfrac{3}{2+\frac{7}{x}} = \frac{2}{3} - \frac{3}{2} = -\frac{5}{6}$

(4) $\lim\limits_{x\to\infty} \dfrac{2x^3-2x^3-x}{2x^2+1} = \lim\limits_{x\to\infty} \dfrac{-x}{2x^2+1} = 0$

(5) $\lim\limits_{x\to\infty} \dfrac{(x^2-1)(x-2)-(x^2+1)(x+2)}{x^2-4}$

$= \lim\limits_{x\to\infty} \dfrac{x^3-2x^2-x+2-x^3-2x^2-x-2}{x^2-4}$

$= \lim\limits_{x\to\infty} \dfrac{-4x^2-2x}{x^2-4} = \lim\limits_{x\to\infty} \dfrac{-4-\frac{2}{x}}{1-\frac{4}{x^2}} = -4$

(6) $\lim\limits_{x\to\infty} \dfrac{\left(\sqrt{x^2-2}-\sqrt{x^2+x}\right)\left(\sqrt{x^2-2}+\sqrt{x^2+x}\right)}{\sqrt{x^2-2}+\sqrt{x^2+x}}$

$= \lim\limits_{x\to\infty} \dfrac{x^2-2-x^2-x}{\sqrt{x^2-2}+\sqrt{x^2+x}}$

$= \lim\limits_{x\to\infty} \dfrac{-2-x}{\sqrt{x^2-2}+\sqrt{x^2+x}}$

$= \lim\limits_{x\to\infty} \dfrac{\frac{-2}{x}-1}{\sqrt{1-\frac{2}{x^2}}+\sqrt{1+\frac{1}{x}}} = -\frac{1}{2}$

(7) $\lim\limits_{x\to\infty} \dfrac{\sqrt{4x^2+1}-\sqrt{x^2+1}}{x}$

$= \lim\limits_{x\to\infty} \dfrac{\sqrt{4+\frac{1}{x^2}}-\sqrt{1+\frac{1}{x^2}}}{1} = \dfrac{2-1}{1} = 1$

(8) $\lim\limits_{x\to\infty} \dfrac{\left(\sqrt{x^2+5x}-x\right)\left(\sqrt{x^2+5x}+x\right)}{\left(\sqrt{x^2+5x}+x\right)}$

$= \lim\limits_{x\to\infty} \dfrac{x^2+5x-x^2}{\sqrt{x^2+5x}+x} = \lim\limits_{x\to\infty} \dfrac{5x}{\sqrt{x^2+5x}+x}$

$= \lim\limits_{x\to\infty} \dfrac{5}{\sqrt{1+\frac{5}{x}}+1} = \dfrac{5}{1+1} = \frac{5}{2}$

(9) $\lim\limits_{x\to\infty} \frac{x\left(\sqrt{4x^2+1}-2x\right)\left(\sqrt{4x^2+1}+2x\right)}{\left(\sqrt{4x^2+1}+2x\right)}$

$= \lim\limits_{x\to\infty} \frac{x(4x^2+1-4x^2)}{\left(\sqrt{4x^2+1}+2x\right)}$

$= \lim\limits_{x\to\infty} \frac{1}{\sqrt{4+\frac{1}{x^2}}+2} = \frac{1}{4}$

(10) By dividing both of numerator and denominator by $x^{17} = (x^5)(x^2)^6 = (x^7)(x^2)^5$, we get :

$$\lim\limits_{x\to\infty} \frac{\left(2-\frac{1}{x}\right)^5\left(1+\frac{3}{x^2}\right)^6}{\left(1+\frac{1}{x}\right)^7\left(1-\frac{5}{x^2}\right)^5} = \frac{(2)^5(1)^6}{(1)^7(1)^5} = 32$$

Another solution :

$$\lim\limits_{x\to\infty} \frac{\left[x\left(2-\frac{1}{x}\right)\right]^5\left[x^2\left(1+\frac{3}{x^2}\right)\right]^6}{\left[x\left(1+\frac{1}{x}\right)\right]^7\left[x^2\left(1-\frac{5}{x^2}\right)\right]^5}$$

$$= \lim\limits_{x\to\infty} \frac{x^5\left(2-\frac{1}{x}\right)^5 \times x^{12}\left(1+\frac{3}{x^2}\right)^6}{x^7\left(1+\frac{1}{x}\right)^7 \times x^{10}\left(1-\frac{5}{x^2}\right)^5}$$

$$= \lim\limits_{x\to\infty} \frac{x^{17}\left(2-\frac{1}{x}\right)^5\left(1+\frac{3}{x^2}\right)^6}{x^{17}\left(1+\frac{1}{x}\right)^7\left(1-\frac{5}{x^2}\right)^5} = \frac{(2)^5\times(1)^6}{(1)^7\times(1)^5}$$

$= 32$

(11) By multiplying both of numerator and denominator by their conjugates :

$$\therefore \lim\limits_{x\to\infty} \frac{\left(\sqrt{x+1}-\sqrt{x-1}\right)\left(\sqrt{x+1}+\sqrt{x-1}\right)\left(\sqrt{4x+1}+\sqrt{4x-1}\right)}{\left(\sqrt{4x+1}-\sqrt{4x-1}\right)\left(\sqrt{x+1}+\sqrt{x-1}\right)\left(\sqrt{4x+1}+\sqrt{4x-1}\right)}$$

$$= \lim\limits_{x\to\infty} \frac{(x+1-x+1)\left(\sqrt{4x+1}+\sqrt{4x-1}\right)}{(4x+1-4x+1)\left(\sqrt{x+1}+\sqrt{x-1}\right)}$$

$$= \lim\limits_{x\to\infty} \frac{\sqrt{4x+1}+\sqrt{4x-1}}{\sqrt{x+1}+\sqrt{x-1}}$$

By dividing both of numerator and denominator by $\sqrt{x}$, we get : $\lim\limits_{x\to\infty} \frac{\sqrt{4+\frac{1}{x}}+\sqrt{4-\frac{1}{x}}}{\sqrt{1+\frac{1}{x}}+\sqrt{1-\frac{1}{x}}} = \frac{2+2}{1+1} = 2$

5

(1) By dividing both of numerator and denominator by x^2 ,

we get : $\lim\limits_{x\to\infty} \left(\frac{3-\frac{5}{x}+\frac{1}{x^2}}{4-\frac{7}{x^2}}\right)^{\frac{1}{x}} = \left(\frac{3}{4}\right)^0 = 1$

(2) $\lim\limits_{x\to\infty} \frac{x+1}{\sqrt{x^2-1}} + \lim\limits_{x\to\infty} a^{\frac{1}{x}}$

$= \lim\limits_{x\to\infty} \left(\frac{1+\frac{1}{x}}{\sqrt{1-\frac{1}{x^2}}}\right) + a^{zero} = 1+1 = 2$

(3) $\lim\limits_{x\to\infty} \left(\frac{2-\frac{3}{x}}{\sqrt[3]{27-\frac{15}{x^2}+\frac{2}{x^3}}}\right) + \lim\limits_{x\to\infty} 8^{\frac{1}{x}}$

$= \frac{2}{3} + 1 = 1\frac{2}{3}$

(4) $\lim\limits_{x\to\infty} \left(\frac{5}{\frac{1}{x^2}-1}\right) - \lim\limits_{x\to\infty} 13^{\frac{1}{x}} = -5-1 = -6$

6

$\because$ The limit is exist and equals 3

$\therefore$ The degree of numerator = the degree of denominator

$\therefore n = 2$ $\qquad \therefore \lim\limits_{x\to\infty} \frac{4ax^2-4x+5}{3-9x+8x^2} = 3$

By dividing both of numerator and denominator by x^2 , we get :

$\lim\limits_{x\to\infty} \frac{4a-\frac{4}{x}+\frac{5}{x^2}}{\frac{3}{x^2}-\frac{9}{x}+8} = 3 \quad \therefore \frac{4a}{8} = 3 \quad \therefore a = 6$

7

$\lim\limits_{x\to\infty} \frac{\sqrt[3]{a+\frac{3}{x^3}}}{\sqrt{4+\frac{7}{x^2}}} = -1$

$\therefore \frac{\sqrt[3]{a}}{2} = -1 \qquad \therefore a = -8$

8

$\lim\limits_{x\to\infty} \frac{\left(\sqrt{ax^2+3bx+5}-2x\right)}{1} \times \frac{\left(\sqrt{ax^2+3bx+5}+2x\right)}{\left(\sqrt{ax^2+3bx+5}+2x\right)}$

$= \lim\limits_{x\to\infty} \frac{ax^2+3bx+5-4x^2}{\sqrt{ax^2+3bx+5}+2x} = 3$

$\because$ The greatest degree at denominator is 1

$\therefore$ The degree of the numerator = 1

$\therefore ax^2 - 4x^2 = 0 \qquad \therefore a = 4$

$\therefore \lim\limits_{x\to\infty} \frac{3bx+5}{\sqrt{ax^2+3bx+5}+2x} = 3$

By dividing both of numerator and denominator by x

$$\therefore \lim_{x \to \infty} \frac{3b + \frac{5}{x}}{\left(\sqrt{a + \frac{3b}{x} + \frac{5}{x^2}} + 2\right)} = 3$$

$\therefore \frac{3b}{\sqrt{a+2}} = 3 \qquad \therefore 3b = 3 \times (2+2) \qquad \therefore b = 4$

9

$$\lim_{x \to \infty} \frac{2 - a x^2}{3 - b x + x^2} = 5$$

$$\therefore \lim_{x \to \infty} \frac{\frac{2}{x^2} - a}{\frac{3}{x^2} - \frac{b}{x} + 1} = 5 \qquad \therefore a = -5$$

$$, \because \lim_{x \to -2} \frac{2 + 5x^2}{3 - b x + x^2} = 2$$

$\therefore \frac{2 + 5(-2)^2}{3 + 2b + (-2)^2} = 2 \qquad \therefore 7 + 2b = 11 \qquad \therefore b = 2$

10

$$\lim_{x \to \infty} \frac{x^2 + 1 - (a x + b)(x + 1)}{x + 1} = 2$$

$$\therefore \lim_{x \to \infty} \frac{x^2 - a x^2 - (a+b) x + 1 - b}{x + 1} = 2$$

, $\because$ the limit is exist and equals 2

$\therefore$ The degree of the numerator

= the degree of the denominator

$\therefore x^2 - a x^2 = 0 \qquad \therefore a = 1$

$, a + b = -2 \qquad \therefore b = -3$

11

$\because$ The limit $= \infty$

$\therefore$ The degree of the numerator is greater than the degree of the denominator.

$\therefore a + 1 =$ zero $\qquad \therefore a = -1$

$, 7 - b =$ zero $\qquad \therefore b = 7$

Third Higher skills

(1) By dividing both of numerator and denominator by x^{-2}, we get: $\lim_{x \to \infty} \frac{1 + \frac{1}{x^2}}{\left(2 + \frac{3}{x}\right)^{-2}} = \frac{1}{2^{-2}} = 4$

(2) $$\lim_{\frac{1}{n} \to \text{zero}} \frac{\left(a + \frac{1}{n}\right)^4 - a^4}{\frac{1}{n}}$$

$$= \lim_{(a + \frac{1}{n}) \to a} \frac{\left(a + \frac{1}{n}\right)^4 - a^4}{\left(a + \frac{1}{n}\right) - a} = 4a^3$$

(3) $$\lim_{\frac{1}{x} \to \text{zero}} \frac{\left(3 + \frac{1}{x}\right)^5 - 243}{\frac{1}{x}}$$

$$= \lim_{(3 + \frac{1}{x}) \to 3} \frac{\left(3 + \frac{1}{x}\right)^5 - 3^5}{\left(3 + \frac{1}{x}\right) - 3} = 5(3)^4 = 405$$

(4) Multiplying by the conjugate , we get :

$$\lim_{x \to \infty} \frac{\left(\sqrt{x^2+2} - x\right)\left(\sqrt{2x^2+1}\right)\left(\sqrt{x^2+2} + x\right)}{\sqrt{x^2+2} + x}$$

$$= \lim_{x \to \infty} \frac{(x^2 + 2 - x^2)\left(\sqrt{2x^2+1}\right)}{\sqrt{x^2+2} + x}$$

$$= \lim_{x \to \infty} \frac{2\left(\sqrt{2x^2+1}\right)}{\sqrt{x^2+2} + x}$$

$$= \lim_{x \to \infty} \frac{2\left(\sqrt{2 + \frac{1}{x^2}}\right)}{\sqrt{1 + \frac{2}{x^2}} + 1} = \sqrt{2}$$

(5) $$\lim_{x \to \infty} \left(\frac{2}{x} - \frac{1}{x^2}\right)\sqrt{4x^2+1}$$

$$= \lim_{x \to \infty} \frac{2\sqrt{4x^2+1}}{x} - \lim_{x \to \infty} \frac{\sqrt{4x^2+1}}{x^2}$$

$$= \lim_{x \to \infty} \frac{2\sqrt{4 + \frac{1}{x^2}}}{1} - \lim_{x \to \infty} \frac{\sqrt{\frac{4}{x^2} + \frac{1}{x^4}}}{1}$$

$= 2 \times 2 - 0 = 4$

(6) By dividing both of numerator and denominator by $x\sqrt{x} = \sqrt{x^3}$,

we get : $$\lim_{x \to \infty} \frac{5 + \frac{16}{\sqrt{x}} - 3\sqrt{\frac{1}{x} - \frac{1}{x^3}}}{\frac{1}{\sqrt{x}} + \sqrt{4 + \frac{1}{x^3}}} = \frac{5}{2}$$

Exercise 16

First Multiple choice questions

(1) a	(2) a	(3) a	(4) b	(5) b	(6) c
(7) d	(8) a	(9) a	(10) b	(11) c	(12) c
(13) c	(14) b	(15) b	(16) c	(17) c	(18) a
(19) d	(20) d	(21) b	(22) d	(23) a	(24) b
(25) c	(26) b	(27) a	(28) a	(29) b	(30) b
(31) b	(32) a	(33) c	(34) c	(35) a	(36) c
(37) b	(38) a	(39) a	(40) a	(41) b	(42) a
(43) d	(44) c	(45) b			

Second Essay questions

1

(1) $\lim\limits_{x \to 0} \dfrac{\frac{\sin 2x}{x}}{\frac{\tan 5x}{x}} = \frac{2}{5}$

(2) $\lim\limits_{x \to 0} \dfrac{\dfrac{\sin \frac{x}{5}}{x}}{\dfrac{\tan \frac{2}{3}x}{x}} = \dfrac{\frac{1}{5}}{\frac{2}{3}} = \frac{3}{10}$

(3) $\lim\limits_{x \to 0} \frac{\sin 3x}{5x} - \lim\limits_{x \to 0} \frac{\sin 2x}{5x} = \frac{3}{5} - \frac{2}{5} = \frac{1}{5}$

(4) $\lim\limits_{x \to 0} \dfrac{\frac{\sin x}{x} + \frac{\tan x}{x}}{\frac{\sin x}{x}} = \frac{1+1}{1} = 2$

(5) $\lim\limits_{x \to 0} \dfrac{\cos 3x}{\frac{\tan 2x}{x}} = \frac{1}{2}$

(6) $\lim\limits_{x \to 0} \frac{5x}{\sin x} - \lim\limits_{x \to 0} \frac{\tan 2x}{x} = 5 - 2 = 3$

(7) $\lim\limits_{x \to 0} \dfrac{3 + \frac{\sin x}{x}}{2 + \frac{\tan 3x}{x}} = \frac{3+1}{2+3} = \frac{4}{5}$

(8) $\lim\limits_{x \to 0} \dfrac{2\left(\frac{\sin x}{x}\right) + 4\left(\frac{\tan 2x}{x}\right)}{2 + \frac{\sin 3x}{x}} = \frac{2+8}{2+3} = 2$

(9) $\lim\limits_{x \to 0} \dfrac{\frac{\sin x}{x} - \frac{3\tan x}{x}}{5\cos x} = \frac{1-3}{5} = \frac{-2}{5}$

(10) $\lim\limits_{x \to 0} \dfrac{3}{5\left(\frac{\tan x}{x}\right) - 2\left(\frac{\sin 2x}{x}\right)} = \frac{3}{5-4} = 3$

(11) By dividing both of numerator and denominator by x, we get: $\lim\limits_{x \to 0} \dfrac{1 + \cos x}{\frac{\sin x}{x} \times \cos x} = \frac{1+1}{1 \times 1} = 2$

(12) $\lim\limits_{x \to 0} \dfrac{\left(\frac{\tan 3x}{x}\right) - \left(\frac{\sin 4x}{x}\right)}{x+2} = \frac{3-4}{2} = -\frac{1}{2}$

(13) $\lim\limits_{x \to 0} \dfrac{x + \frac{\sin 3x}{x}}{5\cos 2x} = \frac{0+3}{5 \times 1} = \frac{3}{5}$

(14) $\lim\limits_{x \to 0} \dfrac{\frac{\tan 2x}{x} + 2\cos x}{\frac{\sin 3x}{x}} = \frac{2+2}{3} = \frac{4}{3}$

(15) $\lim\limits_{x \to 0} \dfrac{2\sin \frac{1}{7}x}{x} = \frac{2}{7}$

(16) $\lim\limits_{x \to 0} \left[\dfrac{\sin \frac{1}{5}x}{x} + \dfrac{\tan \frac{1}{5}x}{x}\right] = \frac{2}{5}$

(17) $\lim\limits_{x \to 0} \left(\frac{3\sin 4x}{5x} + 4\sin 4x\right) = \frac{12}{5}$

(18) $\lim\limits_{(x-3) \to 0} \dfrac{\sin 3(x-3)}{2(x-3)} = \frac{3}{2}$

(19) By dividing both of numerator and denominator by x, we get: $\lim\limits_{x \to 0} \dfrac{\frac{\sin 24x}{x} \times \cos 6x}{\frac{\tan 6x}{x} \times \cos 24x}$

$= \frac{24 \times 1}{6 \times 1} = 4$

(20) The limit $= \dfrac{\frac{1}{2}\cos\left(-2 \times \frac{1}{2} + 1\right)}{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)} = \frac{2}{3}$

(21) $\lim\limits_{x \to 0} \left[\frac{\sin^2 x}{x^2} \times x\right]$

$= \lim\limits_{x \to 0} \frac{\sin^2 x}{x^2} \times \lim\limits_{x \to 0} x = 1 \times 0 = 0$

2

(1) $\frac{1}{5} \lim\limits_{x \to 0} \frac{\sin^2 3x}{x^2} = \frac{1}{5} \times 3^2 = \frac{9}{5}$

(2) $\lim\limits_{x \to 0} \dfrac{\frac{\sin^2 3x}{x^2}}{\frac{\tan^2 2x}{x^2}} = \lim\limits_{x \to 0} \dfrac{\left(\frac{\sin 3x}{x}\right)^2}{\left(\frac{\tan 2x}{x}\right)^2} = \frac{9}{4}$

(3) $\lim\limits_{x\to 0} \dfrac{\dfrac{\sin^2 \frac{x}{2}}{x^2}}{\dfrac{\tan 2x}{x}}$

$= \lim\limits_{x\to 0} \dfrac{\left(\dfrac{\sin \frac{x}{2}}{x}\right)^2}{\left(\dfrac{\tan 2x}{x}\right)} = \frac{1}{8}$

(4) $\lim\limits_{x\to 0} \dfrac{\dfrac{x \sin 2x}{x \times x}}{\dfrac{\sin x}{x} \times \dfrac{\tan 3x}{x}}$

$= \lim\limits_{x\to 0} \dfrac{\left(\dfrac{\sin 2x}{x}\right)}{\left(\dfrac{\sin x}{x}\right)\left(\dfrac{\tan 3x}{x}\right)} = \frac{2}{3}$

(5) $\frac{1}{4} \lim\limits_{x\to 0} \left(\dfrac{\tan 2x}{x}\right)^3 = \frac{8}{4} = 2$

(6) $\lim\limits_{x\to 0} \dfrac{\dfrac{\tan 2x}{x}}{\left(\dfrac{\sin 2x}{x}\right)^4} = \frac{1}{8}$

(7) $\lim\limits_{x\to 0} \dfrac{\dfrac{\sin 4x}{x} \times \dfrac{\tan^2 5x}{x^2}}{\dfrac{\sin x}{x}} = \dfrac{4 \times 25}{1} = 100$

(8) $\lim\limits_{x\to 0} \dfrac{\dfrac{\tan 2x}{x}}{1 + \dfrac{\sin^2 3x}{x^2}} = \lim\limits_{x\to 0} \dfrac{\dfrac{\tan 2x}{x}}{1 + \left(\dfrac{\sin 3x}{x}\right)^2}$

$= \dfrac{2}{1+9} = \dfrac{1}{5}$

(9) $\lim\limits_{x\to 0} \dfrac{1 + \dfrac{\tan^2 2x}{x^2}}{2 + \dfrac{\sin^2 3x}{x^2}} = \lim\limits_{x\to 0} \dfrac{1 + \left(\dfrac{\tan 2x}{x}\right)^2}{2 + \left(\dfrac{\sin 3x}{x}\right)^2}$

$= \dfrac{1+4}{2+9} = \dfrac{5}{11}$

(10) $\lim\limits_{x\to 0} \dfrac{\dfrac{\tan x}{x}}{4 - \dfrac{\sin^2 3x}{x^2}} = \lim\limits_{x\to 0} \dfrac{\dfrac{\tan x}{x}}{4 - \left(\dfrac{\sin 3x}{x}\right)^2}$

$= \dfrac{1}{4-9} = \dfrac{-1}{5}$

(11) $\lim\limits_{x\to 0} \dfrac{2x + \dfrac{\sin 5x}{x}}{1 - \dfrac{\tan 3x^2}{x^2}} = \dfrac{0+5}{1-3} = -\dfrac{5}{2}$

(12) $\lim\limits_{x\to 0} \dfrac{2 + 5\dfrac{\tan x}{x} + 3\dfrac{\sin x}{x}}{\dfrac{\sin 2x}{x} \times \cos 2x}$

$= \dfrac{2+5+3}{2 \times 1} = \dfrac{10}{2} = 5$

(13) $\lim\limits_{x\to 0} \dfrac{\dfrac{\sin 2x}{x} + \dfrac{\sin^2 2x}{x^2}}{\dfrac{\tan^2 3x}{x^2} + 1} = \dfrac{2+4}{9+1} = \dfrac{6}{10} = \dfrac{3}{5}$

(14) $\lim\limits_{x\to 0} \left(\dfrac{\tan 3x^2}{x^2} + \dfrac{\sin^2 5x}{x^2}\right) = 3 + 25 = 28$

(15) $\lim\limits_{x\to 0} \left(\dfrac{\sin 5x^3}{2x^3} + \dfrac{\sin^3 5x}{2x^3}\right) = \dfrac{5}{2} + \dfrac{125}{2} = 65$

(16) $\left(\lim\limits_{x\to 0} \dfrac{2x + \dfrac{\sin 3x}{x}}{2x + \dfrac{\tan 6x}{x}}\right)^4 = \left(\dfrac{1}{2}\right)^4 = \dfrac{1}{16}$

(17) $\lim\limits_{x\to 0} \left(\dfrac{\tan x}{x}\right)^2 + \lim\limits_{x\to 0} \left(\dfrac{\tan 3x}{x}\right)^2$

$+ \lim\limits_{x\to 0} \left(\dfrac{\tan 5x}{x}\right)^2 = 1 + 9 + 25 = 35$

(18) $\lim\limits_{x\to 0} \dfrac{\sin x}{x} \times \lim\limits_{x\to 0} \left(\dfrac{\sin 2x}{x}\right)^2$

$\times \lim\limits_{x\to 0} \left(\dfrac{\sin 4x}{x}\right)^3 = 1 \times 4 \times 64 = 256$

(19) $\lim\limits_{x\to 0} \dfrac{\dfrac{\tan 6x}{x} - \dfrac{\sin(-3x)}{x}}{\cos 5x + \cos 2x} = \dfrac{6-(-3)}{1+1} = \dfrac{9}{2}$

(20) $\lim\limits_{x\to 0} \dfrac{\sin x(\sqrt{x+2} + \sqrt{2})}{(\sqrt{x+2} - \sqrt{2})(\sqrt{x+2} + \sqrt{2})}$

$= \lim\limits_{x\to 0} \dfrac{\sin x(\sqrt{x+2} + \sqrt{2})}{(x+2-2)}$

$= \lim\limits_{x\to 0} \dfrac{\sin x}{x} \times \lim\limits_{x\to 0} \left(\sqrt{x+2} + \sqrt{2}\right)$

$= 1 \times 2\sqrt{2} = 2\sqrt{2}$

(21) $\lim\limits_{(x-3)\to 0} \dfrac{\tan(x-3)}{x-3} \times \dfrac{1}{x^2+3x+9} = \dfrac{1}{27}$

(22) $\lim\limits_{(x-1)\to 0} \dfrac{\sin(x-1)}{x-1} \times \lim\limits_{x\to 1} \dfrac{1}{x+2}$

$= 1 \times \dfrac{1}{3} = \dfrac{1}{3}$

3

(1) $\lim\limits_{x\to 0} \dfrac{3(1-\cos x)}{x} = 3 \times 0 = 0$

(2) $\lim\limits_{x\to 0} \left(\dfrac{-(1-\cos x)}{x} \times \dfrac{x}{\sin x}\right) = 0 \times 1 = 0$

(3) $\lim\limits_{x\to 0} \dfrac{\dfrac{1-\cos x}{x} + \dfrac{\sin x}{x}}{\dfrac{1-\cos x}{x} - \dfrac{\sin x}{x}} = \dfrac{0+1}{0-1} = -1$

(4) $\lim_{x\to 0}\left(\frac{1-\cos 3x}{x}+\frac{1-\cos 4x}{x}\right)$

$=\lim_{3x\to 0}3\left(\frac{1-\cos 3x}{3x}\right)+\lim_{4x\to 0}4\left(\frac{1-\cos 4x}{4x}\right)$

$=0+0=0$

(5) $\lim_{x\to 0}\left(\frac{\sin x}{x}\times\frac{1-\cos x}{x}\right)=1\times 0=0$

(6) $\lim_{x\to 0}\frac{x(1-\cos x)}{\sin^2 3x}$

$=\lim_{x\to 0}\left(\frac{x^2}{\sin^2 3x}\times\frac{1-\cos x}{x}\right)=\frac{1}{9}\times 0=0$

(7) $\lim_{x\to 0}\frac{\sin^2 3x}{4x^2}=\frac{9}{4}$

(8) $\lim_{x\to 0}\frac{4x^2}{\sin^2\frac{1}{2}x}=16$

(9) $\lim_{x\to 0}\frac{3(1-\cos^2 4x)}{8x^2}=\lim_{x\to 0}\frac{3\sin^2 4x}{8x^2}$

$=\frac{3}{8}\lim_{x\to 0}\left(\frac{\sin 4x}{x}\right)^2=\frac{3}{8}\times 16=6$

(10) $\lim_{x\to 0}\frac{\tan^2 x}{x^2}=\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^2=1$

(11) $\lim_{x\to 0}\left(\frac{(1-\cos x)}{x^2}\times\frac{(1+\cos x)}{(1+\cos x)}\right)$

$=\lim_{x\to 0}\left(\frac{1-\cos^2 x}{x^2}\times\frac{1}{1+\cos x}\right)$

$=\lim_{x\to 0}\frac{\sin^2 x}{x^2}\times\lim_{x\to 0}\frac{1}{1+\cos x}=1\times\frac{1}{2}=\frac{1}{2}$

(12) $\lim_{x\to 0}\left(\frac{4(1-\cos x)}{x^2}\times\frac{x^2}{\sin^2 3x}\right)$

$=\lim_{x\to 0}\left(\frac{4(1-\cos x)}{x^2}\times\frac{1+\cos x}{1+\cos x}\times\frac{x^2}{\sin^2 3x}\right)$

$=\lim_{x\to 0}\left(\frac{4\sin^2 x}{x^2}\times\frac{1}{1+\cos x}\times\frac{x^2}{\sin^2 3x}\right)$

$=4\times\frac{1}{2}\times\frac{1}{9}=\frac{2}{9}$

(13) $\lim_{x\to 0}\left(\frac{1-\cos 2x}{x^2}\times\frac{1+\cos 2x}{1+\cos 2x}\times\frac{x}{\tan 2x}\right)$

$=\lim_{x\to 0}\left(\frac{1-\cos^2 2x}{x^2}\times\frac{1}{1+\cos 2x}\times\frac{x}{\tan 2x}\right)$

$=\lim_{x\to 0}\left(\frac{\sin^2 2x}{x^2}\times\frac{1}{1+\cos 2x}\times\frac{x}{\tan 2x}\right)$

$=4\times\frac{1}{2}\times\frac{1}{2}=1$

(14) $\lim_{x\to 0}\frac{1-\cos 3x}{-\sin^2 2x}$

$=\lim_{x\to 0}\left(\frac{1-\cos 3x}{-\sin^2 2x}\times\frac{1+\cos 3x}{1+\cos 3x}\right)$

$=\lim_{x\to 0}\left(\frac{\sin^2 3x}{-\sin^2 2x}\times\frac{1}{1+\cos 3x}\right)$

$=\frac{-9}{4}\times\frac{1}{2}=-\frac{9}{8}$

4

(1) $\lim_{x\to 0}\frac{\sin 5x}{\tan 3x}=\lim_{x\to 0}\frac{\frac{\sin 5x}{x}}{\frac{\tan 3x}{x}}=\frac{5}{3}$

(2) $\lim_{x\to 0}x\left(\frac{1}{\sin 2x}-\frac{1}{\tan 3x}\right)$

$=\lim_{x\to 0}\left(\frac{x}{\sin 2x}-\frac{x}{\tan 3x}\right)=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$

(3) $\lim_{x\to 0}\frac{3x\tan 2x}{\sin^2 3x}=\lim_{x\to 0}\frac{3\left(\frac{\tan 2x}{x}\right)}{\left(\frac{\sin 3x}{x}\right)^2}=\frac{2}{3}$

(4) $\lim_{x\to 0}\frac{6x^2}{\sin 2x\times\tan x}$

$=\lim_{x\to 0}\left[6\times\frac{x}{\sin 2x}\times\frac{x}{\tan x}\right]$

$=6\times\frac{1}{2}\times 1=3$

(5) $\lim_{x\to 0}\frac{\sin x}{\sin 4x}=\lim_{x\to 0}\frac{\frac{\sin x}{x}}{\frac{\sin 4x}{x}}=\frac{1}{4}$

(6) $\lim_{x\to 0}\frac{x}{\sin x}=1$

(7) $\lim_{x\to 0}\frac{x\sin 3x}{\tan^2 2x}=\lim_{x\to 0}\frac{\frac{\sin 3x}{x}}{\left(\frac{\tan 2x}{x}\right)^2}=\frac{3}{4}$

(8) $\lim_{(x-\frac{\pi}{4})\to 0}\frac{\tan 2\left(x-\frac{\pi}{4}\right)}{8\left(x-\frac{\pi}{4}\right)}=\frac{2}{8}=\frac{1}{4}$

(9) $\lim_{(x-\frac{\pi}{4})\to 0}\frac{\frac{\sin 2\left(x-\frac{\pi}{4}\right)}{x-\frac{\pi}{4}}}{\frac{\tan 4\left(x-\frac{\pi}{4}\right)}{x-\frac{\pi}{4}}}=\frac{2}{4}=\frac{1}{2}$

(10) $\lim_{x\to 0}\frac{\sin x}{\tan 3x}=\lim_{x\to 0}\frac{\frac{\sin x}{x}}{\frac{\tan 3x}{x}}=\frac{1}{3}$

(11) $\lim_{(\pi-x)\to 0}\frac{\sin(\pi-x)}{-(\pi-x)}=-1$

(12) $\lim\limits_{(x+\pi)\to 0} \frac{\tan(x+\pi)}{(x+\pi)} = 1$

(13) $\lim\limits_{x\to\frac{\pi}{2}} \frac{2x-\pi}{\sin\left(\frac{\pi}{2}-x\right)} = \lim\limits_{(\frac{\pi}{2}-x)\to 0} \frac{-2\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)}$

$= -2 \lim\limits_{(\frac{\pi}{2}-x)\to 0} \frac{\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)} = -2\times 1 = -2$

(14) $\lim\limits_{(\frac{\pi}{2}-x)\to 0} \frac{\tan\left(\frac{\pi}{2}-x\right)}{2\left(\frac{\pi}{2}-x\right)} = \frac{1}{2}$

(15) $\lim\limits_{x\to 1} \frac{\sin(\pi-\pi x)}{1-x}$

$= \lim\limits_{(1-x)\to 0} \frac{\sin\pi(1-x)}{1-x} = \pi$

(16) $\lim\limits_{(x+1)\to 0} \frac{1+x}{\sin\left(\frac{\pi}{2}+\frac{\pi}{2}x\right)}$

$= \lim\limits_{(x+1)\to 0} \frac{1}{\frac{\sin\frac{\pi}{2}(1+x)}{1+x}} = \frac{2}{\pi}$

(17) $\lim\limits_{x\to 1} (1-x)\cot\left(\frac{\pi}{2}-\frac{\pi}{2}x\right)$

$= \lim\limits_{(1-x)\to 0} \frac{1-x}{\tan\left(\frac{\pi}{2}(1-x)\right)} = \frac{2}{\pi}$

5

(1) $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$ (2) $\lim\limits_{x\to\frac{\pi}{2}} \frac{\sin x}{x} = \frac{2}{\pi}$

(3) $\lim\limits_{x\to\pi} \frac{\sin x}{x} = \frac{0}{\pi} = 0$

6

(1) $\lim\limits_{x\to 5} \frac{\sin(x-5)}{(x-5)(x+5)}$

$= \lim\limits_{x\to 5} \frac{1}{x+5} \times \lim\limits_{(x-5)\to 0} \frac{\sin(x-5)}{x-5} = \frac{1}{10}$

(2) $\lim\limits_{(x-5)\to 0} \left[\frac{\sin(x-5)}{(x-5)}\right]^2 = (1)^2 = 1$

(3) $\lim\limits_{x\to 5} \left[(x-5)\times\frac{\sin(x-5)^2}{(x-5)^2}\right]$

$= \lim\limits_{x\to 5} (x-5) \times \lim\limits_{(x-5)^2\to 0} \frac{\sin(x-5)^2}{(x-5)^2}$

$= 0\times 1 = 0$

Third Higher skills

1

(1) (d) (2) (c) (3) (d) (4) (c) (5) (b)

Instructions to solve 1 :

(1) It is possible to use the given values of "a" to deduce that the limit = 1 at a = 3

(2) $\lim\limits_{x\to 0} x\sec\left(\frac{\pi}{2}+x\right) = \lim\limits_{x\to 0} (-x\csc x)$

$= \lim\limits_{x\to 0} \frac{-x}{\sin x} = -1$

(3) $\lim\limits_{x\to\frac{\pi}{4}} \frac{1-\tan x}{\cos x-\sin x}$ "Multiplying numerator and denominator by cos x"

$= \lim\limits_{x\to\frac{\pi}{4}} \frac{(\cos x-\sin x)}{\cos x(\cos x-\sin x)} = \lim\limits_{x\to\frac{\pi}{4}} \frac{1}{\cos x} = \sqrt{2}$

(4) $\lim\limits_{x\to\sqrt{5}} \frac{\sin(x^2-5)}{\tan(\pi x^2)} = \lim\limits_{x\to\sqrt{5}} \frac{\sin(x^2-5)}{-\tan(-\pi x^2)}$

$= \lim\limits_{x\to\sqrt{5}} \frac{\sin(x^2-5)}{-\tan(5\pi-\pi x^2)}$

$= \lim\limits_{x\to\sqrt{5}} \frac{\sin(x^2-5)}{\tan\left(\pi(x^2-5)\right)} = \frac{1}{\pi}$

(5) Dividing each of numerator and denominator by x

$\therefore \lim\limits_{x\to 0} \frac{\frac{\sin x}{x}+\frac{\sin 2x}{x}+\frac{\sin 3x}{x}+\dots+\frac{\sin 10x}{x}}{\frac{\tan x}{x}+\frac{\tan 2x}{x}+\frac{\tan 3x}{x}+\dots+\frac{\tan 10x}{x}}$

$= 1$

2

(1) By dividing both of numerator and denominator by $\tan^2 x$, we get :

$\lim\limits_{\cot x\to 0} \frac{3-4\cot x+7\cot^2 x}{5\cot^2 x+2} = \frac{3}{2}$

(2) $\lim\limits_{x\to -1} \frac{\sin[(x+1)(x-2)]}{x+1}$

$= \lim\limits_{(x+1)(x-2)\to 0} \frac{\sin[(x+1)(x-2)]}{(x+1)(x-2)}$

$\times \lim\limits_{x\to -1} (x-2) = 1\times(-3) = -3$

(3) $\because x \longrightarrow \infty \qquad \therefore \frac{1}{x} \longrightarrow 0$

$$\therefore \lim_{\frac{1}{x} \to 0} \frac{\tan\frac{4}{x} + \sin\frac{3}{x}}{\frac{1}{x}}$$

$$= \lim_{\frac{1}{x} \to 0} \left[\frac{\tan\frac{4}{x}}{\frac{1}{x}} + \frac{\sin\frac{3}{x}}{\frac{1}{x}}\right] = 4 + 3 = 7$$

(4) $\frac{1}{5} \lim_{\sin x \to 0} \left[\frac{\sin(\sin x)}{(\sin x)}\right] = \frac{1}{5}$

(5) $\lim_{1-\cos x \to 0} \frac{\sin(1-\cos x)}{(1-\cos x)} = 1$

(6) $\lim_{x \to 0} \left(\frac{\tan(\tan 5x)}{\tan 5x} \times \frac{\tan 5x}{4x}\right)$

$$= \lim_{\tan 5x \to 0} \frac{\tan(\tan 5x)}{\tan 5x} \times \lim_{x \to 0} \frac{\tan 5x}{4x}$$

$$= 1 \times \frac{5}{4} = \frac{5}{4}$$

(7) $\lim_{x \to 0} \left(\frac{2\tan x}{x} \times \frac{1}{1+\tan^2 x}\right) = 2 \times \frac{1}{1+0} = 2$

(8) $\lim_{x \to \frac{\pi}{2}} \left(\frac{\tan(\cos x)}{\cos x} \times \frac{\cos x}{2x - \pi}\right)$

$$= \lim_{\cos x \to 0} \frac{\tan(\cos x)}{\cos x} \times \lim_{\left(\frac{\pi}{2} - x\right) \to 0} \frac{\sin\left(\frac{\pi}{2} - x\right)}{-2\left(\frac{\pi}{2} - x\right)}$$

$$= 1 \times \frac{-1}{2} = -\frac{1}{2}$$

(9) $\lim_{x \to \frac{\pi}{2}} \frac{(1-\sin x)(1+\sin x)}{(2x-\pi)(1+\sin x)}$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\cos^2 x}{-2\left(\frac{\pi}{2} - x\right)} \times \frac{1}{1+\sin x}$$

$$= \lim_{\left(\frac{\pi}{2} - x\right) \to 0} \frac{\sin^2\left(\frac{\pi}{2} - x\right)}{-2\left(\frac{\pi}{2} - x\right)^2} \times \frac{\frac{\pi}{2} - x}{1+\sin x}$$

$$= -\frac{1}{2} \times 0 = 0$$

(10) $\lim_{\left(\frac{\pi}{2} - x\right) \to 0} \frac{\frac{\pi}{2} - x}{\tan 2\left(\frac{\pi}{2} - x\right)} = \frac{1}{2}$

(11) $\lim_{x \to \frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2} - x\right)}{-\sin\left(\frac{3\pi}{2} - 3x\right)}$

$$= \lim_{x \to \frac{\pi}{2}} \left(\frac{\sin\left(\frac{\pi}{2} - x\right)}{\frac{\pi}{2} - x} \times \frac{\frac{\pi}{2} - x}{-\sin 3\left(\frac{\pi}{2} - x\right)}\right)$$

$$= 1 \times -\frac{1}{3} = -\frac{1}{3}$$

Exercise 17

First Multiple choice questions

(1) First : b Second : a Third : d

(2) c (3) c (4) b (5) b (6) c

(7) c (8) b (9) a (10) a (11) c (12) b

(13) c (14) b (15) b (16) d (17) a (18) d

(19) b (20) a (21) a (22) c (23) c (24) d

(25) d (26) c (27) c

Second Essay questions

1

$\lim_{x \to 3^-} f(x) = (3)^2 + 1 = 10$

$\lim_{x \to 3^+} f(x) = 3(3) + 1 = 10$

$\because f(3^-) = f(3^+) \qquad \therefore \lim_{x \to 3} f(x) = 10$

2

$f(3^-) = \lim_{x \to 3^-} f(x) = 2 \times 3 - 7 = -1$

$f(3^+) = \lim_{x \to 3^+} \frac{\cancel{(x-3)}(x-4)}{\cancel{(x-3)}} = -1$

$\because f(3^-) = f(3^+) = -1$

$\therefore \lim_{x \to 3} f(x) = -1$

3

$f(0^-) = \lim_{x \to 0^-} f(x) = 3$

$f(0^+) = \lim_{x \to 0^+} f(x) = 1$

$\because f(0^-) \neq f(0^+)$

$\therefore \lim_{x \to 0} f(x)$ does not exist.

4

$f(0^-) = \lim_{x \to 0^-} f(x) = 1$

$f(0^+) = \lim_{x \to 0^+} \frac{5x + \tan 2x}{6x + \sin x}$

$$= \lim_{x \to 0^+} \frac{5 + \frac{\tan 2x}{x}}{6 + \frac{\sin x}{x}} = \frac{5+2}{6+1} = 1$$

$\because f(0^+) = f(0^-) = 1 \quad \therefore \lim_{x \to 0} f(x) = 1$

5

(1) $\lim\limits_{x \to 0^-} f(x) = 0 - 2 = -2$

$\lim\limits_{x \to 0^+} f(x) = 0^2 = 0$

$\because f(0^-) \neq f(0^+)$

$\therefore \lim\limits_{x \to 0} f(x)$ does not exist.

(2) $f(1^-) = f(1^+) = 1^2 = 1$ (has the same definition)

$\therefore \lim\limits_{x \to 1} f(x) = 1$

(3) $f(2^-) = \lim\limits_{x \to 2^-} x^2 = 4$

$f(2^+) = \lim\limits_{x \to 2^+} 2x = 4$

$\because f(2^-) = f(2^+) \qquad \therefore \lim\limits_{x \to 2} f(x) = 4$

6

$f(3^-) = f(3^+) = 0 \qquad \therefore \lim\limits_{x \to 3} f(x) = 0$

7

$f(0^+) = \lim\limits_{x \to 0^+} \left(\frac{x}{\sqrt{x+1} - 1} \times \frac{\sqrt{x+1} + 1}{\sqrt{x+1} + 1} \right)$

$= \lim\limits_{x \to 0^+} \frac{x\left(\sqrt{x+1} + 1\right)}{x + 1 - 1}$

$= \lim\limits_{x \to 0^+} \left(\sqrt{x+1} + 1\right) = 2$

, $f(0^-) = \lim\limits_{x \to 0^-} \frac{2x}{\sin x} = 2$

$\because f(0^+) = f(0^-) \qquad \therefore \lim\limits_{x \to 0} f(x) = 2$

8

$f(4^+) = \lim\limits_{x \to 4^+} \frac{(x-3)^6 - 1}{x - 4} = \lim\limits_{(x-3) \to 1^+} \frac{(x-3)^6 - 1^6}{(x-3) - 1} = 6$

$f(4^-) = \lim\limits_{x \to 4^-} (x + 2) = 6$

$\because f(4^+) = f(4^-) \qquad \therefore \lim\limits_{x \to 4} f(x) = 6$

9

$\because \lim\limits_{x \to 2} f(x) = 7 \qquad \therefore f(2^-) = f(2^+) = 7$

, $\because f(2^-) = 4 + 3m = 7 \qquad \therefore m = 1$

, $\because f(2^+) = 10 + k = 7 \qquad \therefore k = -3$

10

$\because f(0^-) = f(0^+) = 2 \qquad \therefore f(0^-) = a = 2$

11

$\because \lim\limits_{x \to a} |3x + 2| = 14 \qquad \therefore |3a + 2| = 14$

$\therefore 3a + 2 = 14 \qquad \therefore a = 4$

or $3a + 2 = -14 \qquad \therefore a = \frac{-16}{3}$

12

(1) $f\left(\frac{-\pi}{3}^+\right) = \lim\limits_{x \to \left(-\frac{\pi}{3}\right)^+} \frac{3x}{\tan x} = \frac{-\pi}{-\sqrt{3}} = \frac{\pi}{\sqrt{3}}$

, $f\left(\frac{-\pi}{3}^-\right)$ does not exist.

$\therefore \lim\limits_{x \to \frac{\pi}{3}} f(x)$ does not exist.

(2) $f\left(\frac{\pi}{3}^-\right) = 3 \cos \frac{\pi}{3} = \frac{3}{2}$

, $f\left(\frac{\pi}{3}^+\right)$ does not exist.

$\lim\limits_{x \to \frac{\pi}{3}} f(x)$ does not exist.

(3) $f(0^-) = \lim\limits_{x \to 0^-} \frac{3x}{\tan x} = 3 \qquad , f(0^+) = 3$

$\because f(0^-) = f(0^+) = 3 \qquad \therefore \lim\limits_{x \to 0} f(x) = 3$

13

$\because f(x) = x^2 + \frac{\sqrt{x^2 - 6x + 9}}{x - 3} = x^2 + \frac{|x - 3|}{x - 3}$

$\therefore f(x) = \begin{cases} x^2 + 1 & , \quad x > 3 \\ x^2 - 1 & , \quad x < 3 \end{cases}$

$f(3^+) = \lim\limits_{x \to 3^+} (x^2 + 1) = 10$

$f(3^-) = \lim\limits_{x \to 3^-} (x^2 - 1) = 8$

, $\because f(3^+) \neq f(3^-) \qquad \therefore \lim\limits_{x \to 3} f(x)$ does not exist.

14

$f(x) = \frac{x^2 + 2|x|}{x}$

$f(x) = \begin{cases} \frac{x^2 + 2x}{x} & , \quad x > 0 \\ \frac{x^2 - 2x}{x} & , \quad x < 0 \end{cases}$

$= \begin{cases} \frac{x(x+2)}{x} & , \quad x > 0 \\ \frac{x(x-2)}{x} & , \quad x < 0 \end{cases}$

$= \begin{cases} x + 2 & , \quad x > 0 \\ x - 2 & , \quad x < 0 \end{cases}$

$f(0^-) = 0 - 2 = -2$, $f(0^+) = 0 + 2 = 2$

$\because f(0^-) \neq f(0^+)$ $\therefore \lim\limits_{x \to 0} f(x)$ does not exist.

15

$$f(x) = \begin{cases} -x^2 + 2 & , x < 0 \\ 2 & , x > 0 \end{cases}$$

$f(0^-) = \lim\limits_{x \to 0^-} (-x^2 + 2) = 2$, $f(0^+) = 2$

$\because f(0^-) = f(0^+)$ $\therefore \lim\limits_{x \to 0} f(x) = 2$

16

$$f(x) = \begin{cases} \dfrac{(x-1)^2}{-(x-1)} & , x < 1 \\ 6x - 3m & , x > 1 \end{cases}$$

$$= \begin{cases} -(x-1) & , x < 1 \\ 6x - 3m & , x > 1 \end{cases}$$

$\because \lim\limits_{x \to 1} f(x)$ exists.

$\therefore f(1^-) = f(1^+)$ $\therefore -(1-1) = 6(1) - 3m$ $\therefore m = 2$

17

$$(f+g)(x) = \begin{cases} x^2 - 3x + 3 & , x \geq 1 \\ 2x^2 - 5x + 4 & , x < 1 \end{cases}$$

$(f+g)(1^+) = \lim\limits_{x \to 1^+} (x^2 - 3x + 3) = 1$

$(f+g)(1^-) = \lim\limits_{x \to 1^-} (2x^2 - 5x + 4) = 1$

$\because (f+g)(1^+) = (f+g)(1^-)$

$\therefore \lim\limits_{x \to 1} \left[(f+g)(x)\right] = 1$

18

$\because$ The function has a limit when $x = 0$

$\therefore f(0^+) = f(0^-)$

$\therefore \lim\limits_{x \to 0^+} \dfrac{a x}{\sqrt{x+4} - 2} = \lim\limits_{x \to 0^-} \dfrac{\sin 3x}{a x}$

$\therefore \lim\limits_{x \to 0^+} \left(\dfrac{a x}{\sqrt{x+4} - 2} \times \dfrac{\sqrt{x+4} + 2}{\sqrt{x+4} + 2} \right) = \dfrac{3}{a}$

$\therefore \lim\limits_{x \to 0^+} \dfrac{a x \left(\sqrt{x+4} + 2\right)}{x} = \dfrac{3}{a}$

$\therefore 4a = \dfrac{3}{a}$ $\therefore 4a^2 = 3$ $\therefore a = \pm \dfrac{\sqrt{3}}{2}$

19

(1) $f(5^+) = \lim\limits_{x \to 5^+} \sqrt{x - 5} = 0$

$f(5^-)$ does not exist.

$\therefore \lim\limits_{x \to 5} f(x)$ does not exist.

(2) $f(3^+)$ does not exist

$f(3^-) = \lim\limits_{x \to 3^-} \sqrt{3 - x} = 0$

$\therefore \lim\limits_{x \to 3} f(x)$ does not exist

(3) $f(-1^+)$ does not exist

$f(-1^-) = \lim\limits_{x \to (-1^-)} \sqrt{x^2 - 1} = 0$

$\therefore \lim\limits_{x \to -1} f(x)$ does not exist

$f(1^+) = \lim\limits_{x \to 1^+} \sqrt{x^2 - 1} = 0$

$f(1^-)$ does not exist

$\therefore \lim\limits_{x \to 1} f(x)$ does not exist.

(4) $f(-2^-)$ does not exist

$f(-2^+) = \lim\limits_{x \to (-2^+)} \sqrt{4 - x^2} = 0$

$\therefore \lim\limits_{x \to -2} f(x)$ does not exist.

$f(2^+)$ does not exist

$f(2^-) = \lim\limits_{x \to 2^-} \sqrt{4 - x^2} = 0$

$\therefore \lim\limits_{x \to 2} f(x)$ does not exist.

20

(1) $f(0^-) = \lim\limits_{x \to 0^-} \dfrac{(x+2)^6 - 2^6}{x(x+32)}$

$= \lim\limits_{x \to 0^-} \left[\dfrac{1}{x+32} \times \dfrac{(x+2)^6 - 2^6}{x} \right]$

$= \dfrac{1}{32} \times 6 \times 2^5 = 6$

, $f(0^+) = \lim\limits_{x \to 0^+} \dfrac{-12x}{1 - (\sin x + \cos x)^2}$

$= \lim\limits_{x \to 0^+} \dfrac{-12x}{1 - (1 + 2\sin x \cos x)}$

$= \lim\limits_{x \to 0^+} \dfrac{12x}{-2\sin x \cos x} = 6$

, $\because f(0^+) = f(0^-) = 6$ $\therefore \lim\limits_{x \to 0} f(x) = 6$

(2) $f(0^-) = \lim_{x \to 0^-} \left[\frac{1-\cos x}{\sin^2 3x} \times \frac{1+\cos x}{1+\cos x} \right]$

$= \lim_{x \to 0^-} \left[\frac{1-\cos^2 x}{\sin^2 3x} \times \frac{1}{1+\cos x} \right]$

$= \lim_{x \to 0^-} \frac{\sin^2 x}{\sin^2 3x} \times \lim_{x \to 0^-} \frac{1}{1+\cos x}$

$= \frac{1}{9} \times \frac{1}{2} = \frac{1}{18}$

, $f(0^+) = \lim_{x \to 0^+} \frac{1}{2} x^2 \cos 6x \cot^2 3x$

$= \lim_{x \to 0^+} \frac{\frac{1}{2}x^2 \cos 6x}{\tan^2 3x} = \frac{1}{2} \times \frac{1}{9} \times 1 = \frac{1}{18}$

, $\because f(0^-) = f(0^+) = \frac{1}{18}$

$\therefore \lim_{x \to 0} f(x) = \frac{1}{18}$

21

$\because |x| = \begin{cases} x & , x \geq 0 \\ -x & , x < 0 \end{cases}$

(1) $\lim_{x \to 0^+} \frac{\sin|x|}{x} = \lim_{x \to 0^+} \frac{\sin x}{x} = 1$

, $\lim_{x \to 0^-} \frac{\sin|x|}{x} = \lim_{x \to 0^-} \frac{\sin(-x)}{x} = -1$

$\therefore$ The right limit $\neq$ the left limit.

$\therefore \lim_{x \to 0} \frac{\sin|x|}{x}$ is not exist.

(2) $\lim_{x \to 0^+} \frac{\cos|x|}{x+1} = \lim_{x \to 0^+} \frac{\cos x}{x+1} = 1$

, $\lim_{x \to 0^-} \frac{\cos|x|}{x+1} = \lim_{x \to 0^-} \frac{\cos(-x)}{x+1}$

$= \lim_{x \to 0^-} \frac{\cos x}{x+1} = 1$

$\therefore$ The right limit = the left limit = 1

$\therefore \lim_{x \to 0} \frac{\cos|x|}{x+1} = 1$

22

$\lim_{x \to 0} \frac{\sqrt{\sec^2 x - 1}}{x} = \lim_{x \to 0} \frac{\sqrt{\tan^2 x}}{x} = \lim_{x \to 0} \frac{|\tan x|}{x}$

, $\because |\tan x| = \begin{cases} \tan x & , \tan x \geq 0 \\ -\tan x & , \tan x < 0 \end{cases}$

$\therefore \lim_{x \to 0^+} \frac{|\tan x|}{x} = \lim_{x \to 0^+} \frac{\tan x}{x} = 1$

, $\lim_{x \to 0^-} \frac{|\tan x|}{x} = \lim_{x \to 0^-} \frac{-\tan x}{x} = -1$

, $\because$ the right limit $\neq$ the left limit.

$\therefore \lim_{x \to 0} \frac{\sqrt{\sec^2 x - 1}}{x}$ does not exist.

23

$\lim_{x \to 3^+} \left(f(x) + g(x)\right)$

$= \lim_{x \to 3^+} \left(f(x)\right) + \lim_{x \to 3^+} \left(g(x)\right) = 4 + 14 = 18$

, $\lim_{x \to 3^-} \left(f(x) + g(x)\right) = \lim_{x \to 3^-} \left(f(x)\right) + \lim_{x \to 3^-} \left(g(x)\right)$

$= 8 + 10 = 18$

$\therefore \lim_{x \to 3^+} \left(f(x) + g(x)\right) = \lim_{x \to 3^-} \left(f(x) + g(x)\right)$

$\therefore \lim_{x \to 3} \left(f(x) + g(x)\right) = 18$

24

(1) $\because \lim_{x \to 2^-} \frac{1}{x-2} = \frac{1}{0^-} = -\infty$

$\lim_{x \to 2^+} \frac{1}{x-2} = \frac{1}{0^+} = \infty$

$\therefore \lim_{x \to 2} \frac{1}{x-2}$ does not exist.

(2) $\because \lim_{x \to 3^-} f(x) = \frac{1}{0^+} = \infty$

$\lim_{x \to 3^+} f(x) = \frac{1}{0^+} = \infty$

$\therefore \lim_{x \to 3} f(x) = \infty$

(3) $\because \lim_{x \to 1^-} \frac{1}{x-1} = \frac{1}{0^-} = -\infty$

$\lim_{x \to 1^+} (x-2) = -1$

$\therefore \lim_{x \to 1} f(x)$ does not exist.

25

$f(x) = \begin{cases} 1 + \frac{x(x+2)}{-(x+2)} & , -3 < x < -2 \\ 1 + \frac{x(x+2)}{(x+2)} & , -2 < x < 0 \\ 2x + 1 & , 0 < x < 3 \end{cases}$

$= \begin{cases} 1 - x & , -3 < x < -2 \\ 1 + x & , -2 < x < 0 \\ 2x + 1 & , 0 < x < 3 \end{cases}$

(1) $\lim_{x \to -3^-} f(x)$ does not exist.

$\lim_{x \to -3^+} f(x) = 4$

$\therefore \lim_{x \to -3} f(x)$ does not exist.

(2) $\lim_{x \to -2^-} f(x) = 1 - (-2) = 3$

$\lim_{x \to -2^+} f(x) = 1 + (-2) = -1$

$\therefore \lim_{x \to -2} f(x)$ does not exist.

(3) $\lim_{x \to 0^-} f(x) = 1$

$\lim_{x \to 0^+} f(x) = 1$ $\therefore \lim_{x \to 0} f(x) = 1$

(4) $\lim_{x \to 3^-} f(x) = 2(3) + 1 = 7$

$\lim_{x \to 3^+} f(x)$ does not exist.

$\therefore \lim_{x \to 3} f(x)$ does not exist.

26

$f(2^+) = \lim_{x \to 2^+} \frac{(2x-3)^{\frac{1}{3}} - 1}{(3x-5)^{\frac{1}{5}} - 1}$

$= \frac{2}{3} \lim_{x \to 2^+} \left[\frac{(2x-3)^{\frac{1}{3}} - 1^{\frac{1}{3}}}{2(x-2)} \times \frac{3(x-2)}{(3x-5)^{\frac{1}{5}} - 1^{\frac{1}{5}}} \right]$

$= \frac{2}{3} \lim_{(2x-3) \to 1} \frac{(2x-3)^{\frac{1}{3}} - 1^{\frac{1}{3}}}{(2x-3) - 1}$

$\times \lim_{(3x-5) \to 1} \frac{(3x-5) - 1}{(3x-5)^{\frac{1}{5}} - 1^{\frac{1}{5}}}$

$= \frac{2}{3} \times \frac{1}{3} \times 5 = \frac{10}{9}$

, $\because$ the function has a limit when $x = 2$

$\therefore f(2^+) = f(2^-)$

$\therefore \frac{10}{9} = 2k$ $\therefore k = \frac{5}{9}$

27

$\because$ The function has a limit when $x = 3$

$\therefore f(3^-) = f(3^+)$

$\therefore \lim_{x \to 3} \frac{x^2 + ax + b}{(x-3)(x-1)} = 12$

, $\because$ the denominator $= 0$ when $x = 3$

$\therefore$ The numerator $= 0$ when $x = 3$

$\therefore (3)^2 + a(3) + b = 0$ $\therefore b = -3a - 9$

$\therefore \lim_{x \to 3} \frac{x^2 + ax - 3a - 9}{(x-3)(x-1)} = 12$

$\therefore \lim_{x \to 3} \frac{(x^2 - 9) + (ax - 3a)}{(x-3)(x-1)} = 12$

$\therefore \lim_{x \to 3} \frac{(x-3)(x+3) + a(x-3)}{(x-3)(x-1)} = 12$

$\therefore \lim_{x \to 3} \frac{\cancel{(x-3)}(x+3+a)}{\cancel{(x-3)}(x-1)} = 12$

$\therefore \frac{6+a}{2} = 12$ $\therefore a = 18$, $b = -63$

Third Higher skills

(1) (c) (2) (b) (3) (a) (4) (c) (5) (c)
(6) (d) (7) (b) (8) (d) (9) (c) (10) (d)

Instructions to solve :

(1) $\because f(x) = x^2$ $\therefore f(f(x)) = (x^2)^2 = x^4$

$\therefore \lim_{x \to 2} f(f(x)) = \lim_{x \to 2} x^4 = 2^4 = 16$

(2) $\because \lim_{x \to 3} f(x) = 7$

$\therefore$ The curve either passes through the point (3 , 7) or has an open dot at this point

, $\because f$ is an odd function

$\therefore$ The curve either passes through the point (−3 , −7) or has an open dot at this point

$\therefore \lim_{x \to -3} f(x) = -7$

(3) $\because \lim_{x \to 2} f(x) = 5$

$\therefore$ The curve either passes through the point (2 , 5) or has an open dot at this point

, $\because f$ is an even function

$\therefore$ The curve either passes through the point (−2 , 5) or has an open dot at this point

$\therefore \lim_{x \to -2} f(x) = 5$

(4) $\because f$ is a polynomial function and $\lim_{x \to 2} f(x) = 3$

$\therefore$ Curve of the function passes through (2 , 3)

$\therefore$ Curve of f^{-1} passes through (3 , 2)

$\therefore \lim_{x \to 3} f^{-1}(x) = 2$

(5) $\because$ Curve of $f(x+1)$ is the same as the curve $f(x)$ by translation one unit in direction of $\overrightarrow{OX}$

, $\because \text{Lim } f(x) = 5$ at $x \to 2$

$\therefore \text{Lim } f(x+1) = 5$ at $x \to 1$

i.e. $\lim_{x \to 1} f(x+1) = 5$

(6) $\because \lim_{x \to -2} \frac{[f(x)]^3 - 64}{f(x) - 4}$

"Notice that : at $x \to -2$

, then $f(x) \to f(-2)$ *i.e.* $f(x) \to 4$"

$= \lim_{f(x) \to 4} \frac{[f(x)]^3 - (4)^3}{[f(x)] - 4} = 3(4)^2 = 48$

(7) From the graph we find that $f(x) = x - 1$

, $g(x) = -x + 1$

$\therefore \lim_{x \to 1} \frac{f(x)}{g(x)} = \lim_{x \to 1} \frac{x - 1}{-x + 1}$

$= \lim_{x \to 1} \frac{x - 1}{-(x - 1)} = -1$

(8) $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} [h(x) + g(x)]$

$= \lim_{x \to 1^+} h(x) + \lim_{x \to 1^+} g(x)$

$= -1 + 1 = \text{zero}$

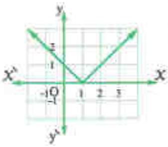
, $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} [h(x) + g(x)]$

$= \lim_{x \to 1^-} h(x) + \lim_{x \to 1^-} g(x)$

$= 2 + (-2) = \text{zero}$

$\therefore \lim_{x \to 1} f(x) = \text{zero}$

(9) Notice that : Curve of $|f(x)|$ is the same curve of $f(x)$ by reflection of the part below x-axis as the opposite graph , from the graph :



$\therefore \lim_{x \to 2} |f(x)| = 1$

(10) $\because f(x) = \frac{4}{3} x + 4$, by exchanging the variables

$\therefore x = \frac{4}{3} y + 4 \quad \therefore y = \frac{3}{4}(x - 4) = \frac{3}{4} x - 3$

$\therefore f^{-1}(x) = \frac{3}{4} x - 3$

$\therefore \lim_{x \to \infty} \frac{f(x)}{f^{-1}(x)} = \lim_{x \to \infty} \frac{\frac{4}{3} x + 4}{\frac{3}{4} x - 3} = \frac{16}{9}$

Exercise 18

First Multiple choice questions

(1) b	(2) a	(3) d	(4) a	(5) d	(6) b
(7) a	(8) c	(9) c	(10) c	(11) c	(12) b
(13) a	(14) c	(15) b	(16) c	(17) c	(18) d
(19) c	(20) a	(21) a	(22) a	(23) a	(24) d
(25) b	(26) b	(27) c	(28) d	(29) a	(30) a
(31) c	(32) c	(33) c	(34) b	(35) b	(36) d
(37) d	(38) b				

Second Essay questions

Exercises on continuity of a function at a point

1

(1) $\because f(-1) = \lim_{x \to -1} (x^2 + x - 3) = -3$

$\therefore$ The function f is continuous at $x = -1$

(2) $\because f(9) = \lim_{x \to 9} \sqrt[3]{x - 1} = 2$

$\therefore$ The function f is continuous at $x = 9$

(3) $\because f(1) = \lim_{x \to 1} \frac{x^2 - 4}{x - 2} = 3$

$\therefore$ The function f is continuous at $x = 1$

, $f(2)$ is undefined

$\therefore$ The function f is not continuous at $x = 2$

(4) $\because f(3) = \lim_{x \to 3} \left(5 - |x - 3|\right) = 5$

$\therefore$ The function f is continuous at $x = 3$

(5) $\because f(4) = \lim_{x \to 4} \left(|x - 4| + |x + 2|\right) = 6$

$\therefore$ The function f is continuous at $x = 4$

, $\because f(-2) = \lim_{x \to -2} \left(|x - 4| + |x + 2|\right) = 6$

$\therefore$ The function f is continuous at $x = -2$

(6) $\because f\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^2 + 3 = 4$

, $f\left(\frac{1^-}{2}\right) = \lim_{x \to \frac{1}{2}^-} (4x^2 + 3) = 4$

, $f\left(\frac{1^+}{2}\right) = \lim_{x \to \frac{1}{2}^+} (5 - 2x) = 4$

$\therefore$ The function f is continuous at $x = \frac{1}{2}$

(7) $\because f(-1) = \lim_{x \to -1} 2x = -2$

$\therefore$ The function f is continuous at $x = -1$

, $\because$ the function f is undefined at $x = 1$

$\therefore$ The function f is not continuous at $x = 1$

(8) * At $x = -2$

$f(-2) = 1$, $f(-2^+) = 1$

, $f(-2^-) = \lim_{x \to (-2^-)} (x+4) = 2$

$\therefore f(-2^-) \neq f(-2^+)$

$\therefore$ The function f is not continuous at $x = -2$

* At $x = -1$

$f(-1) = -2 + 3 = 1$, $f(-1^-) = 1$

, $f(-1^+) = \lim_{x \to (-1^+)} (2x + 3) = 1$

, $f(-1^-) = f(-1^+)$

$\therefore$ The function f is continuous at $x = -1$

(9) $\because f(1) = 4$

, $f(1^+) = \lim_{x \to 1^+} (x^2 + 3) = 4$

, $f(1^-) = \lim_{x \to 1^-} \frac{x^2 + 2x - 3}{x - 1}$

$= \lim_{x \to 1^-} \frac{(x-1)(x+3)}{x-1} = 4$

, $\because f(1) = f(1^+) = f(1^-)$

$\therefore$ The function f is continuous at $x = 1$

(10) $f(x) = \begin{cases} x^2 & , \ x > 0 \\ 2 & , \ x = 0 \\ -x^2 & , \ x < 0 \end{cases}$

, $f(0) = 2$, $f(0^+) = \lim_{x \to 0^+} x^2 = 0$, $f(0^-)$

$= \lim_{x \to 0^-} (-x^2) = 0$

$\therefore f(0) \neq \lim_{x \to 0} f(x)$

$\therefore$ The function f is not continuous at $x = 0$

(11) $f(x) = \begin{cases} 1 & , \ x > 3 \\ 2 & , \ x = 3 \\ -1 & , \ x < 3 \end{cases}$

$f(3) = 2$, $f(3^+) = 1$, $f(3^-) = -1$

$\because f(3) \neq f(3^+) \neq f(3^-)$

$\therefore$ The function f is not continuous at $x = 3$

(12) $f(x) = \begin{cases} \frac{x^2 - x - 2}{x - 2} & , \ x < 2 \\ 3 & , \ x \geq 2 \end{cases}$

$= \begin{cases} \frac{(x-2)(x+1)}{(x-2)} & , \ x < 2 \\ 3 & , \ x \geq 2 \end{cases}$

$= \begin{cases} x + 1 & , \ x < 2 \\ 3 & , \ x \geq 2 \end{cases}$

$\therefore f(2) = 3$, $f(2^+) = \lim_{x \to 2^+} 3 = 3$

, $f(2^-) = \lim_{x \to 2^-} (x + 1) = 3$

, $\because f(2) = f(2^+) = f(2^-)$

$\therefore$ The function f is continuous at $x = 2$

(13) $f(2) = 3 \times 4 + 2 = 14$

, $\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^7 - 2^7}{x^4 - 2^4} = \frac{7}{4} (2)^3 = 14$

$\therefore f(2) = \lim_{x \to 2} f(x)$

$\therefore$ The function f is continuous at $x = 2$

(14) $f(0) = 0$, $f(0^-) = 2 \lim_{x \to 0^-} \sin x = 0$

, $f(0^+) = \lim_{x \to 0^+} \frac{1 - \cos x}{x} = 0$

$\therefore f(0) = f(0^-) = f(0^+)$

$\therefore$ The function f is continuous at $x = 0$

(15) $\because f(2) = \frac{1}{4}$, $f(2^+) = \lim_{x \to 2^+} \left(1 - \frac{3}{x^2}\right) = \frac{1}{4}$

, $f(2^-) = \lim_{x \to 2^-} \frac{\sin(x-2)}{x^2 - 4}$

$= \lim_{x \to 2^-} \frac{\sin(x-2)}{(x-2)} \times \lim_{x \to 2^-} \frac{1}{x+2}$

$= 1 \times \frac{1}{4} = \frac{1}{4}$

$\therefore f(2) = f(2^-) = f(2^+)$

$\therefore$ The function f is continuous at $x = 2$

(16) $f(x) = \begin{cases} -x^2 + 4 & , \ x \leq 0 \\ 4 & , \ x > 0 \end{cases}$

$f(0) = 4$

, $f(0^-) = \lim_{x \to 0^-} (-x^2 + 4) = 4$, $f(0^+) = 4$

$\therefore f(0) = f(0^+) = f(0^-)$

$\therefore$ The function f is continuous at $x = 0$

(17) $f(x)=\begin{cases}\dfrac{(x-1)(x+3)}{|x+3|} & , x\neq -3\\ 2 & , x=-3\end{cases}$

$=\begin{cases}x-1 & , x>-3\\ 2 & , x=-3\\ -x+1 & , x<-3\end{cases}$

* At $x=-3$

$f(-3)=2$, $f(-3^{+})=\lim\limits_{x\to -3^{+}}(x-1)=-4$

, $f(-3^{-})=\lim\limits_{x\to -3^{-}}(-x+1)=4$

$\because f(-3^{+})\neq f(-3^{-})$

$\therefore$ The function f is not continuous at $x=-3$

* At $x=1$

$f(1)=0$, $\lim\limits_{x\to 1}(x-1)=0$

$\because f(1)=\lim\limits_{x\to 1}f(x)$

$\therefore$ The function f is continuous at $x=1$

(18) $f(x)=\begin{cases}-x-3 & , x<-3\\ x+3 & , -3\leq x\leq 2\\ 11-3x & , x>2\end{cases}$

* At $x=-3$

$f(-3)=0$

, $f(-3^{-})=\lim\limits_{x\to -3^{-}}(-x-3)=0$

, $f(-3^{+})=\lim\limits_{x\to -3^{+}}(x+3)=0$

$\because f(-3)=f(-3^{-})=f(-3^{+})$

$\therefore$ The function f is continuous at $x=-3$

* At $x=2$

$f(2)=5$

, $f(2^{-})=\lim\limits_{x\to 2^{-}}(x+3)=5$

, $f(2^{+})=\lim\limits_{x\to 2^{+}}(11-3x)=5$

$\therefore f(2)=f(2^{-})=f(2^{+})$

$\therefore$ The function f is continuous at $x=2$

(19) $f(0)=3$

, $f(0^{+})=\lim\limits_{x\to 0^{+}}\dfrac{x+3}{x+1}=3$

, $f(0^{-})=\lim\limits_{x\to 0^{-}}\dfrac{5x^{2}+\sin^{2}2x}{x\tan 3x}$

$$=\lim_{x\to 0^{-}}\frac{5+\left(\frac{\sin 2x}{x}\right)^{2}}{\frac{\tan 3x}{x}}=\frac{5+4}{3}=3$$

, $\because f(0)=f(0^{+})=f(0^{-})$

$\therefore$ The function f is continuous at $x=0$

(20) $f(\pi)=1$, $\lim\limits_{x\to \pi}f(x)=\lim\limits_{x\to \pi}\dfrac{\sin x}{\pi-x}$

$$=\lim_{(\pi-x)\to 0}\frac{\sin(\pi-x)}{\pi-x}=1$$

$\therefore f(\pi)=\lim\limits_{x\to \pi}f(x)$

$\therefore$ The function f is continuous at $x=\pi$

2

(1) $f(-3)=-3+a$

, $\lim\limits_{x\to -3}f(x)=\lim\limits_{x\to -3}\dfrac{(x+3)(x-1)}{x+3}=-4$

, $\because$ the function f is continuous at $x=-3$

$\therefore -3+a=-4$ $\qquad\therefore a=-1$

(2) $\lim\limits_{x\to 2}f(x)=\lim\limits_{x\to 2}\dfrac{(x-2)(x-3)}{(x-2)(x^{2}+2x+4)}$

$$=\lim_{x\to 2}\frac{x-3}{x^{2}+2x+4}=\frac{-1}{12}$$

, $f(2)=\dfrac{-2}{|a|}$

, $\because$ the function f is continuous at $x=2$

$\therefore \dfrac{-1}{12}=\dfrac{-2}{|a|}$ $\qquad\therefore |a|=24$ $\qquad\therefore a=\pm 24$

(3) $f(1)=k$

, $\lim\limits_{x\to 1}f(x)=\lim\limits_{x\to 1}\left[\dfrac{\sqrt{x+3}-2}{x^{2}-1}\times\dfrac{\sqrt{x+3}+2}{\sqrt{x+3}+2}\right]$

$$=\lim_{x\to 1}\frac{x+3-4}{(x+1)(x-1)\left(\sqrt{x+3}+2\right)}$$

$$=\lim_{x\to 1}\frac{1}{(x+1)\left(\sqrt{x+3}+2\right)}=\frac{1}{8}$$

$\because$ The function f is continuous at $x=1$

$\therefore k=\dfrac{1}{8}$

(4) $f(0)=a+1$

, $\lim\limits_{x\to 0}f(x)=\lim\limits_{x\to 0}\dfrac{\sin^{2}2x}{x\tan 3x}$

$$=\lim_{x\to 0}\frac{\left(\frac{\sin 2x}{x}\right)^{2}}{\frac{\tan 3x}{x}}=\frac{4}{3}$$

$\because$ The function f is continuous at $x=0$

$\therefore a+1=\dfrac{4}{3}$ $\qquad\therefore a=\dfrac{1}{3}$

(5) $f(-2) = -8$

, $f(-2^+) = \lim_{x \to -2^+} (aX + b) = -2a + b$

$\because$ The function f is continuous at $X = -2$

$\therefore -2a + b = -8$ (1)

$f(5) = 13$, $f(5^-) = \lim_{x \to 5^-} (aX + b) = 5a + b$

$\because$ The function f is continuous at $X = 5$

$\therefore 5a + b = 13$ (2)

From (1) and (2) : $\therefore a = 3$, $b = -2$

(6) $f(2) = 3$

, $f(2^+) = \lim_{x \to 2^+} (a + bX) = a + 2b$

, $f(2^-) = \lim_{x \to 2^-} (b - aX^2) = b - 4a$

$\because$ The function f is continuous at $X = 2$

$\therefore a + 2b = 3$ (1)

, $b - 4a = 3$ (2)

From (1) and (2) : $\therefore a = -\frac{1}{3}$, $b = \frac{5}{3}$

(7) $f(0) = k$

$$, \lim_{x \to 0} f(X) = \lim_{x \to 0} \left[\frac{\cos 2X - 1}{X^2} \times \frac{\cos 2X + 1}{\cos 2X + 1}\right]$$

$$= \lim_{x \to 0} \left[\frac{\cos^2 2X - 1}{X^2} \times \frac{1}{\cos 2X + 1}\right]$$

$$= \lim_{x \to 0} \frac{-\sin^2 2X}{X^2} \times \lim_{x \to 0} \frac{1}{\cos 2X + 1}$$

$$= -\lim_{x \to 0} \left(\frac{\sin 2X}{X}\right)^2 \times \frac{1}{2} = -2$$

, $\because$ the function f is continuous at $X = 0$

$\therefore k = -2$

(8) $f(c^-) = 2 - c^2$, $f(c^+) = c$

$\because$ The function f is continuous at $X = c$

$\therefore 2 - c^2 = c$ $\qquad \therefore c^2 + c - 2 = 0$

$\therefore (c - 1)(c + 2) = 0$ $\qquad \therefore c = 1$ or $c = -2$

(9) $f(0) = 0$

, $\lim_{x \to 0} f(X) = \lim_{x \to 0} \frac{1 - \cos X}{kX} = 0 \times \frac{1}{k}$

, $\because$ the function f is continuous at $X = 0$

$\therefore \frac{1}{k} \times 0 = 0$ when $k \in \mathbb{R} - \{0\}$

3

(1) $\lim_{x \to 3} \frac{X^2 - X - 6}{X - 3} = \lim_{x \to 3} \frac{(X + 2)(X - 3)}{X - 3} = 5$

Redefine : $f(X) = \begin{cases} \frac{X^2 - X - 6}{X - 3} & , X \neq 3 \\ 5 & , X = 3 \end{cases}$

(2) $\lim_{x \to 1} \frac{X^3 - 1}{X^2 - 3X + 2}$

$= \lim_{x \to 1} \frac{(X - 1)(X^2 + X + 1)}{(X - 1)(X - 2)} = -3$

Redefine : $f(X) = \begin{cases} \frac{X^3 - 1}{X^2 - 3X + 2} & , X \neq 1 \\ -3 & , X = 1 \end{cases}$

(3) $f(1^+) = \lim_{x \to 1^+} (X^3 + 2X) = 3$

, $f(1^-) = \lim_{x \to 1^-} (5X - 1) = 4$

$\because f(1^+) \neq f(1^-)$

$\therefore$ The function cannot redefine to be continuous at $X = 1$

(4) $f(2^+) = \lim_{x \to 2^+} (X^2 + 1) = 5$

, $f(2^-) = \lim_{x \to 2^-} \frac{(X - 2)(X + 2)}{X - 2} = 4$

, $\because f(2^+) \neq f(2^-)$

$\therefore$ The function cannot redefine to be continuous at $X = 2$

(5) $f(0^+) = \lim_{x \to 0^+} \frac{3X + 1 - \cos X}{5X}$

$= \lim_{x \to 0^+} \left(\frac{3}{5} + \frac{1 - \cos X}{5X}\right) = \frac{3}{5}$

, $f(0^-) = \lim_{x \to 0^-} \frac{3\cos X}{5} = \frac{3}{5}$, $\lim_{x \to 0} f(X) = \frac{3}{5}$

Redefine : $f(X) = \begin{cases} \frac{3X + 1 - \cos X}{5X} & , X > 0 \\ \frac{3}{5} & , X = 0 \\ \frac{3}{5}\cos X & , X < 0 \end{cases}$

(6) $f(X) = \begin{cases} \frac{X - 3}{X - 3} & , X > 3 \\ \frac{-(X - 3)}{X - 3} & , X < 3 \end{cases} = \begin{cases} 1 & , X > 3 \\ -1 & , X < 3 \end{cases}$

, $f(3^+) = 1$, $f(3^-) = -1$, $f(3^+) \neq f(3^-)$

$\therefore$ The function cannot redefine to be continuous at $X = 3$

(7) $\lim\limits_{x\to 8} \frac{x^2-64}{\sqrt[3]{x}-2} = \lim\limits_{x\to 4} \frac{x^2-8^2}{x^{\frac{1}{3}}-8^{\frac{1}{3}}}$

$= 2 \times 3 \times 8^{2-\frac{1}{3}} = 192$

Redefine : $f(x) = \begin{cases} \frac{x^2-64}{\sqrt[3]{x}-2} & , x \neq 8 \\ 192 & , x = 8 \end{cases}$

(8) $f(x) = \begin{cases} \frac{x+x}{x} & , x>0 \\ \frac{-x+x}{x} & , x<0 \end{cases} = \begin{cases} 2 & , x>0 \\ 0 & , x<0 \end{cases}$

$f(0^+) = 2$, $f(0^-) = 0$

, $\because f(0^+) \neq f(0^-)$

$\therefore$ The function cannot redefine to be continuous at $x = 0$

(9) $\lim\limits_{x\to 5} \left[\frac{2}{x-5} - \frac{12}{(x-5)(x+1)}\right]$

$= \lim\limits_{x\to 5} \frac{2(x+1)-12}{(x-5)(x+1)} = \lim\limits_{x\to 5} \frac{2x-10}{(x-5)(x+1)}$

$= \lim\limits_{x\to 5} \frac{2(x-5)}{(x-5)(x+1)} = \frac{1}{3}$

Redefine :

$f(x) = \begin{cases} \frac{2}{x-5} - \frac{12}{x^2-4x-5} & , x \neq 5 \\ \frac{1}{3} & , x = 5 \end{cases}$

(10) $f(x) = \begin{cases} \frac{\sin x}{x} & , x>0 \\ \frac{-\sin x}{x} & , x<0 \end{cases}$

$\because f(0^+) = \lim\limits_{x\to 0^+} \frac{\sin x}{x} = 1$

, $f(0^-) = \lim\limits_{x\to 0^-} \frac{-\sin x}{x} = -1$

, $\because f(0^+) \neq f(0^-)$

$\therefore$ The function cannot redefine to be continuous at $x = 0$

Exercises on continuity of a function on an interval

4

(1) $\because$ The function is polynomial

$\therefore f$ is continuous on $\mathbb{R}$

(2) $\because$ The function is polynomial

$\therefore f$ is continuous on $\mathbb{R}$

(3) $\because$ The function is rational , $x-3=0$

$\therefore x = 3$

$\therefore$ The function f is continuous on $\mathbb{R} - \{3\}$

(4) $\because$ The function is rational , $x^2-5x+6=0$

$\therefore (x-3)(x-2)=0$ $\quad \therefore x=3$ or $x=2$

$\therefore f$ is continuous on $\mathbb{R} - \{2 , 3\}$

(5) $\because$ The function is rational , $x^4-13x^2+36=0$

$\therefore (x^2-9)(x^2-4)=0$

$\therefore x = \pm 3$ or $x = \pm 2$

$\therefore f$ is continuous on $\mathbb{R} - \{3 , -3 , 2 , -2\}$

(6) $\because$ The function is rational , $x^2+x+1=0$

$\therefore$ There are no zeroes for the denominator.

$\therefore f$ is continuous on $\mathbb{R}$

(7) $\because$ The function is rational , $x^3-x=0$

$\therefore x(x^2-1)=0$ $\quad \therefore x=0$ or $x=\pm 1$

$\therefore f$ is continuous on $\mathbb{R} - \{0 , 1 , -1\}$

(8) $\because$ The function is in the form of absolute of polynomial

$\therefore f$ is continuous on $\mathbb{R}$

(9) $\because$ Each of the two functions x^3 and $\sin 2x$ is continuous on $\mathbb{R}$

$\therefore f$ is continuous on $\mathbb{R}$

(10) $\because$ Each of the two functions $\sin x$ and $3\cos(x+1)$ is continuous on $\mathbb{R}$

$\therefore f$ is continuous on $\mathbb{R}$

(11) $\because$ Each of the two functions $(3x+4)^2$ and $\sin 2x$ is continuous on $\mathbb{R}$

$\therefore f$ is continuous on $\mathbb{R}$

(12) $\because$ The function is rational , $|x|-2=0$

$\therefore |x| = 2$ $\quad \therefore x = \pm 2$

$\therefore f$ is continuous on $\mathbb{R} - \{2 , -2\}$

(13) $\because$ The function is rational , $|x|+1=0$

$\therefore |x| = -1$ (refused) $\quad \therefore f$ is continuous on $\mathbb{R}$

(14) $\because x-2>0$ $\quad \therefore x>2$

$\therefore f$ is continuous $]2 , \infty[$

(15) $\because$ The function x^2 , $(x+1)$ and $\sin 3x$ are continuous on $\mathbb{R}$

$\therefore f$ is continuous on $\mathbb{R} - \{-1\}$

(16) $\because X-4\geq 0$ $\therefore X\geq 4$ $\therefore X\in[4,\infty[$

$\therefore f$ is continuous on the interval $[4,\infty[-\{6\}$

(17) $\because X-3\geq 0$ $\therefore X\geq 3$ $\therefore X\in[3,\infty[$

$\therefore f$ is continuous on

the interval $[3,\infty[-\{2,-2\}$

i.e. f is continuous on the interval $[3,\infty[$

(18) $\because X+2>0$ $\therefore X>-2$ $\therefore X\in]-2,\infty[$

$\therefore f$ is continuous on the interval $]-2,\infty[-\{0\}$

(19) $\because X+2\geq 0$ $\therefore X\geq -2$

$\therefore X\in[-2,\infty[$, $X+2\neq 1$ $\therefore X\neq -1$

$\therefore f$ is continuous on the interval $[-2,\infty[-\{-1\}$

(20) $\because X+2\geq 0$ $\therefore X\geq -2$ $\therefore X\in[-2,\infty[$

, $3-X\geq 0$ $\therefore 3\geq X$ $\therefore X\in]-\infty,3]$

$\therefore f$ is continuous on the interval $[-2,3]$

(21) $\because 25-X^2>0$ $\therefore X^2<25$

$\therefore |X|<5$ $-5<X<5$

$\therefore f$ is continuous on the interval $]-5,5[$

(22) $\because 5-|X|\geq 0$ $\therefore |X|\leq 5$ $\therefore -5\leq X\leq 5$

$\therefore f$ is continuous on the interval $[-5,5]$

(23) $\because \sin X=0$ when $X=\pi n$

$\therefore f$ is continuous on $\mathbb{R}-\{X: X=\pi n, n\in\mathbb{Z}\}$

(24) $\because \cos X=0$ when $X=\frac{\pi}{2}+\pi n$

$\therefore f$ is continuous on $\mathbb{R}-\left\{X: X=\frac{\pi}{2}+\pi n, n\in\mathbb{Z}\right\}$

(25) $\because$ Each of the two functions $\sin 3X$ and $\cos 5X$ is continuous on $\mathbb{R}$

, $\because 1-\cos X=0$ $\therefore \cos X=1$

$\therefore X=2\pi n$

$\therefore f$ is continuous on $\mathbb{R}-\{X: X=2\pi n, n\in\mathbb{Z}\}$

(26) $\because 1+\sin X=0$ $\therefore \sin X=-1$

$\therefore X=\frac{3\pi}{2}+2\pi n$ $\therefore f$ is continuous on

$\mathbb{R}-\left\{X: X=\frac{3\pi}{2}+2\pi n, n\in\mathbb{Z}\right\}$

(27) $\because$ Each of the two functions $\sin^2 X$ and $\cos X$ is continuous on $\mathbb{R}$

, $\because X^2-9=0$ $\therefore X=\pm 3$

$\therefore f$ is continuous on $\mathbb{R}-\{3,-3\}$

(28) f is continuous on $\mathbb{R}-\left\{X: X=\frac{\pi}{2}+n\pi, n\in\mathbb{Z}\right\}$

(29) $\because \tan X$ is continuous on

$\mathbb{R}-\left\{X: X=\frac{\pi}{2}+n\pi, n\in\mathbb{Z}\right\}$, $X^2-9=0$

$\therefore X=\pm 3$

$\therefore f$ is continuous on $\mathbb{R}-\left\{3,-3,\frac{\pi}{2}+n\pi\right\}$

when $n\in\mathbb{Z}$

5

(1) **First :**

$\because f(X)=X^2+1$ is polynomial

$\therefore f$ is continuous on $]-\infty,1[$

, $f(X)=2X$ is polynomial

$\therefore f$ is continuous on $]1,\infty[$

Second :

$\therefore f(1)=2$, $f(1^-)=\lim\limits_{X\to 1^-}(X^2+1)=2$

, $f(1^+)=\lim\limits_{X\to 1^+}2X=2$

$\because f(1)=f(1^-)=f(1^+)$

$\therefore f$ is continuous at $X=1$

$\therefore f$ is continuous on $\mathbb{R}$

(2) **First :**

$\because f(X)=X^2$ is polynomial

$\therefore f$ is continuous on $]-\infty,3[$

, $\because f(X)=5X-4$ is polynomial

$\therefore f$ is continuous on $]3,\infty[$

Second :

$\therefore f(3)=11$, $f(3^-)=\lim\limits_{X\to 3^-}X^2=9$

, $f(3^+)=\lim\limits_{X\to 3^+}(5X-4)=11$

$\therefore f$ is not continuous at $X=3$

$\therefore f$ is continuous on $\mathbb{R}-\{3\}$

(3) **First :**

$\because f(X)=X^2-3X+2$ is polynomial

$\therefore f$ is continuous on $]-\infty,3[$

, $\because f(X)=2$ is polynomial

$\therefore f$ is continuous on $]3,4[$

$\because f(X)=6-X^2$ is polynomial

$\therefore f$ is continuous on $]4,\infty[$

Second :

$\therefore f(3)=2$, $f(3^-)=\lim\limits_{X\to 3^-}(X^2-3X+2)=2$

, $f(3^+)=2$

$\therefore f(3)=f(3^-)=f(3^+)$

$\therefore f$ is continuous at $X=3$

Third :

$\therefore f(4) = 2 , f(4^-) = 2$

$, f(2^+) = \lim\limits_{x \to 4^+} (6 - x^2) = -10$

$\because f(4^+) \neq f(4^-)$

$\therefore f$ is not continuous at $x = 4$

$\therefore f$ is continuous on $\mathbb{R} - \{4\}$

(4) First :

$\because f(x) = 1 + \sin x$ is continuous on $\left]0 , \frac{\pi}{2}\right[$

$, f(x) = 1 - \cos 2x$ is continuous on $\left]\frac{\pi}{2} , \infty\right[$

Second :

$f(0) = 1 , f(0^+) = \lim\limits_{x \to 0^+} (1 + \sin x) = 1$

$\therefore f(0) = f(0^+)$

$\therefore f$ is continuous from the right at $x = 0$

Third :

$f\left(\frac{\pi}{2}\right) = 2 , f\left(\frac{\pi^-}{2}\right) = \lim\limits_{x \to \frac{\pi}{2}^-} (1 + \sin x) = 2$

$, f\left(\frac{\pi^+}{2}\right) = \lim\limits_{x \to \frac{\pi}{2}^+} (1 - \cos 2x) = 2$

$\therefore f\left(\frac{\pi}{2}\right) = f\left(\frac{\pi^-}{2}\right) = f\left(\frac{\pi^+}{2}\right)$

$\therefore f$ is continuous at $x = \frac{\pi}{2}$

$\therefore f$ is continuous on the interval $[0 , \infty[$

(5) First :

$f(x) = \sin x$ is continuous on $\left]\frac{-\pi}{4} , \frac{3\pi}{4}\right[$

$, f(x) = \cos x$ is continuous on $\left]\frac{3\pi}{4} , 2\pi\right[$

Second :

$f\left(\frac{-\pi}{4}\right) = \frac{-\sqrt{2}}{2}$

$, f\left(\frac{-\pi^+}{4}\right) = \lim\limits_{x \to \left(\frac{-\pi}{4}\right)^+} \sin x = \frac{-\sqrt{2}}{2}$

$\because f\left(\frac{-\pi}{4}\right) = f\left(\frac{-\pi^+}{4}\right)$

$\therefore f$ is continuous from right at $x = -\frac{\pi}{4}$

Third :

$f\left(\frac{3\pi}{4}\right) = \frac{-\sqrt{2}}{2}$

$, f\left(\frac{3\pi^+}{4}\right) = \lim\limits_{x \to \left(\frac{3\pi}{4}\right)^+} \cos x = \frac{-\sqrt{2}}{2}$

$, f\left(\frac{3\pi^-}{4}\right) = \lim\limits_{x \to \left(\frac{3\pi}{4}\right)^-} \sin x = \frac{\sqrt{2}}{2}$

$\because f\left(\frac{3\pi^+}{4}\right) \neq f\left(\frac{3\pi^-}{4}\right)$

$\therefore f$ is not continuous at $x = \frac{3\pi}{4}$

Fourth :

$f(2\pi) = 1$

$f(2\pi^-) = \lim\limits_{x \to 2\pi^-} \cos x = 1$

$\because f(2\pi) = f(2\pi^-)$

$\therefore f$ is continuous from the left at $x = 2\pi$

$\therefore f$ is continuous on

the interval $\left[-\frac{\pi}{4} , 2\pi\right] - \left\{\frac{3\pi}{4}\right\}$

(6) First :

$\because f(x) = \frac{x \tan x + \sin^2 3x}{5x^2}$ is continuous

on $\left]-\frac{\pi}{4} , 0\right[$

$, f(x) = 2\cos 2x$ is continuous on $\left]0 , \frac{\pi}{4}\right[$

Second :

$f(0) = 2$

$, f(0^-) = \lim\limits_{x \to 0^-} \frac{\frac{\tan x}{x} + \left(\frac{\sin 3x}{x}\right)^2}{5} = \frac{1 + 9}{5} = 2$

$, f(0^+) = \lim\limits_{x \to 0^+} (2\cos 2x) = 2$

$, \because f(0^+) = f(0^-) = f(0)$

$\therefore f$ is continuous at $x = 0$

$\therefore f$ is continuous on $\left]-\frac{\pi}{4} , \frac{\pi}{4}\right[$

(7) $f(x) = \frac{(x+3)^4 - 81}{x}$ is continuous on $]-\infty , 0[$

and $]0 , \infty[$

$, f(0) = 108$

$, \lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \frac{(x+3)^4 - 3^4}{x} = 4(3)^3 = 108$

$\because f(0) = \lim\limits_{x \to 0} f(x)$

$\therefore f$ is continuous at $x = 0$

$\therefore f$ is continuous on $\mathbb{R}$

(8) $\because f(x) = \begin{cases} 5 + x^2 & , \quad x \leq 0 \\ 5 & , \quad x > 0 \end{cases}$

$f(x) = 5 + x^2$ is polynomial

and continuous on $]-\infty , 0[$

$f(x) = 5$ is polynomial and continuous on $]0 , \infty[$

$, f(0) = 5 , f(0^+) = 5 , f(0^-) = 5$

$\because f(0) = f(0^+) = f(0^-)$

$\therefore f$ is continuous at $x = 0$

$\therefore f$ is continuous on $\mathbb{R}$

6

$\because f$ is continuous on $[-\pi , \pi]$

$\therefore f$ is continuous at $x=0$ $\quad \therefore f(0^-)=f(0^+)$

$\therefore \lim_{x\to 0^-} \frac{x+\sin 2x}{\sin \frac{1}{2}x} = \lim_{x\to 0^+}(x+k)$

$\therefore \lim_{x\to 0^-} \frac{1+\frac{\sin 2x}{x}}{\frac{\sin \frac{1}{2}x}{x}} = k \quad \therefore \frac{1+2}{\frac{1}{2}} = k \quad \therefore k=6$

7

$\because f$ is continuous on $\mathbb{R}$

$\therefore f$ is continuous at $x=-1$

$\therefore f(-1^-)=f(-1^+)$

$\therefore \lim_{x\to -1^-}(4x) = \lim_{x\to -1^+}(ax+b)$

$\therefore -4=-a+b$ (1)

, $\because f$ is continuous at $x=3$

$\therefore f(3^-)=f(3^+)$

$\therefore \lim_{x\to 3^-}(ax+b) = \lim_{x\to 3^+}(-2x)$

$\therefore 3a+b=-6$ (2)

From (1) and (2) : $\therefore a=-\frac{1}{2}$, $b=-\frac{9}{2}$

8

$\because f$ is continuous on $\mathbb{R}$

$\therefore f(x)$ has a limit at $x=9$

, $\because$ the denominator $=0$ when $x=9$

$\therefore \sqrt{9-a}-1=0 \quad \therefore \sqrt{9-a}=1 \quad \therefore a=8$

, $\lim_{x\to 9} f(x) = f(9)$

$\therefore \lim_{x\to 9} \frac{\sqrt{x-8}-1}{\sqrt{x}-3} = 9-b$

$\therefore \lim_{x\to 9} \left[\frac{\sqrt{x-8}-1}{\sqrt{x}-3} \times \frac{\sqrt{x-8}+1}{\sqrt{x-8}+1} \times \frac{\sqrt{x}+3}{\sqrt{x}+3}\right] = 9-b$

$\therefore \lim_{x\to 9} \frac{(x-8-1)(\sqrt{x}+3)}{(x-9)(\sqrt{x-8}+1)} = 9-b$

$\therefore 3=9-b \quad \therefore b=6$

Third Higher skills

1

(1) First : $\therefore f(x)=\begin{cases} x+1 & , x<1 \\ 3x-1 & , 1\le x\le 2 \\ 4x-x^2 & , x>2 \end{cases}$

$\therefore f(x)=x+1$ is polynomial and continuous on $]-\infty , 1[$

, $f(x)=3x-1$ is polynomial and continuous on $]1 , 2[$

, $f(x)=4x-x^2$ is polynomial and continuous on $]2 , \infty[$

Second :

$f(1)=2$, $f(1^-)=\lim_{x\to 1^-}(x+1)=2$

, $f(1^+)=\lim_{x\to 1^+}(3x-1)=2$

$\because f(1^-)=f(1^+)=f(1)$

$\therefore f$ is continuous at $x=1$

Third :

$f(2)=5$, $f(2^-)=\lim_{x\to 2^-}(3x-1)=5$

$f(2^+)=\lim_{x\to 2^+}(4x-x^2)=4$

$\because f(2^-)\neq f(2^+)$

$\therefore f$ is not continuous at $x=2$

$\therefore f$ is continuous on $\mathbb{R}-\{2\}$

(2) $f(x)=\begin{cases} x^2+2x & , x<-2 \\ \sqrt{4-x^2} & , -2\le x\le 2 \\ x^2+2x & , x>2 \end{cases}$

First :

$f(x)=x^2+2x$ is polynomial and continuous on $]-\infty , -2[$

, $f(x)=\sqrt{4-x^2}$ is continuous on $]-2 , 2[$

, $f(x)=x^2+2x$ is polynomial and continuous on $]2 , \infty[$

Second :

$f(-2)=0$, $f(-2^+)=\lim_{x\to -2^+}\sqrt{4-x^2}=0$

, $f(-2^-)=\lim_{x\to -2^-}(x^2+2x)=0$

$\because f(-2) = f(-2^+) = f(-2^-)$

$\therefore f$ is continuous at $x = -2$

Third :

$\because f(2) = 0$, $f(2^-) = \lim_{x \to 2^-} \sqrt{4 - x^2} = 0$

, $f(2^+) = \lim_{x \to 2^+} (x^2 + 2x) = 8$

$\because f(2^-) \neq f(2^+)$

$\therefore f$ is not continuous at $x = 2$

$\therefore f$ is continuous on $\mathbb{R} - \{2\}$

2

$\because$ The function is continuous on $\mathbb{R}$

$\therefore$ The equation $(x^2 + a x + 9 = 0)$ has no solution in $\mathbb{R}$

$\therefore b^2 - 4ac < 0$ $\qquad \therefore a^2 - 4 \times 1 \times 9 < 0$

$\therefore a^2 < 36$ $\qquad \therefore |a| < 6$

$\therefore -6 < a < 6$ $\qquad \therefore a \in]-6 , 6[$

3

$\because$ The function is continuous on $\mathbb{R}$

$\therefore$ The equation $(x^2 + 6x + a = 0)$ has no solution in $\mathbb{R}$

$\therefore b^2 - 4ac < 0$ $\qquad \therefore 36 - 4 \times 1 \times a < 0$

$\therefore -4a < -36$ $\qquad \therefore a > 9$ $\qquad \therefore a \in]9 , \infty[$

4

$\because f(x) = \begin{cases} -x & , x < 0 \\ 1 & , x \geq 0 \end{cases}$

$\therefore f(0) = 1$, $f(0^+) = \lim_{x \to 0^+} (1) = 1$

, $f(0^-) = \lim_{x \to 0^-} (-x) = 0$, $\because f(0^+) \neq f(0^-)$

$\therefore$ The function f is discontinuous at $x = 0$

, $\because g(x) = \begin{cases} 1 & , x < 0 \\ x & , x \geq 0 \end{cases}$

$\therefore f(0) = 0$, $f(0^+) = \lim_{x \to 0^+} (x) = 0$

, $f(0^-) = \lim_{x \to 0^-} (1) = 1$ $\quad \therefore f(0^+) \neq (0^-)$

$\therefore$ The function g is not continuous at $x = 0$

, $\because (f \cdot g)(x) = \begin{cases} -x & , x < 0 \\ x & , x \geq 0 \end{cases}$

$\therefore (f \cdot g)(0) = \text{zero}$, $(f \cdot g)(0^+) = \lim_{x \to 0^+} (x) = \text{zero}$

, $(f \cdot g)(0^-) = \lim_{x \to 0^-} (-x) = \text{zero}$

$\therefore (f \cdot g)(0^+) = (f \cdot g)(0^-) = (f \cdot g)(0)$

$\therefore$ The function $(f \cdot g)$ is continuous at $x = 0$

5

(1) $f(4) = 1$, $\lim_{x \to 4} f(x) = -1$

, $\because f(4) \neq \lim_{x \to 4} f(x)$

$\therefore f$ is not continuous at $x = 4$

(2) $g(4) = -6$

, $\lim_{x \to 4} g(x) = \lim_{x \to 4} (4x - 10) = 6$

, $\because g(4) \neq \lim_{x \to 4} g(x)$

$\therefore$ g is not continuous at $x = 4$

(3) Putting : $h(x) = f(x) \cdot g(x)$

$= \begin{cases} -4x + 10 & , x \neq 4 \\ -6 & , x = 4 \end{cases}$

, $\because h(4) = -6$

, $\lim_{x \to 4} h(x) = \lim_{x \to 4} (-4x + 10) = -6$

, $\because h(4) = \lim_{x \to 4} h(x)$

$\therefore$ h is continuous at $x = 4$

(4) Putting : $n(x) = |f(x)| = \begin{cases} 1 & , x \neq 4 \\ 1 & , x = 4 \end{cases}$

, $\because n(4) = \lim_{x \to 4} n(x) = 1$

$\therefore$ n is continuous at $x = 4$

(5) Putting : $m(x) = g(x) - 6 f(x)$

$= \begin{cases} 4x - 4 & , x \neq 4 \\ -12 & , x = 4 \end{cases}$

, $\because m(4) = -12$, $\lim_{x \to 4} m(x) = 12$

, $\because m(4) \neq \lim_{x \to 4} m(x)$

$\therefore$ m is not continuous at $x = 4$

(6) Putting : $k(x) = g\left(f(x)\right) = \begin{cases} -14 & , x \neq 4 \\ -6 & , x = 4 \end{cases}$

, $\because k(4) = -6$, $\lim_{x \to 4} k(x) = -14$

, $\because k(4) \neq \lim_{x \to 4} k(x)$

$\therefore$ k is not continuous at $x = 4$

UNIT
4

Answers of "Unit Four"

Exercise 19

First Multiple choice questions

(1) c	(2) c	(3) a	(4) a	(5) d	(6) a
(7) c	(8) a	(9) b	(10) a	(11) b	(12) c
(13) a	(14) b	(15) b	(16) b	(17) c	(18) a
(19) a	(20) b	(21) c	(22) c	(23) b	(24) c
(25) d	(26) a	(27) b	(28) d	(29) d	(30) c
(31) b	(32) b	(33) d	(34) b	(35) d	(36) a
(37) d	(38) a	(39) c	(40) b	(41) c	(42) d
(43) b	(44) b	(45) d	(46) b	(47) d	

Second Essay questions

1

$\because m(\angle Z) = 180^\circ - (80^\circ + 60^\circ) = 40^\circ$

$\therefore \frac{X}{\sin 80^\circ} = \frac{y}{\sin 60^\circ} = \frac{10}{\sin 40^\circ}$

$\therefore X = \frac{10 \sin 80^\circ}{\sin 40^\circ} \simeq 15$ cm. , $y = \frac{10 \sin 60^\circ}{\sin 40^\circ} \simeq 13$ cm.

2

$\because m(\angle C) = 180^\circ - (112^\circ + 33^\circ) = 35^\circ$

$\therefore \frac{b}{\sin 33^\circ} = \frac{19}{\sin 35^\circ}$

$\therefore b = \frac{19 \sin 33^\circ}{\sin 35^\circ} \simeq 18.04$ cm. , $2r = \frac{19}{\sin 35^\circ}$

$\therefore r = \frac{19}{2 \sin 35^\circ} \simeq 16.56$ cm.

3

(1) $\because m(\angle L) = 180^\circ - (100^\circ + 40^\circ) = 40^\circ$

$\therefore \frac{\ell}{\sin 40^\circ} = \frac{68.4}{\sin 100^\circ}$

$\therefore \ell = \frac{68.4 \sin 40^\circ}{\sin 100^\circ} \simeq 44.64$ cm.

(2) $2r = \frac{68.4}{\sin 100^\circ}$ $\therefore r \simeq 34.73$ cm.

(3) Area of Δ LMN $= \frac{1}{2} \ell \, m \sin N$

$= \frac{1}{2} \times 44.64 \times 68.4 \sin 40^\circ$

$\simeq 981.34 \text{ cm}^2$

4

$m(\angle M) = 180^\circ - (18^\circ 52' + 44^\circ 17') = 116^\circ 51'$

$\therefore \frac{MN}{\sin 18^\circ 52'} = \frac{LM}{\sin 44^\circ 17'} = \frac{35}{\sin 116^\circ 51'}$

$\therefore MN = \frac{35 \sin 18^\circ 52'}{\sin 116^\circ 51'} \simeq 12.7$ cm.

$LM = \frac{35 \sin 44^\circ 17'}{\sin 116^\circ 51'} \simeq 27.4$ cm. and $r = \frac{35}{2 \sin 116^\circ 51'}$

$\therefore$ The area of the circumcircle

$= \pi \left(\frac{35}{2 \sin 116^\circ 51'} \right)^2 \simeq 1208.67 \text{ cm}^2$

5

$m(\angle B) = 180^\circ - (40^\circ + 80^\circ) = 60^\circ$

$\because \angle C$ is the greatest angle in measure

$\therefore$ c is the length of the greatest side

$\therefore \frac{c}{\sin 80^\circ} = \frac{10}{\sin 60^\circ}$ $\therefore c = \frac{10 \sin 80^\circ}{\sin 60^\circ} \simeq 11$ cm.

6

$\because m(\angle B) = 180^\circ - (100^\circ + 15^\circ) = 65^\circ$

$\therefore \angle B$ is the smallest angle in measure and hence

$\therefore$ b is the length of the smallest side

$\therefore \frac{b}{\sin 15^\circ} = \frac{4.5}{\sin 65^\circ}$ $\therefore b = \frac{4.5 \sin 15^\circ}{\sin 65^\circ} \simeq 1.3$ cm.

7

$r = \frac{7\sqrt{3}}{2 \sin 60^\circ} = 7$ cm.

$\therefore$ The area of the circle $= \frac{22}{7} \times 7^2 = 154 \text{ cm}^2$

The circumference of the circle $= 2 \times \frac{22}{7} \times 7 = 44$ cm.

8

$\because m(\angle C) = 180^\circ - (60^\circ + 45^\circ) = 75^\circ$

$\therefore \frac{a}{\text{Sin } 60^\circ} = \frac{b}{\text{Sin } 45^\circ} = \frac{c}{\text{Sin } 75^\circ}$

$\therefore \frac{a}{\frac{\sqrt{3}}{2}} = \frac{b}{\frac{\sqrt{2}}{2}} = \frac{c}{\frac{\sqrt{6}+\sqrt{2}}{4}}$

$\therefore a : b : c = \frac{\sqrt{3}}{2} : \frac{\sqrt{2}}{2} : \frac{\sqrt{6}+\sqrt{2}}{4}$ (multiply by $2\sqrt{2}$)

$= \sqrt{6} : 2 : \sqrt{3} + 1$

9

$\because r = \frac{13}{2 \sin 53^\circ 8'} \simeq 8.1$ cm. , $\frac{13}{\sin 53^\circ 8'} = \frac{15}{\sin C}$

$\therefore \sin C = \frac{15 \sin 53^\circ 8'}{13}$

$\therefore m(\angle C) = 67^\circ 23' 9''$ or $112^\circ 36' 51''$

10

$\because \frac{8}{\sin 35^\circ} = \frac{6}{\sin B}$ $\therefore \sin B = \frac{6 \sin 35^\circ}{8}$

$\therefore m(\angle B) \simeq 25^\circ\ 28'\ 45''$

and the another solution is refused.

11

$\because m(\angle C) = 180^\circ - (57^\circ\ 13' + 64^\circ\ 18') = 58^\circ\ 29'$

$, \because \frac{c}{\sin C} = \frac{a+b+c}{\sin A + \sin B + \sin C}$

$\therefore \frac{8.7}{\sin 58^\circ\ 29'} = \frac{\text{Perimeter of } \Delta ABC}{\sin 57^\circ\ 13' + \sin 58^\circ\ 29' + \sin 64^\circ\ 18'}$

$\therefore$ Perimeter of $\Delta ABC \simeq 26.5$ cm.

12

$\because m(\angle A) = 180^\circ - (45^\circ + 60^\circ) = 75^\circ$

$\therefore \frac{a}{\sin 75^\circ} = \frac{b}{\sin 45^\circ} = \frac{c}{\sin 60^\circ} = 40$

$\therefore a = 40 \sin 75^\circ \simeq 38.6$ cm.

$b = 40 \sin 45^\circ \simeq 28.3$ cm.

and $c = 40 \sin 60^\circ \simeq 34.6$ cm.

The area of $\Delta ABC = \frac{1}{2} \times 38.6 \times 28.3 \sin 60^\circ$

$\simeq 473 \text{ cm}^2$.

The perimeter of $\Delta ABC = 38.6 + 28.3 + 34.6$

$= 101.5 \simeq 102$ cm.

13

$\because \Delta ABC$ is isosceles $\therefore m(\angle B) = m(\angle C) = 30^\circ$

$\therefore \frac{c}{\sin C} = 2r$ $\therefore \frac{c}{\sin 30^\circ} = 24$

$\therefore c = 24 \sin 30^\circ = 12$ cm. $\therefore b = 12$ cm.

$\therefore$ The area of $\Delta ABC = \frac{1}{2} \times 12 \times 12 \times \sin 120^\circ$

$\simeq 62.4 \text{ cm}^2$.

14

$m(\angle C) = 180^\circ - (15^\circ + 15^\circ) = 150^\circ$

$\therefore \frac{a}{\sin 15^\circ} = \frac{b}{\sin 15^\circ} = \frac{c}{\sin 150^\circ}$

$\therefore$ One of the ratios $= \frac{a+b+c}{\sin 15^\circ + \sin 15^\circ + \sin 150^\circ}$

$= \frac{25}{\sin 15^\circ + \sin 15^\circ + \sin 150^\circ} \simeq 24.57$

$\therefore 2r \simeq 24.57$ $\therefore r = 12.285$ cm.

The area of the circle $= \pi\,(12.285)^2 \simeq 474 \text{ cm}^2$.

15

$\because m(\angle C) = 180^\circ - (44^\circ + 66^\circ) = 70^\circ$

$\therefore \frac{a}{\sin 44^\circ} = \frac{b}{\sin 66^\circ} = \frac{c}{\sin 70^\circ}$

$= \frac{40}{\sin 44^\circ + \sin 66^\circ + \sin 70^\circ}$

$\therefore a \simeq 10.9$ cm. , $b \simeq 14.3$ cm. $c \simeq 14.8$ cm.

16

$m(\angle A) = 60^\circ \div 3 = 20^\circ$

$m(\angle C) = 180^\circ - (20^\circ + 60^\circ) = 100^\circ$

$\therefore \frac{a}{\sin 20^\circ} = \frac{12}{\sin 100^\circ}$ $\therefore a = \frac{12 \sin 20^\circ}{\sin 100^\circ} \simeq 4.2$ cm.

$\therefore$ The area of $\Delta ABC = \frac{1}{2} \times 4.2 \times 12 \sin 60^\circ \simeq 22 \text{ cm}^2$.

17

$\because m(\angle A) = 180^\circ - (82^\circ + 56^\circ) = 42^\circ$

$, \because$ the area of $\Delta ABC = \frac{1}{2}\, a\, b \sin C$

$\therefore 450 = \frac{1}{2}\, a\, b \sin 56^\circ$ $\therefore b = \frac{900}{a \sin 56^\circ}$

$, \because \frac{a}{\sin 42^\circ} = \frac{b}{\sin 82^\circ}$ $\therefore \frac{a}{\sin 42^\circ} = \frac{900}{a \sin 56^\circ \sin 82^\circ}$

$\therefore a^2 = \frac{900 \sin 42^\circ}{\sin 56^\circ \sin 82^\circ}$ $\therefore a \simeq 27$ cm.

18

$\because 43.2 = \frac{1}{2}\, AB \times 12 \times 0.6$ $\therefore AB = 12$ cm.

$\because m(\angle A) \approx 36^\circ\ 52'$ (and the other solution is refused because the triangle is acute-angled triangle)

$\therefore m(\angle B) = m(\angle C) = \frac{180^\circ - 36^\circ\ 52'}{2} = 71^\circ\ 34'$

$\therefore \frac{BC}{\sin 36^\circ\ 52'} = \frac{12}{\sin 71^\circ\ 34'}$

$\therefore BC = \frac{12 \times \sin 36^\circ\ 52'}{\sin 71^\circ\ 34'} \simeq 7.6$ cm.

19

$\because \frac{7}{\sin 60^\circ} = \frac{8}{\sin B}$ $\therefore m(\angle B) \simeq 81^\circ\ 47'\ 12''$

(and the other solution is refused because the triangle is acute-angled triangle)

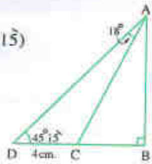
$\therefore m(\angle C) = 180^\circ - (60^\circ + 81^\circ\ 47'\ 12'') = 38^\circ\ 12'\ 48''$

$\therefore \frac{7}{\sin 60^\circ} = \frac{\text{Perimeter of } \Delta ABC}{\sin 60^\circ + \sin 81^\circ\ 47'\ 12'' + \sin 38^\circ\ 12'\ 48''}$

$\therefore$ Perimeter of $\Delta ABC \simeq 20$ cm.

20

In Δ ACD :



$m(\angle ACD) = 180° - (18° + 45°\ 15') = 116°\ 45'$

$\therefore \frac{AD}{\sin 116°\ 45'} = \frac{4}{\sin 18°}$

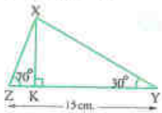
$\therefore AD = \frac{4 \sin 116°\ 45'}{\sin 18°} \simeq 11.6$ cm.

In Δ ABD : $\frac{AB}{\sin 45°\ 15'} = \frac{11.6}{\sin 90°}$

$\therefore AB = \frac{11.6 \sin 45°\ 15'}{\sin 90°} \simeq 8$ cm.

21

In Δ XYZ : $m(\angle X) = 180° - (30° + 70°) = 80°$



$\frac{XY}{\sin 70°} = \frac{15}{\sin 80°}$

$\therefore XY = \frac{15 \sin 70°}{\sin 80°} \simeq 14.31$ cm.

In Δ XYK : $\frac{XK}{\sin 30°} = \frac{14.31}{\sin 90°}$

$\therefore XK = \frac{14.31 \sin 30°}{\sin 90°} = 7.16$ cm.

22

$\tan A = \frac{1}{2\sqrt{2}}$



$\therefore \sin A = \frac{1}{3}$

$\because \frac{8}{\frac{1}{3}} = \frac{20}{\sin C}$ $\qquad \therefore \sin C = \frac{5}{6}$

$\therefore m(\angle C) \simeq 56°\ 27'$ or $m(\angle C) \simeq 123°\ 33'$

$\because \angle C$ is an obtuse angle $\qquad \therefore m(\angle C) \simeq 123°\ 33'$

23

$\because \tan C = \frac{4}{3}$ $\qquad \therefore m(\angle C) \simeq 53°\ 7'\ 48''$

$\therefore m(\angle A) = 180° - (30° + 53°\ 7'\ 48'') = 96°\ 52'\ 12''$

$\therefore \frac{a}{\sin 96°\ 52'\ 12''} = \frac{5}{\sin 30°} = \frac{c}{\sin 53°\ 7'\ 48''}$

$\therefore a = \frac{5 \sin 96°\ 52'\ 12''}{\sin 30°} \simeq 10$ cm.

, $c = \frac{5 \sin 53°\ 7'\ 48''}{\sin 30°} \simeq 8$ cm.

, area of Δ ABC $= \frac{1}{2} a\, c \sin B$

$= \frac{1}{2} \times 10 \times 8 \times \sin 30° = 20\ cm^2$

24

$\frac{x + y + z}{\sin X + \sin Y + \sin Z} = 2r$ $\quad \therefore r = \frac{56.88}{2 \times 2.37} = 12$ cm.

25

$\because \frac{a}{\sin 60°} = \frac{b}{\sin 45°} = \frac{a + b}{\sin 60° + \sin 45°}$

$\therefore \frac{a}{\sin 60°} = \frac{b}{\sin 45°} = \frac{\sqrt{6} + 2}{\sin 60° + \sin 45°}$

$\therefore a = \sqrt{6}$ cm. , $b = 2$ cm.

26

$\sin A : \sin B : \sin C = 2 : 4 : 5$ $\quad \therefore a : b : c = 2 : 4 : 5$

$\therefore a = 2\,m$, $b = 4\,m$ and $c = 5\,m$

$\because c - b = 5\,m - 4\,m = m$ $\qquad \therefore m = 3$

$\therefore a = 6$ cm. and $b = 12$ cm.

27

$\because m(\angle A) : m(\angle B) : m(\angle C) = 3 : 4 : 3$

$\therefore m(\angle A) = 3\,k$, $m(\angle B) = 4\,k$, $m(\angle C) = 3\,k$

$\because m(\angle A) + m(\angle B) + m(\angle C) = 180°$

$\therefore 3\,k + 4\,k + 3\,k = 180°$ $\qquad \therefore k = 18°$

$\therefore \frac{5}{\sin 54°} = \frac{b}{\sin 72°} = \frac{c}{\sin 54°} = \frac{\text{Perimeter of } \Delta ABC}{\sin 54° + \sin 72° + \sin 54°}$

$\therefore$ Perimeter of Δ ABC $\simeq 15.9$ cm.

28

$\because m(\angle A) + m(\angle B) + m(\angle C) = 180°$

$\therefore m + 3\,m + 5\,m = 180°$ $\qquad \therefore m = 20°$

$\therefore \frac{a}{\sin 20°} = \frac{b}{\sin 60°} = \frac{c}{\sin 100°} = \frac{16}{\sin 20° + \sin 60° + \sin 100°}$

$\because \angle A$ is the smallest angle in measure.

$\therefore$ a is the smallest side in length. $\qquad \therefore a \simeq 2.5$ cm.

29

$\because m(\angle A) + \frac{3}{2} m(\angle A) + 2\, m(\angle A) = 180°$

$\therefore m(\angle A) = 40°$, $m(\angle B) = 60°$, $m(\angle C) = 80°$

, $\because \frac{a}{\sin 40°} = \frac{b}{\sin 60°} = \frac{c}{\sin 80°} = 20$

$\therefore a = 20 \sin 40°$, $b = 20 \sin 60°$

$\therefore$ Area of Δ ABC $= \frac{1}{2} \times 20 \sin 40° \times 20 \sin 60° \times \sin 80°$

$\simeq 110\ cm^2$

30

$\because 6 \sin A = 4 \sin B = 3 \sin C$

$\therefore \frac{\sin A}{2} = \frac{\sin B}{3} = \frac{\sin C}{4}$

Put $a = 2\,m$, $b = 3\,m$, $c = 4\,m$

$\therefore 2\,m + 3\,m + 4\,m = 45$ $\quad \therefore m = 5$

$\therefore a = 10$ cm. and $c = 20$ cm.

31

$\frac{a+b-c+a+c-b}{3+5} = \frac{a+c-b+b+c-a}{5+7}$

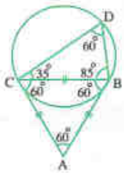
$= \frac{a+b-c+b+c-a}{3+7}$

$\therefore \frac{2a}{8} = \frac{2c}{12} = \frac{2b}{10}$ $\quad \therefore \frac{a}{4} = \frac{c}{6} = \frac{b}{5}$

$\therefore a : b : c = 4 : 5 : 6$

$\therefore \sin A : \sin B : \sin C = 4 : 5 : 6$

32



$\because \overline{AB}$ and $\overline{AC}$ are two tangent segments to the circle at B and C

$\therefore AB = AC$

, $\because m(\angle A) = 60°$

$\therefore \Delta ABC$ is an equilateral triangle

, $\because$ the area of the triangle

$= \frac{1}{2} \times AB \times AC \times \sin(\angle A)$

$\therefore 9\sqrt{3} = \frac{1}{2} \times (AB)^2 \times \sin 60°$ $\quad \therefore 9\sqrt{3} = \frac{\sqrt{3}}{4} \times (AB)^2$

$\therefore AB = 6$ cm. $\quad \therefore BC = 6$ cm.

, $\because m(\angle D) = m(\angle BCA) = 60°$

(inscribed angle and tangency angle subtended by the same arc $\overset{\frown}{BC}$)

$\therefore m(\angle BCD) = 180° - (85° + 60°) = 35°$

$\therefore$ In ΔBCD : $\frac{BD}{\sin 35°} = \frac{DC}{\sin 85°} = \frac{6}{\sin 60°}$

$\therefore BD = \frac{6 \sin 35°}{\sin 60°} \simeq 4$ cm. , $DC = \frac{6 \sin 85°}{\sin 60°} \simeq 7$ cm.

$\therefore$ The perimeter of the triangle DBC

$= 6 + 4 + 7 = 17$ cm.

33

In ΔABC : $m(\angle C) = \frac{1}{2}\, m(\angle AMB) = 40°$

$\therefore m(\angle ABC) = 180° - (40° + 85°) = 55°$

$\therefore \frac{5}{\sin 40°} = \frac{b}{\sin 55°} = \frac{a}{\sin 85°} = 2r$

$= \frac{\text{Perimeter of } \Delta ABC}{\sin 40° + \sin 55° + \sin 85°}$

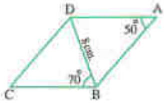
$\therefore$ Perimeter of $\Delta ABC \simeq 19.12$ cm.

, $r = 3.89$ cm

$\therefore$ The area of the circle $M = \pi r^2 = \pi (3.89)^2$

$\simeq 47.5\ cm^2$.

34



$\because$ ABCD is a parallelogram

$\therefore m(\angle C) = 50°$

In ΔBDC :

$m(\angle BDC) = 180° - (50° + 70°) = 60°$

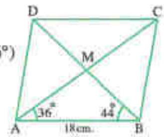
$\therefore \frac{8}{\sin 50°} = \frac{BC}{\sin 60°} = \frac{DC}{\sin 70°}$

$\therefore BC \simeq 9$ cm. , $DC \simeq 9.8$ cm.

$\therefore$ The perimeter of the parallelogram $= 2(BC + CD)$

$\simeq 38$ cm.

35



In ΔABM :

$m(\angle AMB) = 180° - (44° + 36°)$

$= 100°$

$\therefore \frac{18}{\sin 100°} = \frac{AM}{\sin 44°}$

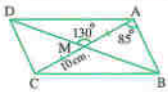
$\therefore AM = \frac{18 \times \sin 44°}{\sin 100°}$

$\therefore AC = 2(AM) \simeq 25.39$ cm.

, the area of $\square$ ABCD $= 2 \times \frac{1}{2} \times 18 \times 25.39 \sin 36°$

$\simeq 269\ cm^2$.

36



In ΔABM :

$m(\angle M) = 50°$

, $m(\angle B) = 180° - (85° + 50°) = 45°$

$\therefore \frac{10}{\sin 45°} = \frac{BM}{\sin 85°}$

$\therefore BM = \frac{10 \sin 85°}{\sin 45°} \simeq 14.1$ cm.

$\therefore BD = 2\,BM = 28.2$ cm.

, area of $\square$ ABCD

$= 4 \times \frac{1}{2} \times 10 \times 14.1 \sin 50° \simeq 216\ cm^2$.

37



$\because \overline{AD} \parallel \overline{BC}$

$\therefore m(\angle ADC) + m(\angle DCB) = 180°$

$\therefore m(\angle DCB) = 180° - 120° = 60°$

$\therefore m(\angle DCA) = 60° - 23°\ 25' = 36°\ 35'$

In ΔADC : $\frac{20}{\sin 36°\ 35'} = \frac{AC}{\sin 120°}$

$\therefore AC = \frac{20 \sin 120°}{\sin 36°\ 35'} = 29$ cm.

In ΔABC : $m(\angle A) = 180° - (62° + 23°\ 25') = 94°\ 35'$

$\therefore \frac{29}{\sin 62°} = \frac{BC}{\sin 94°\ 35'}$

$\therefore BC = \frac{29 \sin 94°\ 35'}{\sin 62°} = 33$ cm.

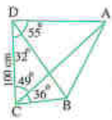
$\therefore$ The area of the trapezium

$= \frac{1}{2} AC \times BC \times \sin 23°\ 25'$

$+ \frac{1}{2} AC \times AD \times \sin 23°\ 25'$

$= \frac{1}{2} \times 29\ (33 + 20) \sin 23°\ 25' \simeq 305\ cm^2.$

38



In ΔBDC :

$m(\angle DBC) = 180° - (32° + 85°) = 63°$

$\therefore \frac{BD}{\sin 85°} = \frac{100}{\sin 63°}$

$\therefore BD = \frac{100 \sin 85°}{\sin 63°} = 112$ cm.

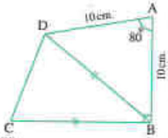
, in ΔADC :

$m(\angle DAC) = 180° - (49° + 87°) = 44°$

$\therefore \frac{AC}{\sin 87°} = \frac{100}{\sin 44°}$

$\therefore AC = \frac{100 \sin 87°}{\sin 44°} = 144$ cm.

39



In ΔABD :

$\because AB = AD$

$\therefore m(\angle ABD) = m(\angle ADB)$

$= \frac{180° - 80°}{2} = 50°$

$\therefore \frac{10}{\sin 50°} = \frac{BD}{\sin 80°}$ $\quad \therefore BD = \frac{10 \sin 80°}{\sin 50°}$

, $\because m(\angle ABC) = 90°$

$\therefore m(\angle DBC) = 90° - 50° = 40°$

$\therefore$ Area of $\Delta ABD = \frac{1}{2} \times 10 \times 10 \times \sin 80° \simeq 49\ cm^2.$

, area of $\Delta DBC = \frac{1}{2} \times \frac{10 \sin 80°}{\sin 50°} \times \frac{10 \sin 80°}{\sin 50°} \times \sin 40°$

$\simeq 53\ cm^2.$

$\therefore$ Area of $ABCD = 49 + 53 = 102\ cm^2.$

40

(1) $\because \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Multiplying the first ratio by 3

and the second ratio by 4 , then by subtracting :

$\therefore \frac{3a - 4b}{3 \sin A - 4 \sin B} = \frac{c}{\sin C}$

(2) $\because \frac{a}{\sin A} = \frac{b}{\sin B}$ $\quad \therefore b = \frac{a \sin B}{\sin A}$

$\therefore$ The area of the triangle ABC $= \frac{1}{2}$ a b sin C

$= \frac{a^2 \sin B \sin C}{2 \sin A}$

(3) $\because \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2r$ $\quad \therefore \sin C = \frac{c}{2r}$

$\therefore$ The area of $\Delta ABC = \frac{1}{2} a b \times \frac{c}{2r} = \frac{abc}{4r}$

Third Higher skills

1

(1) (d)	(2) (c)	(3) (c)	(4) (b)
(5) (d)	(6) (c)	(7) (d)	(8) (b)
(9) (c)	(10) (b)	(11) (c)	

Instructions to solve 1 :

(1) $\because \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2r = 6$

$\therefore \frac{a}{\sin A} \times \frac{b}{\sin B} \times \frac{c}{\sin C} = 6 \times 6 \times 6$

$\therefore \frac{abc}{\sin A \sin B \sin C} = 216$

(2) $\because \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2r$

$\therefore a \csc A = b \csc B = c \csc C = 2r$

$\therefore a \csc A + b \csc B + \csc C = 6r$

(3) $\because \frac{a + b + c}{\sin A + \sin B + \sin C} = 2r$

$\therefore \frac{\sin B + \sin C + \sin A}{\sin A + \sin B + \sin C} = 2r$

$\therefore 2r = 1$ $\quad \therefore r = \frac{1}{2}$

$\therefore$ The circumference of the circumcircle of $\Delta ABC = 2\pi r = 2\pi \times \frac{1}{2} = \pi$

(4) $\because \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2r$

$\therefore \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{1}{2r}$

Multiply up and down the first ratio by a , the second ratio by b and third ratio by c , and by adding the antecedents and consequents

$\therefore \frac{a \sin A + b \sin B + c \sin C}{a^2 + b^2 + c^2} = \frac{1}{2r}$

(5) $\because \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 10$

$\therefore a = 10 \sin A$, $b = 10 \sin B$

, $\because$ the area of $\Delta ABC = 24$ $\therefore \frac{1}{2} ab \sin c = 24$

$\therefore \frac{1}{2} \times 10 \sin A \times 10 \sin B \times \sin C = 24$

$\therefore \sin A \sin B \sin C = \frac{24}{50} = \frac{12}{25}$

$\therefore \sin A \sin B \sin \left(180^\circ - (A + B)\right) = \frac{12}{25}$

$\therefore \sin A \sin B \sin (A + B) = \frac{12}{25}$

(6) $\because BC = \sqrt{(6)^2 + (8)^2} = 10$ cm. $\therefore BD = 4$ cm.

In ΔABD : $\frac{BD}{\sin \theta} = \frac{AD}{\sin B}$ $\therefore \frac{4}{\sin \theta} = \frac{AD}{0.6}$

$\therefore AD \sin \theta = 2.4$ (1)

In ΔACD : $\frac{DC}{\sin (90^\circ - \theta)} = \frac{AD}{\sin C}$ $\therefore \frac{6}{\cos \theta} = \frac{AD}{0.8}$

$\therefore AD \cos \theta = 4.8$ (2)

Divide (2) by (1) : $\therefore \frac{\cos \theta}{\sin \theta} = \frac{4.8}{2.4}$

$\therefore \cot \theta = 2$

(7) $\because CD = 2\,DB$

$\therefore$ Area of $\Delta ACD = 2$ area of ΔABD

$\therefore \frac{1}{2} \times AD \times AC \times \sin 60^\circ$

$= 2 \times \frac{1}{2} \times AB \times AD \times \sin 45^\circ$

$\therefore \frac{AB}{AC} = \frac{\left(\frac{\sqrt{3}}{2}\right)}{\left(2 \times \frac{1}{\sqrt{2}}\right)} = \frac{\sqrt{6}}{4}$

, $\because \frac{AB}{AC} = \frac{\sin C}{\sin B}$ $\therefore \frac{\sin B}{\sin C} = \frac{4}{\sqrt{6}} = \frac{2\sqrt{6}}{3}$

(8) In ΔABD : $\because \frac{BD}{\sin A} = \frac{AD}{\sin 45^\circ}$

$\therefore \frac{BD}{\sin A} = \frac{4\sqrt{2}}{\left(\frac{1}{\sqrt{2}}\right)} = 8$

In ΔBDC : $\because \frac{BD}{\sin C} = \frac{BC}{\sin 30^\circ}$ $\therefore \frac{BD}{\sin C} = 2\,BC$

, $\because \sin A = \sin C$ $\therefore 2\,BC = 8$ $\therefore BC = 4$ cm.

(9) In ΔCDE : $\because \tan (\angle DEC) = \frac{3}{4}$ $\therefore \sin C = \frac{4}{5}$

, in ΔABC : $\because \frac{AB}{\sin C} = 2r$

$\therefore 2r = \frac{6}{\left(\frac{4}{5}\right)}$ $\therefore r = 3.75$ cm.

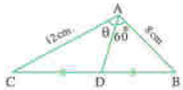
(10) In ΔABD : $\frac{BD}{\sin 60^\circ} = \frac{8}{\sin (\angle ADB)}$

$\therefore BD \times \sin (\angle ADB) = 8 \sin 60^\circ = 4\sqrt{3}$ (1)

In ΔADC : $\frac{CD}{\sin \theta} = \frac{12}{\sin (\angle ADC)}$

$\therefore CD \times \sin (\angle ADC) = 12 \sin \theta$ (2)

From (1) , (2) :

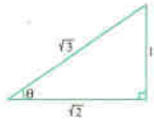


$\because BD = CD$

, $\sin (\angle ADB)$

$= \sin (\angle ADC)$

$\therefore 12 \sin \theta = 4\sqrt{3}$



$\therefore \sin \theta = \frac{1}{\sqrt{3}}$

$\therefore \tan \theta = \frac{1}{\sqrt{2}}$

$\therefore \tan (\angle DAC) = \frac{1}{\sqrt{2}}$

(11) In ΔABC : $\because \frac{AB}{\sin 70^\circ} = \frac{6}{\sin 40^\circ} = 2r$

$\therefore AB = 8.77$ cm. , $r = 4.667$ cm.

$\therefore$ Area of the shaded region

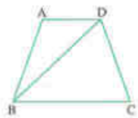
$=$ Area of the circle $-$ area of triangle ABC

$= \pi r^2 - \frac{1}{2} a c \sin B$

$= \pi \times (4.667)^2 - \frac{1}{2} \times 6 \times 8.77 \times \sin 70^\circ$

$= 43.7 \text{ cm}^2$

2



In ΔABD :

$\frac{AD}{\sin (\angle ABD)} = \frac{BD}{\sin A}$

, in ΔDBC : $\frac{BC}{\sin (\angle BDC)} = \frac{BD}{\sin C}$

, $\because \angle A$ and $\angle C$ are supplementary

«ABCD is a cyclic quadrilateral»

$\therefore \sin A = \sin C$

$\therefore \frac{AD}{\sin (\angle ABD)} = \frac{BC}{\sin (\angle BDC)}$

$\therefore AD \sin (\angle BDC) = BC \sin (\angle ABD)$

3

(1) $\because \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$\therefore \frac{a+b+c}{\sin A + \sin B + \sin C} = \frac{a}{\sin A}$

$\therefore \sin A + \sin B + \sin C = \frac{2 S \sin A}{a}$

$= \frac{2 S \sin A \times bc}{a\, b\, c}$

, $\because \Delta = \frac{1}{2}\, bc \sin A$ $\qquad \therefore bc \sin A = 2\,\Delta$

$\therefore \sin A + \sin B + \sin C = \frac{4 S \Delta}{a\, b\, c}$

(2) $\because \frac{1}{c \sin A} = \frac{\frac{1}{2} b}{\frac{1}{2} b\, c \sin A}$, $\frac{1}{a \sin B} = \frac{\frac{1}{2} c}{\frac{1}{2} a\, c \sin B}$

, $\frac{1}{b \sin C} = \frac{\frac{1}{2} a}{\frac{1}{2} b \sin C}$

By adding : $\therefore \frac{1}{c \sin A} + \frac{1}{a \sin B} + \frac{1}{b \sin C}$

$= \frac{\frac{1}{2} b}{\frac{1}{2} b\, c \sin A} + \frac{\frac{1}{2} c}{\frac{1}{2} a\, c \sin B} + \frac{\frac{1}{2} a}{\frac{1}{2} b\, a \sin C}$

$= \frac{\frac{1}{2}(b + c + a)}{\Delta} = \frac{S}{\Delta}$

4

The area of the circle $= \pi r^2$ (1)

$r = \frac{a}{2 \sin A} = \frac{b}{2 \sin B}$ (2)

From (1) and (2) we get :

The area of the circle $= \frac{\pi a b}{4 \sin A \sin B}$

Exercise 20

First Multiple choice questions

(1) c	(2) a	(3) b	(4) b	(5) d	(6) a
(7) a	(8) c	(9) a	(10) b	(11) b	(12) d
(13) d	(14) d	(15) b	(16) a	(17) b	(18) c
(19) d	(20) b	(21) c	(22) b	(23) c	(24) d
(25) b	(26) d	(27) c	(28) b	(29) b	(30) b
(31) b	(32) b	(33) c	(34) b	(35) a	

Second Essay questions

1

$\because z^2 = (13)^2 + (16)^2 - 2 \times 13 \times 16 \cos 95^\circ \simeq 461.26$

$\therefore z \simeq 21.5$ cm.

2

$\because b^2 = (3)^2 + (5)^2 - 2 \times 3 \times 5 \cos 36^\circ\, 21' \simeq 9.8$

$\therefore b \simeq 3$ cm.

3

$\because \cos A = \frac{(5.8)^2 + (3.4)^2 - (7.6)^2}{2\,(5.8)\,(3.4)}$

$\therefore m(\angle A) \simeq 108^\circ\, 34'$

, $\cos B = \frac{(7.6)^2 + (3.4)^2 - (5.8)^2}{2\,(7.6)\,(3.4)}$

$\therefore m(\angle B) \simeq 46^\circ\, 20'$

$\therefore m(\angle C) = 180^\circ - (108^\circ\, 34' + 46^\circ\, 20') = 25^\circ\, 6'$

4

$\because \cos B = \frac{(13)^2 + (15)^2 - (14)^2}{2\,(13)\,(15)}$

$\therefore m(\angle B) = 59^\circ\, 29'$

$\therefore$ The area of $\Delta ABC = \frac{1}{2} \times 13 \times 15 \times \sin 59^\circ\, 29'$

$\simeq 84\ cm^2$

5

$\because X$ is the smallest side in length

$\therefore \angle X$ is the smallest angle in measure.

$\therefore \cos X = \frac{(27)^2 + (24)^2 - (18)^2}{2 \times 27 \times 24}$ $\qquad \therefore m(\angle X) \simeq 40^\circ\, 48'$

$\because r = \frac{18}{2 \sin 40^\circ\, 48'}$

$\therefore$ The area of the circle $= \pi \left(\frac{18}{2 \sin 40^\circ\, 48'}\right)^2$

$\simeq 596\ cm^2$

6

(1) $c^2 = (7)^2 + (9)^2 - 2\,(7)\,(9) \cos 96^\circ\, 23' \simeq 144$

$\therefore c \simeq 12$ cm.

(2) The area of $\Delta ABC = \frac{1}{2} \times 7 \times 9 \times \sin 96^\circ\, 23'$

$\simeq 31\ cm^2$

(3) $r = \frac{c}{2 \sin C} = \frac{12}{2 \sin (96^\circ\, 23')} \simeq 6$ cm.

7

$c = 52 - (13 + 17) = 22$ cm.

$\because c$ is the greatest side in length

$\therefore \angle C$ is the greatest angle in measure.

$\therefore \cos C = \frac{(13)^2 + (17)^2 - (22)^2}{2\,(13)\,(17)}$

$\therefore m(\angle C) \simeq 93^\circ\ 22'$

, the area of $\Delta ABC = \frac{1}{2} \times 13 \times 17 \times \sin 93^\circ\ 22'$

$\simeq 110\ cm^2$

8

$\because$ X is the greatest side in length

$\therefore \angle X$ is the greatest angle in measure.

$\therefore \cos X = \frac{(18)^2 + (10)^2 - (24.5)^2}{2 \times 18 \times 10}$

$\therefore m(\angle X) \simeq 119^\circ\ 19'$

$\because r = \frac{24.5}{2 \sin 119^\circ\ 19'} \simeq 14$ cm.

$\therefore$ The circumference of the circumcircle of Δ XYZ

$= 2 \times \frac{22}{7} \times 14 = 88$ cm.

9

Let $X = 4$ m , $y = 5$ m and $z = 6$ m

$\because$ X is the smallest side in length.

$\therefore \angle X$ is the smallest angle in measure.

$\therefore \cos X = \frac{(5\ m)^2 + (6\ m)^2 - (4\ m)^2}{2 \times 5\ m \times 6\ m} = \frac{3}{4}$

$\therefore m(\angle X) \simeq 41^\circ\ 25'$

10

$\because X : y : z = \sin X : \sin Y : \sin Z = 7 : 8 : 12$

, let $X = 7$ m. , $y = 8$ m. , $z = 12$ m.

, $\because$ z is the greatest side in length.

$\therefore \angle Z$ is the greatest angle in measure.

$\therefore \cos Z = \frac{(7\ m)^2 + (8\ m)^2 - (12\ m)^2}{2 \times 7\ m \times 8\ m} = \frac{-31}{112}$

$\therefore m(\angle Z) \simeq 106^\circ\ 4'$

11

$\because c^2 = (4)^2 + (5)^2 - 2 \times 4 \times 5 \times -\frac{1}{2} = 61$

$\therefore c \simeq 7.8$ cm.

$\because \cos C = -\frac{1}{2}$ $\quad\therefore m(\angle C) = 120^\circ$

$\therefore$ The area of the triangle $= \frac{1}{2} \times 4 \times 5 \times \sin 120^\circ$

$= 5\sqrt{3}\ cm^2$

12

$\tan B = \frac{3}{4}$ $\quad\therefore \sin B = \frac{3}{5}$, $\cos B = \frac{4}{5}$

$\therefore$ The area of $\Delta ABC = \frac{1}{2} \times 16 \times 18 \times \frac{3}{5} = 86.4\ cm^2$

, $b^2 = (16)^2 + (18)^2 - 2 \times 16 \times 18 \times \frac{4}{5} = 119.2$

$\therefore b \simeq 11$ cm.

$\therefore$ The perimeter of $\Delta ABC = 16 + 18 + 11 = 45$ cm.

13

$\because 2 \sin A = 3 \sin B = 4 \sin C$ $\quad\therefore \frac{\sin A}{6} = \frac{\sin B}{4} = \frac{\sin C}{3}$

$\therefore a : b : c = 6 : 4 : 3$

Let $a = 6$ m , $b = 4$ m and $c = 3$ m

$\because$ c is the smallest side in length

$\therefore \angle C$ is the smallest angle in measure.

$\therefore \cos C = \frac{(6\ m)^2 + (4\ m)^2 - (3\ m)^2}{2 \times 6\ m \times 4\ m} = \frac{43}{48}$

$\therefore m(\angle C) \simeq 26^\circ\ 23'$

14

$\because a : b : c = 3 : 4 : 5$

Let $a = 3\,k$, $b = 4\,k$, $c = 5\,k$

$\therefore \cos C = \frac{(3\,k)^2 + (4\,k)^2 - (5\,k)^2}{2 \times 3\,k \times 4\,k} =$ zero

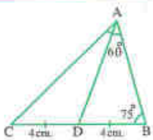
$\therefore m(\angle C) = 90^\circ$

$\because 3\,k + 4\,k + 5\,k = 24$ $\quad\therefore k = 2$

$\therefore a = 6$ cm. , $b = 8$ cm. , $c = 10$ cm.

$\therefore$ The area of $\Delta ABC = \frac{1}{2} \times 6 \times 8 = 24\ cm^2$

15



In ΔABC : $\because \frac{8}{\sin 60^\circ} = \frac{AC}{\sin 75^\circ}$

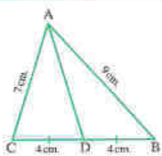
$\therefore AC = \frac{8 \sin 75^\circ}{\sin 60^\circ} \simeq 8.92$ cm.

$\because m(\angle C) = 45^\circ$

$\therefore (AD)^2 = (8.92)^2 + (4)^2 - 2 \times 8.92 \times 4 \cos 45^\circ \simeq 45$

$\therefore AD \simeq 6.7$ cm.

16



In ΔABC :

$\because \cos B = \frac{(8)^2 + (9)^2 - (7)^2}{2 \times 8 \times 9} = \frac{2}{3}$

$\therefore m(\angle B) \simeq 48^\circ\ 11'$

In Δ ABD :

$(AD)^2 = (9)^2 + (4)^2 - 2 \times 9 \times 4 \cos 48° 11' = 49$

$\therefore$ AD = 7 cm.

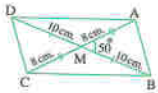
In Δ ABC : $r = \frac{7}{2 \times \sin 48° 11'} = 4.7$ cm.

17

In Δ AMB :

$\because (AB)^2 = (8)^2 + (10)^2 - 2 \times 8 \times 10 \cos 50°$

$\therefore$ AB = 8 cm.



In Δ AMD :

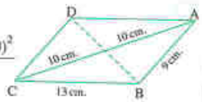
$\therefore m(\angle AMD) = 180° - 50°$

$= 130°$

$\therefore (AD)^2 = (8)^2 + (10)^2 - 2 \times 8 \times 10 \cos 130°$

$\therefore$ AD $\simeq$ 16 cm.

18



In Δ ABC :

$\because \cos B = \frac{(9)^2 + (13)^2 - (20)^2}{2\,(9)\,(13)}$

$\therefore m(\angle B) \simeq 129° 52'$

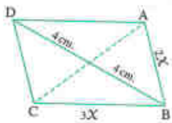
$\therefore m(\angle BAD) = 180° - 129° 52' \simeq 50° 8'$

In Δ ABD :

$(BD)^2 = (9)^2 + (13)^2 - 2\,(9)\,(13) \cos 50° 8'$

$\therefore$ BD $\simeq$ 10 cm.

19



Let AB = 2 X , BC = 3 X

, half of the perimeter of the parallelogram = 10 cm.

$\therefore 2X + 3X = 10 \quad \therefore X = 2$

$\therefore$ AB = 4 cm. , BC = 6 cm.

In Δ ABD :

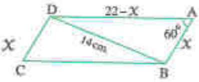
$\because \cos A = \frac{(4)^2 + (6)^2 - (8)^2}{2\,(4)\,(6)} \quad \therefore m(\angle A) \simeq 104° 29'$

$\therefore m(\angle ABC) = 180° - 104° 29' = 75° 31'$

$\therefore (AC)^2 = (4)^2 + (6)^2 - 2\,(4)\,(6) \cos 75° 31'$

$\therefore$ AC $\simeq$ 6.3 cm.

20



AB + AD = 22 cm.

Let AB = X $\therefore$ AD = 22 − X

In Δ ABD :

$(BD)^2 = X^2 + (22 - X)^2 - 2X(22 - X) \cos 60°$

$\therefore 196 = 3X^2 - 66X + 484$

$\therefore 3X^2 - 66X + 288 = 0 \quad \therefore X^2 - 22X + 96 = 0$

$\therefore (X - 6)(X - 16) = 0 \quad \therefore X = 6$ or $X = 16$ (refused)

$\therefore$ AB = 6 cm. and AD = 16 cm.

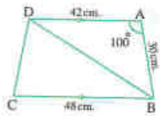
$\therefore \frac{6}{\sin(\angle ADB)} = \frac{14}{\sin 60°} \quad \therefore \sin(\angle ADB) = \frac{6 \sin 60°}{14}$

$\therefore m(\angle ADB) \simeq 21° 47'$

$\therefore$ The area of ▱ ABCD = 2 × the area of Δ ABD

$= 2 \times \frac{1}{2} \times 6 \times 16 \sin 60° \simeq 83 \text{ cm}^2$.

21



In Δ ABD :

$\because (BD)^2 = (30)^2 + (42)^2 - 2 \times 30 \times 42 \cos 100°$

$\therefore$ BD $\simeq$ 55.7 cm.

$, \cos(\angle ADB) = \frac{(55.7)^2 + (42)^2 - (30)^2}{2 \times 55.7 \times 42} \simeq 0.85$

$, \because \overline{AD} \,/\!/\, \overline{BC}$

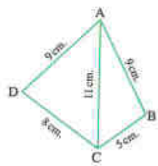
$\therefore m(\angle DBC) = m(\angle ADB)$ «Alternate angles»

$\therefore \cos(\angle DBC) = 0.85$

$\therefore (CD)^2 = (55.7)^2 + (48)^2 - 2 \times 55.7 \times 48 \times 0.85$

$\therefore$ CD $\simeq$ 29.3 cm.

22



In Δ ABC :

$\cos B = \frac{(9)^2 + (5)^2 - (11)^2}{2 \times 9 \times 5} = -\frac{1}{6}$

, in Δ ADC :

$\cos D = \frac{(9)^2 + (8)^2 - (11)^2}{2 \times 9 \times 8} = \frac{1}{6}$

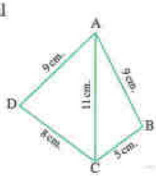
$\therefore \cos B = -\cos D$

$\therefore m(\angle B) + m(\angle D) = 180°$

$\therefore$ The figure is a cyclic quadrilateral.

23

$\because$ ABCD is a cyclic quadrilateral



$\therefore \cos B = -\cos D$

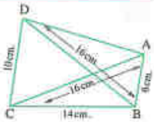
$\therefore \frac{(9)^2+(5)^2-(AC)^2}{2\times 9\times 5} = \frac{-(9)^2-(8)^2+(AC)^2}{2\times 9\times 8}$

$\therefore \frac{106-(AC)^2}{5} = \frac{-145+(AC)^2}{8}$

$\therefore 848 - 8(AC)^2 = -725 + 5(AC)^2$

$\therefore 13(AC)^2 = 1573 \quad \therefore (AC)^2 = 121 \quad \therefore AC = 11$ cm.

24



In Δ ABC :

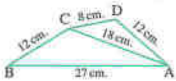
$\cos(\angle BAC) = \frac{(6)^2+(16)^2-(14)^2}{2\times 6\times 16} = \frac{1}{2}$

In Δ BCD : $\cos(\angle BDC) = \frac{(10)^2+(16)^2-(14)^2}{2\times 10\times 16} = \frac{1}{2}$

$\therefore m(\angle BAC) = m(\angle BDC)$ and they are drawn on the same base.

$\therefore$ ABCD is a cyclic quadrilateral.

25



In Δ ADC : $\cos(\angle DAC)$

$= \frac{(12)^2+(18)^2-(8)^2}{2\times 12\times 18} = \frac{101}{108}$

$\therefore m(\angle DAC) \simeq 20° 45'$

, in Δ CAB : $\cos(\angle CAB) = \frac{(27)^2+(18)^2-(12)^2}{2(27)(18)} = \frac{101}{108}$

$\therefore m(\angle DAC) = m(\angle CAB)$

$\therefore \overrightarrow{AC}$ bisects $\angle$ BAD

$\therefore$ The area of the figure ABCD

= The area of Δ ADC + the area of Δ ACB

$= \frac{1}{2}\times 12\times 18\times \sin 20° 45' + \frac{1}{2}\times 18\times 27 \times \sin 20° 45' \simeq 124\ cm^2$.

26



In Δ ADB :

$AB = \sqrt{(10)^2-(8)^2} = 6$ cm.

, $\cos B = \frac{3}{5}$

$\therefore m(\angle B) \simeq 53° 8'$

In Δ DBC : $DB = \frac{1}{2}$ DC

$\therefore DC = 20$ cm. , $BC = 10\sqrt{3}$ cm.

In Δ ABC : $m(\angle ABC) = 90° + 53° 8' = 143° 8'$

$\therefore (AC)^2 = (6)^2 + \left(10\sqrt{3}\right)^2 - 2(6)\left(10\sqrt{3}\right)\cos 143° 8'$

$\therefore AC \simeq 22$ cm.

27

$\because c^2 = (3b)^2 + b^2 - 2\times 3b\times b\cos 60°$

$= 10b^2 - 6b^2 \times \frac{1}{2} = 7b^2$

$\therefore c = \sqrt{7}\,b$

$\therefore \cos B = \frac{7b^2+9b^2-b^2}{2\times\sqrt{7}b\times 3b} = \frac{15}{6\sqrt{7}} = \frac{5}{2\sqrt{7}}$

$\therefore m(\angle B) \simeq 19° 6'$

$\therefore m(\angle A) = 180° - (19° 6' + 60°) = 100° 54'$

28

$\because 10\sqrt{3} = \frac{1}{2}\times 5c\times \sin 120° \quad \therefore c = 8$ cm.

$\therefore b^2 = (5)^2 + (8)^2 - 2\times 5\times 8\cos 120° = 129$

$\therefore b \simeq 11.36$ cm.

$\therefore \cos A = \frac{(11.36)^2+(8)^2-(5)^2}{2\times 11.36\times 8} \quad \therefore m(\angle A) \simeq 22° 24'$

29

$\because b : c = 3 : 4$, let $b = 3m$, $c = 4m$

, $\because$ the area of $\Delta ABC = 64\ cm^2$

$\therefore \frac{1}{2}\times 3m\times 4m\times \sin 30° = 64$

$\therefore m = \frac{8}{\sqrt{3}} \qquad \therefore b = \frac{24}{\sqrt{3}}$ cm. , $c = \frac{32}{\sqrt{3}}$ cm.

$\therefore a^2 = \left(\frac{32}{\sqrt{3}}\right)^2 + \left(\frac{24}{\sqrt{3}}\right)^2 - 2\times\frac{32}{\sqrt{3}}\times\frac{24}{\sqrt{3}}\cos 30°$

$\therefore a \simeq 9.5$ cm.

$\therefore$ The perimeter of Δ ABC $\simeq 41.8$ cm.

30

$\because$ The area of $\Delta ABC = \frac{1}{2}ab\sin C$

$\therefore 20 = \frac{1}{2}\times 6\times 10\sin C \qquad \therefore \sin C = \frac{2}{3}$

$\therefore m(\angle C) \simeq 138° 11'$ «$\angle$ C is obtuse»

$\therefore c^2 = (6)^2 + (10)^2 - 2\times 6\times 10\cos 138° 11'$

$\therefore c \simeq 15$ cm.

31

Let : $\sin A = m$, $\sin B = \frac{3}{2}m$ and $\sin C = 2m$

, $\because \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$\therefore \frac{a}{m} = \frac{b}{\frac{3}{2}m} = \frac{c}{2m} = \frac{c-a}{2m-m} = \frac{4}{m}$

$\therefore a = 4$ cm. $\qquad \therefore b = 6$ cm. , $c = 8$ cm.

$\therefore \cos A = \frac{(6)^2+(8)^2-(4)^2}{2\times 6\times 8} \qquad \therefore m(\angle A) \simeq 28° 57'$

32

$\because \sin A : \sin B : \sin C = 3 : 5 : 7$

$\therefore a : b : c = 3 : 5 : 7$

and let : $a = 3$ m , $b = 5$ m and $c = 7$ m

$\therefore \cos A : \cos B : \cos C$

$= \frac{(5\text{ m})^2 + (7\text{ m})^2 - (3\text{ m})^2}{2 \times 5\text{ m} \times 7\text{ m}} : \frac{(3\text{ m})^2 + (7\text{ m})^2 - (5\text{ m})^2}{2 \times 3\text{ m} \times 7\text{ m}}$

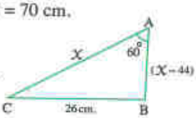
$: \frac{(3\text{ m})^2 + (5\text{ m})^2 - (7\text{ m})^2}{2 \times 3\text{ m} \times 5\text{ m}} = \frac{13}{14} : \frac{11}{14} : -\frac{1}{2} = 13 : 11 : -7$

33

$\because$ The perimeter of Δ ABC = 70 cm.

, $a = 26$ cm.

$\therefore b + c = 70 - 26 = 44$ cm.



Let $b = X$

$\therefore c = 44 - X$

$\therefore (26)^2 = X^2 + (44 - X)^2 - 2X(44 - X)\cos 60°$

$\therefore X^2 - 44X + 420 = 0$ $\qquad \therefore (X - 30)(X - 14) = 0$

$\therefore X = 30$ or $X = 14$ $\qquad \therefore bc = 30 \times 14 = 420$

$\therefore$ The area of Δ ABC

$= \frac{1}{2} b c \sin A = \frac{1}{2} \times 420 \times \sin 60° = 105\sqrt{3}\text{ cm}^2$

34

$\because 34 = 12 + (6 + c) + c \quad \therefore c = 8$ cm. , $b = 14$ cm.

$\because$ The smallest angle is opposite to the smallest side

$\therefore \cos C = \frac{(12)^2 + (14)^2 - (8)^2}{2 \times 12 \times 14}$

$\therefore m(\angle C) = 34° 46' 19''$

$\therefore$ The area of Δ ABC $= \frac{1}{2} \times 12 \times 14 \sin 34° 46' 19''$

$= 47.9\text{ cm}^2$

35

$\because X < 10$

$\therefore$ The greatest angle is opposite to the side of length 14 cm.

$\therefore \cos 120° = \frac{(10)^2 + X^2 - (14)^2}{2 \times 10 \times X}$

$\therefore -\frac{1}{2} = \frac{X^2 - 96}{20X}$ $\qquad \therefore X^2 + 10X - 96 = 0$

$\therefore (X + 16)(X - 6) = 0$

$\therefore X = 6$ or $X = -16$ (refused)

36

$\because \cos C = \frac{a^2 + b^2 - c^2}{2ab}$

$\therefore \cos 120° = \frac{(b-2)^2 + b^2 - (b+2)^2}{2(b-2)(b)}$

$\therefore -\frac{1}{2} = \frac{b^2 - 4b + 4 + b^2 - b^2 - 4b - 4}{2(b-2)(b)}$

$\therefore -1 = \frac{b^2 - 8b}{b(b-2)}$ $\qquad \therefore -1 = \frac{b(b-8)}{b(b-2)}$

$\therefore 2 - b = b - 8$ $\qquad \therefore 2b = 10$

$\therefore b = 5$ $\qquad \therefore a = 3$, $c = 7$

37

$\because (a + b + c)(a + b - c) = 3ab$

$\therefore (a + b)^2 - c^2 = 3ab$

$\therefore a^2 + b^2 + 2ab - c^2 = 3ab$

$\therefore a^2 + b^2 - c^2 = ab$

Dividing both sides by $2ab$ $\quad \therefore \frac{a^2 + b^2 - c^2}{2ab} = \frac{1}{2}$

$\therefore \cos C = \frac{1}{2}$ $\qquad \therefore m(\angle C) = 60°$

38

$\because (a + b + c)(a + b - c) = kab$

$\therefore (a + b)^2 - c^2 = kab$

$\therefore a^2 + b^2 + 2ab - c^2 = kab$

$\therefore a^2 + b^2 - c^2 = ab(k - 2)$

$\therefore \frac{a^2 + b^2 - c^2}{2ab} = \frac{(k-2)ab}{2ab}$ $\qquad \therefore \cos C = \frac{k-2}{2}$

$\because 0 < m(\angle C) < 180°$ $\qquad \therefore -1 < \cos C < 1$

$\therefore -1 < \frac{k-2}{2} < 1$ $\qquad \therefore -2 < k - 2 < 2$

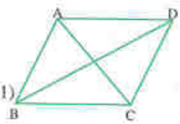
$\therefore 0 < k < 4$ $\qquad \therefore k \in]0 , 4[$

when $k = 1$ $\qquad \therefore \cos C = -\frac{1}{2}$

$\therefore m(\angle C) = 120°$

39

In Δ ABC :



$(AC)^2 = (AB)^2 + (BC)^2$

$- 2(AB)(BC)\cos(\angle ABC)$ (1)

, in Δ BCD :

$(BD)^2 = (BC)^2 + (DC)^2 - 2(BC)(DC)\cos(\angle DCB)$

, $\because$ DC = AB

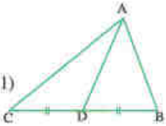
, $\cos(\angle DCB) = -\cos(\angle ABC)$

$\therefore (BD)^2 = (BC)^2 + (AB)^2 + 2\,(BC)\,(AB)\cos(\angle ABC)$ (2)

Adding (1) and (2) :

$\therefore (AC)^2 + (BD)^2 = 2\,(AB)^2 + 2\,(BC)^2$

40



In Δ ABD :

$(AB)^2 = (AD)^2 + (BD)^2$

$\quad - 2\,(AD)\,(BD)\cos(\angle ADB)$ (1)

In Δ ADC :

$(AC)^2 = (AD)^2 + (CD)^2 - 2\,(AD)\,(CD)\cos(\angle ADC)$ (2)

, $\because \cos(\angle ADB) = -\cos(\angle ADC)$, CD = BD

By adding (1) and (2) :

$\therefore (AB)^2 + (AC)^2 = 2\,(AD)^2 + 2\,(BD)^2$

$\therefore (5)^2 + (8)^2 = 2\,(AD)^2 + 2\,(6)^2$

$\therefore AD = \frac{\sqrt{34}}{2}$ cm.

41

$\cot B + \cot C = \frac{\cos B}{\sin B} + \frac{\cos C}{\sin C}$

$= \frac{a^2 + c^2 - b^2}{2\,a\,c\sin B} + \frac{a^2 + b^2 - c^2}{2\,a\,b\sin C}$

$= \frac{a^2 + c^2 - b^2}{4\,\Delta} + \frac{a^2 + b^2 - c^2}{4\,\Delta} = \frac{2\,a^2}{4\,\Delta} = \frac{a^2}{2\,\Delta}$

42

$\because b^2 = c^2 - 2\,c\,a + a^2 + c\,a$

$\therefore b^2 = c^2 + a^2 - c\,a$

, $\because b^2 = c^2 + a^2 - 2\,a\,c\cos B$

$ca = 2\,a\,c\cos B$

$\therefore \cos B = \frac{1}{2}$ $\qquad \therefore m(\angle B) = 60°$

43

In Δ XYZ : $\because X^2 = y^2 + z^2 - 2\,y\,z\cos X$

, $\because X^2 = y^2 + z^2 - y\,z\cot X$

$\therefore 2\cos X = \cot X$ $\qquad \therefore 2\cos X = \frac{\cos X}{\sin X}$

$\therefore 2\cos X\sin X - \cos X = 0$

$\therefore \cos X\,(2\sin X - 1) = 0$

$\therefore \cos X = 0$ $\qquad \therefore m(\angle X) = 90°$

or $\sin X = \frac{1}{2}$ $\qquad \therefore m(\angle X) = 30°$ or $150°$

$\therefore m(\angle X) = 90°$, $30°$ or $150°$

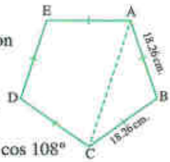
44

$\because \cos B = \frac{a^2 + c^2 - b^2}{2\,a\,c}$ $\qquad \therefore \frac{c}{2\,a} = \frac{a^2 + c^2 - b^2}{2\,a\,c}$

$\therefore c^2 = a^2 + c^2 - b^2$ $\qquad \therefore a^2 = b^2$ $\qquad \therefore a = b$

$\therefore \Delta$ ABC is an isosceles triangle.

45



$\because$ ABCDE is a regular pentagon

$\therefore m(\angle B) = 108°$

$\therefore (AC)^2 = (18.26)^2 + (18.26)^2$

$\quad - 2\,(18.26)\,(18.26) \times \cos 108°$

$\therefore AC \simeq 29.5$ cm.

46

The answer of Ziad is wrong because the sine rule makes the sine of the acute or obtuse angle always positive , although the angle is obtuse , and using the cosine rule in Karim's answer confirm that.

Third Higher skills

1

(1) (b)	(2) (d)	(3) (b)	(4) (b)
(5) (b)	(6) (a)	(7) (c)	(8) (d)
(9) (b)	(10) (b)	(11) (a)	

Instructions to solve 1 :

(1) $\because AB = \sqrt{(3-0)^2 + (4-1)^2} = 3\sqrt{2}$

, $BC = \sqrt{(1-3)^2 + (3-4)^2} = \sqrt{5}$

, $AC = \sqrt{(1-0)^2 + (3-1)^2} = \sqrt{5}$

$\therefore \cos(\angle ACB) = \frac{(\sqrt{5})^2 + (\sqrt{5})^2 - (3\sqrt{2})^2}{2 \times \sqrt{5} \times \sqrt{5}} = \frac{-4}{5}$

(2) * If the length of $\overline{AB}$ is known , workout m ($\angle$ ACB) , m ($\angle$ DCE) and length of $\overline{DE}$

* If the area of Δ ABC , workout m ($\angle$ ACB) , m ($\angle$ DCE) and length of $\overline{DE}$

* If the perimeter of Δ ABC , workout the length of $\overline{AB}$, m ($\angle$ ACB) , m ($\angle$ DCE) and length of $\overline{DE}$

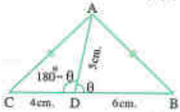
$\therefore$ The length of $\overline{DE}$ can be calculated if any of the previous is known.

(3) $\because (AB)^2 = (5)^2 + (6)^2 - 2 \times 5 \times 6 \cos\theta$ (1)

, $(AC)^2 = (5)^2 + (4)^2 - 2 \times 5 \times 4 \cos(180^\circ - \theta)$ (2)

, $\because AB = AC$

, from (1) , (2) :



$\therefore 25 + 36 - 60 \cos\theta$

$= 25 + 16 + 40 \cos\theta$

$\therefore 100 \cos\theta = 20 \qquad \therefore \cos\theta = \frac{1}{5}$

$\therefore (AB)^2 = 25 + 36 - 60 \times \frac{1}{5} = 49$

$\therefore AB = 7$ cm.

(4) $\because$ ABCD is a cyclic quadrilateral

$\therefore$ coa $A = -\cos C$

$\therefore$ In ΔABD :

$(BD)^2 = (3)^2 + (2)^2 - 2 \times 3 \times 2 \cos A$ (1)

, in ΔCBD :

$(BD)^2 = (3)^2 + (4)^2 - 2 \times 3 \times 4 \cos C$ (2)

From (1) , (2) :

$\therefore 9 + 4 - 12 \cos A = 9 + 16 + 24 \cos A$

$\therefore 36 \cos A = -12 \qquad \therefore \cos A = -\frac{1}{3}$

(5) $\because \overline{AC}$, $\overline{DB}$ are two intersecting chords inside the circle at E

$\therefore AE \times EC = DE \times EB$

$\therefore 3 \times 4 = 2 \times EB \qquad \therefore EB = 6$ cm.

$\therefore \cos(\angle BAE) = \frac{(3)^2 + (8)^2 - (6)^2}{2 \times 3 \times 8} = \frac{37}{48}$

$\therefore m(\angle BAE) = 39^\circ\, 34'$

(6) In ΔABC :

$\because (AB)^2 = (6)^2 + (12)^2 - 2 \times 6 \times 12 \times \cos(0.4)^{rad}$

$\therefore AB \simeq 6.9$ cm. $\quad , \because \theta^{rad} = \frac{L}{r}$

$\therefore$ Length of $\widehat{BD} = 0.4^{rad} \times 6 = 2.4$ cm.

$\therefore$ Perimeter of the shaded part

$= 6.9 + 2.4 + 6 = 15.3$ cm.

(7) $\because \cos A = \frac{b^2 + c^2 - a^2}{2\,bc}$

$\therefore b^2 + c^2 - a^2 = 2\,bc \cos A$

$\therefore (b^2 + c^2 - a^2) \tan A = 2\,bc \cancel{\cos A} \times \frac{\sin A}{\cancel{\cos A}}$

$= 2\,bc \sin A$

$= 4 \times \left(\frac{1}{2}\, b\, c \sin A\right)$

$= 4 \times 12 = 48$

(8) $\left(1 + \frac{a}{c} + \frac{b}{c}\right)\left(1 + \frac{c}{b} - \frac{a}{b}\right)$

$= \left(\frac{c + a + b}{c}\right)\left(\frac{b + c - a}{b}\right)$

$= \frac{(b + c)^2 - a^2}{bc} = \frac{b^2 + c^2 - a^2 + 2\,bc}{bc}$

$= \frac{b^2 + c^2 - a^2}{bc} + 2$

$= 2 \cos A + 2 = 2\left(\frac{1}{2}\right) + 2 = 3$

(9) $\because \frac{a^3 + b^3 + c^3}{a + b + c} = a^2 \quad \therefore a^3 + b^3 + c^3 = a^3 + ba^2 + ca^2$

$\therefore b^3 + c^3 = ba^2 + ca^2$

$\therefore (b + c)(b^2 - bc + c^2) = a^2 (b + c)$

$\therefore a^2 = b^2 - bc + c^2 \quad \therefore b^2 + c^2 - a^2 = bc$

$\therefore \frac{b^2 + c^2 - a^2}{2\,b\,c} = \frac{1}{2} \quad \therefore \cos A = \frac{1}{2}$

$\therefore m(\angle A) = 60^\circ$

(10) Let the length of the small square = L cm.

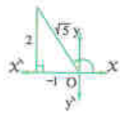
$\therefore$ The length of the big square = 3 L cm.

$\therefore XE = \sqrt{L^2 + L^2}$

$= \sqrt{2}$ L cm.

, $BX = \sqrt{L^2 + (3L)^2}$

$= \sqrt{10}$ L cm.



, BE = 4 L cm.

$\therefore \cos(\angle BXE) = \frac{(\sqrt{2}\,L)^2 + (\sqrt{10}\,L)^2 - (4\,L)^2}{2 \times \sqrt{2}\,L \times \sqrt{10}\,L}$

$= \frac{-4\,L^2}{4\sqrt{5}\,L^2} = \frac{-1}{\sqrt{5}}$

$\therefore \sin(\angle BXE) = \frac{2}{\sqrt{5}}$

(11) $\because BC = \sqrt{(60)^2 + (80)^2} = 100$ cm.

, $\because$ perimeter of ΔACD = perimeter of ΔABD

$\therefore AC + CD = AB + BD$ (because $\overline{AD}$ is common)

$\therefore 80 + CD = 60 + (100 - CD)$

$\therefore 2\,CD = 80 \qquad \therefore CD = 40$ cm. , $BD = 60$ cm.

$\therefore (AD)^2 = (60)^2 + (60)^2 - 2\,(60)\,(60) \cos B$

, $\because \cos B = \frac{60}{100} = 0.6$

$\therefore (AD)^2 = 3600 + 3600 - 2 \times 3600 \times 0.6$

$\therefore AD = 24\sqrt{5}$ cm.

2

$\because \frac{a^2+c^2-b^2}{2a^2c}=\frac{b^2+c^2-a^2}{2b^2c}$

$\therefore a^2b^2+b^2c^2-b^4=a^2b^2+a^2c^2-a^4$

$\therefore b^2c^2-a^2c^2=b^4-a^4$

$\therefore c^2(b^2-a^2)=(b^2-a^2)(b^2+a^2)$

$\therefore (b^2-a^2)\left[c^2-(b^2+a^2)\right]=0$

$\therefore a^2=b^2$ or $c^2=a^2+b^2$ $\qquad \therefore a=b$

i.e. The triangle ABC is isosceles.

or the triangle ABC is right-angled at C

3



$\because m(\angle AEB)=m(\angle AEC)$

$=m(\angle BEC)=\frac{360°}{3}=120°$

In Δ ABE :

$(AB)^2=36+100-2\times 6\times 10\cos 120°$

$\therefore AB=14$ cm.

Let $EC=x$

$\therefore$ In Δ AEC :

$(AC)^2=10^2+x^2-2\times 10\times x\cos 120°$

$\therefore (AC)^2=100+x^2+10x$

In Δ BEC :

$(BC)^2=36+x^2-2\times 6\times x\cos 120°$

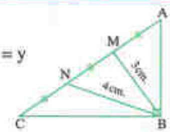
$\therefore (BC)^2=36+x^2+6x$

From Δ ABC : $(AC)^2=(AB)^2+(BC)^2$

$\therefore 100+x^2+10x=196+36+x^2+6x$

$\therefore 4x=132 \qquad \therefore x=33$ cm. $\qquad \therefore EC=33$ cm.

4



Let $AM=MN=NC=x$, $AB=y$

Fom Δ ABM

$\cos A=\frac{y^2+x^2-9}{2xy}$

$\therefore \frac{y}{3x}=\frac{y^2+x^2-9}{2xy}$

$\therefore 2y^2=3y^2+3x^2-27 \qquad \therefore y^2+3x^2=27 \quad (1)$

, in Δ ABN : $\cos A=\frac{y^2+4x^2-16}{2\times y\times 2x}$

$\therefore \frac{y}{3x}=\frac{y^2+4x^2-16}{4xy}$

$\therefore 4y^2=3y^2+12x^2-48 \quad \therefore -y^2+12x^2=48 \; (2)$

By adding (1) , (2) : $\therefore 15x^2=75$

$\therefore x^2=5 \qquad \therefore x=\sqrt{5}$ cm.

By substitution in (1) : $\therefore y^2+15=27$

$\therefore y^2=12 \quad \therefore y=2\sqrt{3}$ cm.

$\therefore BC=\sqrt{\left(3\sqrt{5}\right)^2-12}=\sqrt{33}$ cm.

$\therefore$ The perimeter of Δ ABC = AB + BC + AC

$=2\sqrt{3}+\sqrt{33}+3\sqrt{5}$

$\simeq 16$ cm.

5

(1) $\frac{\cos A}{a}=\frac{b^2+c^2-a^2}{2bac} \quad (1)$

, $\frac{\cos B}{b}=\frac{a^2+c^2-b^2}{2acb} \quad (2)$

, $\frac{\cos C}{c}=\frac{a^2+b^2-c^2}{2abc} \quad (3)$

From (1) , (2) and (3) :

$\therefore \frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}$

$=\frac{b^2+c^2-a^2+a^2+c^2-b^2+a^2+b^2-c^2}{2abc}$

$=\frac{a^2+b^2+c^2}{2abc}$

(2) $\because a^2+b^2+c^2=b^2+c^2-2bc\cos A+c^2+a^2$

$-2ca\cos B+a^2+b^2-2ab\cos C$

$\therefore a^2+b^2+c^2=2a^2+2b^2+2c^2-2bc\cos A$

$-2ca\cos B-2ab\cos C$

$\therefore a^2+b^2+c^2=2(bc\cos A+ca\cos B+ab\cos C)$

(3) $\frac{\tan A}{\tan C}=\frac{\sin A}{\cos A}\times\frac{\cos C}{\sin C}=\frac{\sin A}{\sin C}\times\frac{\cos C}{\cos A}$

$=\frac{a}{c}\times\frac{a^2+b^2-c^2}{2ab}\times\frac{2bc}{b^2+c^2-a^2}$

$=\frac{a^2+b^2-c^2}{b^2+c^2-a^2}$

Exercise 21

First Multiple choice questions

(1) d (2) c (3) c

(4) First : a Second : d (5) a (6) b

(7) a (8) a (9) a (10) b (11) d (12) c

(13) d (14) d (15) c (16) c

UNIT 4

Second Essay questions

1

$m(\angle M) = 180^\circ - (33^\circ\ 16' + 44^\circ\ 19') = 102^\circ\ 25'$

$\therefore \frac{l}{\sin 33^\circ\ 16'} = \frac{17}{\sin 102^\circ\ 25'} = \frac{n}{\sin 44^\circ\ 19'}$

$\therefore l \simeq 9.5$ cm. , $n \simeq 12.2$ cm.

2

$m(\angle A) = 80^\circ$, $m(\angle B) = 40^\circ$, $m(\angle C) = 60^\circ$

$\therefore \frac{BC}{\sin 80^\circ} = \frac{AC}{\sin 40^\circ} = \frac{9}{\sin 60^\circ}$

$\therefore BC \simeq 10.2$ cm. , $AC \simeq 6.7$,

The area of $\Delta ABC = \frac{1}{2}\ BC \times CA \times \sin 60^\circ \simeq 30\ cm^2$.

3

$m(\angle Z) = 180^\circ - (75^\circ\ 12' + 48^\circ\ 15') = 56^\circ\ 33'$

$\therefore \frac{YZ}{\sin 75^\circ\ 12'} = \frac{XZ}{\sin 48^\circ\ 15'} = \frac{40}{\sin 56^\circ\ 33'}$

$\therefore YZ \simeq 46.4$ cm. , $XZ \simeq 35.8$ cm.

The height $= YZ \times \sin Y \simeq 34.6$ cm.

4

$a^2 = (6)^2 + (6)^2 - 2 \times 6 \times 6 \times \cos 153^\circ\ 12'$

$\therefore a \simeq 11.67$ cm.

, $m(\angle B) = m(\angle C) = \frac{180^\circ - 153^\circ\ 12'}{2} = 13^\circ\ 24'$

5

$m(\angle M) = 1.2^{rad} \simeq 68^\circ\ 45'$

$\therefore m^2 = (12.5)^2 + (7.25)^2 - 2 \times 12.5 \times 7.25 \cos 68^\circ\ 45'$

$\therefore m \simeq 11.96$ cm.

$\therefore \cos L = \frac{(11.96)^2 + (7.25)^2 - (12.5)^2}{2 \times 11.96 \times 7.25}$

$\therefore m(\angle L) \simeq 76^\circ\ 53'$ $\qquad \therefore m(\angle N) = 34^\circ\ 22'$

6

$(LN)^2 = (48.5)^2 + (46)^2 - 2 \times 48.5 \times 46 \times (-0.6)$

$\therefore LN \simeq 84.53$ cm. and $\cos L = \frac{(48.5)^2 + (84.53)^2 - (46)^2}{2 \times 48.5 \times 84.53}$

$\therefore m(\angle L) \simeq 25^\circ\ 48'$, $m(\angle M) \simeq 126^\circ\ 52'$

$\therefore m(\angle N) = 27^\circ\ 20'$

7

(1) $\cos A = \frac{(27)^2 + (24)^2 - (15)^2}{2 \times 27 \times 24}$ $\quad \therefore m(\angle A) \simeq 33^\circ\ 33'$

, $\cos B = \frac{(15)^2 + (27)^2 - (24)^2}{2 \times 15 \times 27}$ $\quad \therefore m(\angle B) \simeq 62^\circ\ 11'$

, $m(\angle C) = 180^\circ - (62^\circ\ 11' + 33^\circ\ 33') \simeq 84^\circ\ 16'$

(2) $\cos A = \frac{(35)^2 + (17)^2 - (28)^2}{2 \times 35 \times 17}$

$\therefore m(\angle A) \simeq 52^\circ\ 10'$

, $\cos B = \frac{(28)^2 + (17)^2 - (35)^2}{2 \times 28 \times 17}$

$\therefore m(\angle B) \simeq 99^\circ\ 11'$, $m(\angle C) = 28^\circ\ 39'$

8

$\cos A = \frac{(14)^2 + (15)^2 - (13)^2}{2 \times 14 \times 15}$ $\quad \therefore m(\angle A) \simeq 53^\circ\ 8'$

, $\cos B = \frac{(15)^2 + (13)^2 - (14)^2}{2 \times 15 \times 13}$ $\quad \therefore m(\angle B) \simeq 59^\circ\ 29'$

$\therefore m(\angle C) = 67^\circ\ 23'$

9

$\cos A = \frac{(8)^2 + (4)^2 - (5)^2}{2\ (8)\ (4)}$ $\quad \therefore m(\angle A) \simeq 30^\circ\ 45'$

, $\cos B = \frac{(5)^2 + (4)^2 - (8)^2}{2\ (5)\ (4)}$ $\quad \therefore m(\angle B) \simeq 125^\circ\ 6'$

$\therefore m(\angle C) = 180^\circ - (30^\circ\ 45' + 125^\circ\ 6') \simeq 24^\circ\ 9'$

10

$\frac{10}{\sin A} = \frac{9}{\sin 57^\circ} = \frac{c}{\sin C}$

$\therefore \sin A = \frac{10 \sin 57^\circ}{9}$

$\therefore m(\angle A) = 68^\circ\ 44'$ or $m(\angle A) = 111^\circ\ 16'$

$\therefore m(\angle C) = 54^\circ\ 16'$ or $m(\angle C) = 11^\circ\ 44'$

, $\because \frac{9}{\sin 57^\circ} = \frac{c}{\sin C}$ $\quad \therefore c = 8.7$ cm. or $c = 2.2$ cm.

11

$\frac{4}{\sin 50^\circ} = \frac{3}{\sin B} = \frac{c}{\sin C}$ $\quad \therefore \sin B = \frac{3 \sin 50^\circ}{4}$

$\therefore m(\angle B) = 35^\circ\ 4'$ or $m(\angle B) = 144^\circ\ 56'$ (refused)

$\therefore m(\angle C) = 94^\circ\ 56'$

$\therefore \frac{c}{\sin 94^\circ\ 56'} = \frac{4}{\sin 50^\circ}$ $\quad \therefore c \simeq 5.2$ cm.

12

$\frac{12}{\sin 116^\circ} = \frac{10}{\sin A} = \frac{b}{\sin B}$ $\therefore \sin A = \frac{10 \sin 116^\circ}{12}$

$\therefore m(\angle A) = 48^\circ\ 3\acute{0}$ or $m(\angle A) = 131^\circ\ 3\acute{0}$ (refused)

, $m(\angle B) = 180^\circ - (48^\circ\ 3\acute{0} + 116^\circ) = 15^\circ\ 3\acute{0}$

$\therefore \frac{b}{\sin 15^\circ\ 3\acute{0}} = \frac{12}{\sin 116^\circ}$ $\therefore b = 3.6$ cm.

13

(1) $\because \angle A$ is obtuse , $a > b$

$\therefore$ There is a unique solution.

$\therefore \frac{15}{\sin 120^\circ} = \frac{10}{\sin B}$ $\therefore \sin B = \frac{10 \sin 120^\circ}{15}$

$\therefore m(\angle B) = 35^\circ$

, $m(\angle C) = 180^\circ - (35^\circ + 120^\circ) = 25^\circ$

$\therefore \frac{c}{\sin 25^\circ} = \frac{15}{\sin 120^\circ}$ $\therefore c = 7.3$ cm.

(2) $\because \angle C$ is obtuse , $c > a$

$\therefore$ There is a unique solution.

$\therefore \frac{16}{\sin 115^\circ} = \frac{4}{\sin A}$ $\therefore \sin A = \frac{4 \sin 115^\circ}{16}$

$\therefore m(\angle A) = 13^\circ$

, $m(\angle B) = 180^\circ - (115^\circ + 13^\circ) = 52^\circ$

, $\frac{16}{\sin 115^\circ} = \frac{b}{\sin 52^\circ}$ $\therefore b = 13.9$ cm.

(3) $\because \angle A$ is obtuse , $a < b$

$\therefore$ The conditions don't satisfy the existence of any triangle at all.

(4) $\because \angle A$ is acute , $h = 28 \sin 42^\circ \simeq 18.7$ cm.

, $18.7 < 20 < 28$

$\therefore$ There are two solutions to the triangle.

$\therefore \frac{20}{\sin 42^\circ} = \frac{28}{\sin B}$ $\therefore \sin B = \frac{28 \sin 42^\circ}{20}$

$\therefore m(\angle B) = 70^\circ$ or $m(\angle B) = 110^\circ$

$\therefore m(\angle C) = 180^\circ - (70^\circ + 42^\circ) = 68^\circ$

or $m(\angle C) = 180^\circ - (110^\circ + 42^\circ) = 28^\circ$

, $\because \frac{c}{\sin C} = \frac{20}{\sin 42^\circ}$

$\therefore c = 27.7$ cm. or $c = 14$ cm.

(5) $\because \angle A$ is acute , $h = 7 \sin 60^\circ \simeq 6.1$ cm.

$\therefore a < h$

$\therefore$ The conditions don't satisfy the existence of any triangle at all.

(6) $\because \angle A$ is acute , $a > c$

$\therefore$ There is a unique solution.

$\therefore \frac{12}{\sin 27^\circ} = \frac{7}{\sin C}$ $\therefore m(\angle C) = 15^\circ$

$\therefore m(\angle B) = 180^\circ - (15^\circ + 27^\circ) = 138^\circ$

$\therefore \frac{b}{\sin 138^\circ} = \frac{12}{\sin 27^\circ}$ $\therefore b \simeq 17.7$ cm.

(7) $\because \angle B$ is acute , $h = 4\sqrt{3} \sin 60^\circ = 6$ cm.

$\because h = b$

$\therefore$ There is a unique right-angled triangle.

$\therefore \frac{6}{\sin 60^\circ} = \frac{4\sqrt{3}}{\sin A}$ $\therefore m(\angle A) = 90^\circ$

$\therefore m(\angle C) = 180^\circ - (90^\circ + 60^\circ) = 30^\circ$

$\therefore \frac{c}{\sin 30^\circ} = \frac{6}{\sin 60^\circ}$ $\therefore c \simeq 3.5$ cm.

(8) $\because \angle A$ is acute , $h = 8 \sin 47^\circ \simeq 5.9$ cm.

$\therefore h < a < b$

$\therefore$ There are two solutions to the triangle

$\therefore \frac{6}{\sin 47^\circ} = \frac{8}{\sin B}$

$\therefore m(\angle B) \simeq 77^\circ$ or $m(\angle B) \simeq 103^\circ$

$\therefore m(\angle C) = 180^\circ - (47^\circ + 77^\circ) \simeq 56^\circ$

or $m(\angle C) = 180^\circ - (47^\circ + 103^\circ) \simeq 30^\circ$

, $\because \frac{c}{\sin C} = \frac{6}{\sin 47^\circ}$

$\therefore c \simeq 6.8$ cm. or $c \simeq 4.1$ cm.

14

$\because \angle B$ is acute , $h = 42 \sin 58^\circ \simeq 35.62$ cm.

$\therefore$ There is no solution to the triangle if :

$b < 35.62$ cm.

15

$\because m(\angle A) = 110^\circ$

$\therefore m(\angle B) = m(\angle C) = \frac{180^\circ - 110^\circ}{2} = 35^\circ$

$\therefore \frac{8}{\sin 110^\circ} = \frac{b}{\sin 35^\circ} = \frac{c}{\sin 35^\circ}$ $\therefore b = c \simeq 4.9$ cm.

16

$\because m(\angle B) \simeq 53^\circ\ \grave{8}$ and $m(\angle C) \simeq 22^\circ\ 3\tilde{7}$

$\therefore m(\angle A) = 104^\circ\ 1\tilde{5}$

$\therefore \frac{21}{\sin 104^\circ\ 1\tilde{5}} = \frac{b}{\sin 53^\circ\ \grave{8}} = \frac{c}{\sin 22^\circ\ 3\tilde{7}}$

$\therefore b \simeq 17.3$ cm. and $c \simeq 8.3$ cm.

$C = \frac{1}{2} a\, c \sin B$

$\sin 120^\circ$ $\quad \therefore c = 8$ cm.

$- 2 \times 5 \times 8 \cos 120^\circ$

$\therefore \cos A = \frac{(11.36)^2 + (8)^2 - (5)^2}{2 \times 11.36 \times 8}$

$24''$ $\quad \therefore m(\angle C) = 37^\circ\ 36'$

$0^\circ = 48^\circ$

$0^\circ = 60^\circ$

$0^\circ = 72^\circ$

$\frac{c}{\sin C} = \frac{a + b + c}{\sin A + \sin B + \sin C}$

$= \frac{c}{\sin 72^\circ} = \frac{50}{2.56}$

$\simeq 16.9$ cm. , $c \simeq 18.6$ cm.

$\sin B = 4$ m and $\sin C = 6$ m

$= \frac{a + b + c}{13 \text{ m}}$

$\frac{52}{13}$

16 cm. and $c = 24$ cm.

$\frac{(24)^2 - (12)^2}{16 \times 24}$

$3'$

$\frac{(24)^2 - (16)^2}{12 \times 24}$

$0'$ $\quad \therefore m(\angle C) = 117^\circ\ 17'$

$\overline{C} = 28$

$5''$, $m(\angle B) \simeq 63^\circ\ 14'$

$1'$

$1' \simeq 26$ cm.

$\frac{5}{\sin C} = 16$

, $\because a > c$ $\quad \therefore \angle C$ is acute

$\therefore m(\angle C) \simeq 18^\circ\ 12'\ 36''$

$\therefore m(\angle B) = 180^\circ - (82^\circ + 18^\circ\ 12'\ 36'') = 79^\circ\ 47'\ 24''$

$\therefore b = 16 \sin 79^\circ\ 47'\ 24'' \simeq 15.7$ cm.

22

$\because 2 \times \frac{22}{7} \times r = 44$ $\quad \therefore 2r = 14$

$\therefore \frac{7}{\sin A} = \frac{b}{\sin 40^\circ} = \frac{c}{\sin C} = 14$

$\therefore b = 14 \sin 40^\circ \simeq 9$ cm.

, $\because b > a$ $\quad \therefore \angle A$ is acute

, $\because \sin A = \frac{7}{14}$ $\quad \therefore m(\angle A) = 30^\circ$

$\therefore m(\angle C) = 180^\circ - (40^\circ + 30^\circ) = 110^\circ$

$\therefore c = 14 \sin 110^\circ \simeq 13$ cm.

23

$m(\angle Y) = 180^\circ - (82^\circ + 56^\circ) = 42^\circ$

$\therefore \frac{X}{\sin 82^\circ} = \frac{y}{\sin 42^\circ} = \frac{z}{\sin 56^\circ} = m$

$\therefore X = m \sin 82^\circ$, $y = m \sin 42^\circ$

$\therefore 900 = \frac{1}{2}(m \sin 82^\circ)(m \sin 42^\circ) \sin 56^\circ$

$\therefore m \simeq 57$

"and the negative solution is refused"

$\therefore X \simeq 56$ cm. , $y \simeq 38$ cm. , $z \simeq 47$ cm.

24

$m(\angle C) = 180^\circ - (35^\circ + 75^\circ) = 70^\circ$

$\therefore \frac{a}{\sin 35^\circ} = \frac{b}{\sin 75^\circ} = \frac{c}{\sin 70^\circ} = \frac{a + 3c}{\sin 35^\circ + 3 \sin 70^\circ}$

$= \frac{25}{\sin 35^\circ + 3 \sin 70^\circ}$

$\therefore a \simeq 4.2$ cm. , $b \simeq 7.1$ cm. , $c \simeq 6.9$ cm.

25

$\because \frac{a}{\sin A} = 2r$ $\quad \therefore \frac{13}{\sin A} = 16$

$\therefore m(\angle A) \simeq 54^\circ\ 20'$ or $125^\circ\ 40'$

, when $m(\angle A) = 54^\circ\ 20'$

$\therefore m(\angle C) = 83^\circ\ 40'$

$\therefore \frac{13}{\sin 54^\circ\ 20'} = \frac{b}{\sin 42^\circ} = \frac{c}{\sin 83^\circ\ 40'}$

$\therefore b \simeq 10.71$ cm. , $c \simeq 15.9$ cm.

, when $m(\angle A) = 125^\circ\ 40'$

$\therefore m(\angle C) = 12^\circ\ 20'$

$\therefore \frac{13}{\sin 125^\circ\ 40'} = \frac{b}{\sin 42^\circ} = \frac{c}{\sin 12^\circ\ 20'}$

$\therefore b \simeq 10.71$ cm. , $c \simeq 3.42$ cm.

26

(1) $\cos A = \frac{(7.36)^2 \ldots}{2 \ldots}$

$\therefore m(\angle A) \simeq 24$

, $\cos B = \frac{(3.2)^2 \ldots}{2}$

$\therefore m(\angle B) \simeq 99$

(2) $c^2 = (12)^2 + (21$

$\therefore c \simeq 25.078$ cm

, $\because \frac{25.078}{\sin 95^\circ} = \frac{1}{\sin}$

$\therefore m(\angle A) \simeq 28$

(3) The triangle car

$a + c = b$

(4) $h = 10 \sin 42^\circ =$

$\therefore$ There are tw

$\therefore \frac{7}{\sin 42^\circ} = \frac{10}{\sin}$

$\therefore m(\angle B) \simeq 72$

$\therefore m(\angle C) \simeq 65$

or $m(\angle B) \simeq 10$

$\therefore m(\angle C) \simeq 30$

Third Higher

(1) (a) $\quad$ (2) (c

Instructions solve :

(1) $\because \frac{a}{\sin A} = \frac{b}{\sin B}$

$\therefore \sin B = \frac{40}{39}$

$\therefore$ The conditio

$\therefore$ Number of t

(2) $\because 8 \sin 40^\circ = b$

$\therefore$ Two triangle

(3) $\because 8 \sin 40^\circ = b$

$\therefore$ No triangle

$\frac{+ (6.4)^2 - (3.2)^2}{7.63 \times 6.4}$

° 24̀

$\frac{(6.4)^2 - (7.63)^2}{\times 3.2 \times 6.4}$

° 52̀ ∴ m (∠ C) ≈ 55° 44̀

$^2 - 2 \times 12 \cos 95°$

.

$\frac{2}{A} = \frac{21}{\sin B}$

° 28̀ , m (∠ B) ≈ 56° 32̀

not be formed , because :

6.69 cm. ∴ h < a < b

solutions to the triangle.

$3 = \frac{c}{\sin C}$

° 55̀

° 5̀ , c = 9.5 cm.

7° 5̀

° 55̀ , c = 5.4 cm.

skills

) (3) (a)

$\therefore \frac{3}{\left(\frac{5}{13}\right)} = \frac{8}{\sin B}$

refused)

ns does not satisfy any triangle

iangles = zero

sin A ∴ b sin A < a < b

s can be drawn.

sin A ∴ a < b sin A

an be drawn.

Answers of Life Applications on Unit Four

1

$\because$ m (∠ B) = 180° − (108° + 52°) = 20°

$\therefore \frac{AC}{\sin 20°} = \frac{160}{\sin 52°} = \frac{BC}{\sin 108°}$

∴ AC ≈ 69 km. , BC ≈ 193 km.

i.e. The distance between A and C ≈ 69 km.

and the distance between B and C ≈ 193 km.

2

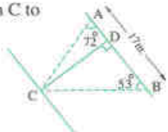
In Δ ABC :

(1) m (∠ C) = 180° − (53° + 72°) = 55°

$\therefore \frac{17}{\sin 55°} = \frac{AC}{\sin 53°}$ ∴ AC ≈ 16.57 m.

∴ The distance between the two signs A and C ≈ 17 m.

(2) Draw a perpendicular from C to $\overline{AB}$, and it is $\overline{CD}$



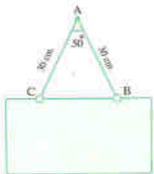
∴ In Δ ACD :

$\frac{CD}{\sin 72°} = \frac{16.57}{\sin 90°}$

∴ CD ≈ 15.76 m.

∴ The distance between the two edges ≈ 15.76 m.

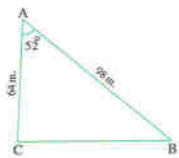
3



$(BC)^2 = (30)^2 + (30)^2 - 2\,(30)\,(30) \cos 50°$

∴ BC ≈ 25 cm.

4



$(BC)^2 = (98)^2 + (64)^2 - 2\,(98)\,(64) \cos 52°$

∴ BC ≈ 77.3 m.

∴ The length of the fence

= 98 + 64 + 77.3 ≈ 239 m.

17

The area of $\Delta ABC = \frac{1}{2} a\, c \sin B$

$\therefore 10\sqrt{3} = \frac{1}{2} \times 5\, c \sin 120^\circ \qquad \therefore c = 8$ cm.

$\therefore b^2 = (5)^2 + (8)^2 - 2 \times 5 \times 8 \cos 120^\circ$

$\therefore b = 11.36$ cm. $\qquad \therefore \cos A = \frac{(11.36)^2 + (8)^2 - (5)^2}{2 \times 11.36 \times 8}$

$\therefore m(\angle A) \simeq 22^\circ\, 24'$ $\qquad \therefore m(\angle C) = 37^\circ\, 36'$

18

$m(\angle A) = \frac{4}{15} \times 180^\circ = 48^\circ$

$m(\angle B) = \frac{5}{15} \times 180^\circ = 60^\circ$

$m(\angle C) = \frac{6}{15} \times 180^\circ = 72^\circ$

$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \frac{a+b+c}{\sin A + \sin B + \sin C}$

$\therefore \frac{a}{\sin 48^\circ} = \frac{b}{\sin 60^\circ} = \frac{c}{\sin 72^\circ} = \frac{50}{2.56}$

$\therefore a \simeq 14.5$ cm. , $b \simeq 16.9$ cm. , $c \simeq 18.6$ cm.

19

Let $\sin A = 3\,m$, $\sin B = 4\,m$ and $\sin C = 6\,m$

$\therefore \frac{a}{3\,m} = \frac{b}{4\,m} = \frac{c}{6\,m} = \frac{a+b+c}{13\,m}$

$\therefore \frac{a}{3} = \frac{b}{4} = \frac{c}{6} = \frac{52}{13}$

$\therefore a = 12$ cm. , $b = 16$ cm. and $c = 24$ cm.

$\therefore \cos A = \frac{(16)^2 + (24)^2 - (12)^2}{2 \times 16 \times 24}$

$\therefore m(\angle A) \simeq 26^\circ\, 23'$

$\therefore \cos B = \frac{(12)^2 + (24)^2 - (16)^2}{2 \times 12 \times 24}$

$\therefore m(\angle B) \simeq 36^\circ\, 20'$ $\qquad \therefore m(\angle C) = 117^\circ\, 17'$

20

$\frac{21}{\sin A} = \frac{25}{\sin B} = \frac{c}{\sin C} = 28$

$\therefore m(\angle A) \simeq 48^\circ\, 35'$, $m(\angle B) \simeq 63^\circ\, 14'$

$\therefore m(\angle C) \simeq 68^\circ\, 11'$

$\therefore c = 28 \sin 68^\circ\, 11' \simeq 26$ cm.

21

$\frac{a}{\sin 82^\circ} = \frac{b}{\sin B} = \frac{5}{\sin C} = 16$

$\therefore a \simeq 15.8$ cm. $\qquad$, $\because a > c$ $\qquad \therefore \angle C$ is acute

, $\because \sin C = \frac{5}{16}$ $\qquad \therefore m(\angle C) \simeq 18^\circ\, 12'\, 36''$

$\therefore m(\angle B) = 180^\circ - (82^\circ + 18^\circ\, 12'\, 36'') = 79^\circ\, 47'\, 24''$

$\therefore b = 16 \sin 79^\circ\, 47'\, 24'' \simeq 15.7$ cm.

22

$\because 2 \times \frac{22}{7} \times r = 44$ $\qquad \therefore 2\,r = 14$

$\therefore \frac{7}{\sin A} = \frac{b}{\sin 40^\circ} = \frac{c}{\sin C} = 14$

$\therefore b = 14 \sin 40^\circ \simeq 9$ cm.

, $\because b > a$ $\qquad \therefore \angle A$ is acute

, $\because \sin A = \frac{7}{14}$ $\qquad \therefore m(\angle A) = 30^\circ$

$\therefore m(\angle C) = 180^\circ - (40^\circ + 30^\circ) = 110^\circ$

$\therefore c = 14 \sin 110^\circ \simeq 13$ cm.

23

$m(\angle Y) = 180^\circ - (82^\circ + 56^\circ) = 42^\circ$

$\therefore \frac{X}{\sin 82^\circ} = \frac{y}{\sin 42^\circ} = \frac{z}{\sin 56^\circ} = m$

$\therefore X = m \sin 82^\circ$, $y = m \sin 42^\circ$

$\therefore 900 = \frac{1}{2}(m \sin 82^\circ)(m \sin 42^\circ) \sin 56^\circ$

$\therefore m \simeq 57$

"and the negative solution is refused"

$\therefore X \simeq 56$ cm. , $y \simeq 38$ cm. , $z \simeq 47$ cm.

24

$m(\angle C) = 180^\circ - (35^\circ + 75^\circ) = 70^\circ$

$\therefore \frac{a}{\sin 35^\circ} = \frac{b}{\sin 75^\circ} = \frac{c}{\sin 70^\circ} = \frac{a + 3\,c}{\sin 35^\circ + 3 \sin 70^\circ}$

$= \frac{25}{\sin 35^\circ + 3 \sin 70^\circ}$

$\therefore a \simeq 4.2$ cm. , $b \simeq 7.1$ cm. , $c \simeq 6.9$ cm.

25

$\because \frac{a}{\sin A} = 2\,r$ $\qquad \therefore \frac{13}{\sin A} = 16$

$\therefore m(\angle A) \simeq 54^\circ\, 20'$ or $125^\circ\, 40'$

, when $m(\angle A) = 54^\circ\, 20'$

$\therefore m(\angle C) = 83^\circ\, 40'$

$\therefore \frac{13}{\sin 54^\circ\, 20'} = \frac{b}{\sin 42^\circ} = \frac{c}{\sin 83^\circ\, 40'}$

$\therefore b \simeq 10.71$ cm. , $c \simeq 15.9$ cm.

, when $m(\angle A) = 125^\circ\, 40'$

$\therefore m(\angle C) = 12^\circ\, 20'$

$\therefore \frac{13}{\sin 125^\circ\, 40'} = \frac{b}{\sin 42^\circ} = \frac{c}{\sin 12^\circ\, 20'}$

$\therefore b \simeq 10.71$ cm. , $c \simeq 3.42$ cm.

26

(1) $\cos A = \frac{(7.36)^2 + (6.4)^2 - (3.2)^2}{2 \times 7.63 \times 6.4}$

$\therefore m(\angle A) = 24° 24'$

, $\cos B = \frac{(3.2)^2 + (6.4)^2 - (7.63)^2}{2 \times 3.2 \times 6.4}$

$\therefore m(\angle B) \simeq 99° 52'$ $\therefore m(\angle C) = 55° 44'$

(2) $c^2 = (12)^2 + (21)^2 - 2 \times 12 \cos 95°$

$\therefore c \simeq 25.078$ cm.

, $\because \frac{25.078}{\sin 95°} = \frac{12}{\sin A} = \frac{21}{\sin B}$

$\therefore m(\angle A) \simeq 28° 28'$, $m(\angle B) = 56° 32'$

(3) The triangle can not be formed , because :

$a + c = b$

(4) $h = 10 \sin 42° \simeq 6.69$ cm. $\therefore h < a < b$

$\therefore$ There are two solutions to the triangle.

$\therefore \frac{7}{\sin 42°} = \frac{10}{\sin B} = \frac{c}{\sin C}$

$\therefore m(\angle B) \simeq 72° 55'$

$\therefore m(\angle C) \simeq 65° 5'$, $c = 9.5$ cm.

or $m(\angle B) \simeq 107° 5'$

$\therefore m(\angle C) \simeq 30° 55'$, $c = 5.4$ cm.

Third Higher skills

(1) (a) (2) (c) (3) (a)

Instructions solve :

(1) $\because \frac{a}{\sin A} = \frac{b}{\sin B}$ $\therefore \frac{3}{\left(\frac{5}{13}\right)} = \frac{8}{\sin B}$

$\therefore \sin B = \frac{40}{39}$ (refused)

$\therefore$ The conditions does not satisfy any triangle

$\therefore$ Number of triangles = zero

(2) $\because 8 \sin 40° = b \sin A$ $\therefore b \sin A < a < b$

$\therefore$ Two triangles can be drawn.

(3) $\because 8 \sin 40° = b \sin A$ $\therefore a < b \sin A$

$\therefore$ No triangle can be drawn.

Answers of Life Applications on Unit Four

1

$\because m(\angle B) = 180° - (108° + 52°) = 20°$

$\therefore \frac{AC}{\sin 20°} = \frac{160}{\sin 52°} = \frac{BC}{\sin 108°}$

$\therefore AC \simeq 69$ km. , $BC \simeq 193$ km.

i.e. The distance between A and C = 69 km.

and the distance between B and C = 193 km.

2

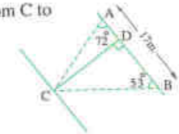
In ΔABC :

(1) $m(\angle C) = 180° - (53° + 72°) = 55°$

$\therefore \frac{17}{\sin 55°} = \frac{AC}{\sin 53°}$ $\therefore AC \simeq 16.57$ m.

$\therefore$ The distance between the two signs A and C $\simeq 17$ m.

(2) Draw a perpendicular from C to $\overline{AB}$, and it is $\overline{CD}$



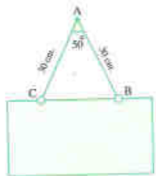
$\therefore$ In ΔACD :

$\frac{CD}{\sin 72°} = \frac{16.57}{\sin 90°}$

$\therefore CD \simeq 15.76$ m.

$\therefore$ The distance between the two edges $\simeq 15.76$ m.

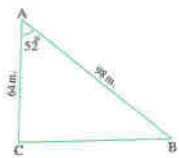
3



$(BC)^2 = (30)^2 + (30)^2 - 2\,(30)\,(30) \cos 50°$

$\therefore BC \simeq 25$ cm.

4



$(BC)^2 = (98)^2 + (64)^2 - 2\,(98)\,(64) \cos 52°$

$\therefore BC \simeq 77.3$ m.

$\therefore$ The length of the fence $= 98 + 64 + 77.3 \simeq 239$ m.

5



$(AB)^2 = (24)^2 + (10)^2$

$- 2 (24) (10) \cos 75°$

$\therefore AB \simeq 23.5$ km.

6

$(AC)^2 = (9)^2 + (15)^2 - 2 (9) (15) \cos 120°$

$\therefore AC = 21$ km.

, $\because$ time $= \frac{\text{distance}}{\text{velocity}}$

$\therefore t_1 = \frac{15}{36} \times 60 = 25$ minutes.

, $t_2 = \frac{9}{36} \times 60 = 15$ minutes.

, $t_3 = \frac{21}{42} \times 60 = 30$ minutes.

$\therefore$ The total time $= 25 + 15 + 30 = 70$ minutes.

7

$\because$ The octagon is regular.

$\therefore$ The measure of each angle of it $= 135°$

, $m (\angle BHC) = m (\angle CHD) = \frac{135°}{6} = 22.5°$

$\therefore (HB)^2 = (6)^2 + (6)^2 - 2 (6) (6) \cos 135°$

$\therefore HB \simeq 11.1$ metre.

, $(HC)^2 = (11.1)^2 + (6)^2 - 2 (11.1) (6) \cos 112.5°$

$\therefore HC \simeq 14.5$ metre.

, $(HD)^2 = (14.5)^2 + (6)^2 - 2 (14.5) (6) \cos 90°$

$\therefore HD \simeq 15.7$ metre.